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Comparison Four Methods for Estimating the Fatigue Life Distribution Parameters through Simulation

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Abstract: ***In this paper we estimated the parameters for two parameter-Birnbaum−Saunders distribution. The paper has estimated the parameters (shape and scale) using (Maximum likelihood, Modified moment, Lindley (1980) Bayesian approximation technique and Shrinkage Bayesian) methods, and then computing the value of the above-mentioned. We consider the Bayesian estimators for the unknown parameters of the Birnbaum_Saunders distribution under the reference prior. The Bayesian estimators cannot be obtained in closed forms. An approximate Bayesian approach is proposed using the idea of Lindley to obtain the Bayesian estimators. We then calculated and estimated all previous parameters, and compared the numerical estimation using statistical indicators mean absolute percentage error among the four considered estimation methods. Results are compared using Monte Carlo simulations studies carried out showed that the Shrinkage method gave us the best estimator.***

Keywords: Birnbaum–Saunders distribution, Maximum likelihood estimator, Modified moment, Bayes estimates, Lindley's approximation , Shrinkage Bayesian estimator method, Mean absolute percentage error and Monte Carlo simulations

1. Introduction

"The two- parameter Birnbaum – Saunders distribution was originally proposed by Birnbaum – Saunders as a fatigue time distribution for fatigue failure caused under cyclic loading"; [5]." It was also assumed that the failure is due to the development and growth of a dominant crack past the critical value ";[14].

The random variable T is said to follow a BS distribution with parameters α and β, denoted as BS(α, β)
if its cumulative distribution function (CDF) is given by

$$F(t,\alpha,\beta) = \Phi\left(\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right]\right) \quad , \ t > 0,\ \alpha,\beta > 0 \qquad (1)$$

where Φ(.)is the standard normal CDF, α and β are the shape and the scale parameters, respectively. Additionally β is the median of the distribution: F $F_T(\beta)$ = Φ(0) =0.5. It is noteworthy that the reciprocal property holds for the BS distribution: $T^{-1} \sim (\alpha, \beta^{-1})$.

The corresponding density function of (1) is

$$f(t;,\alpha,\beta) = \begin{cases} \dfrac{t^{\frac{-3}{2}}(t+\beta)\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right]}{2\alpha\sqrt{\beta}\sqrt{2\pi}} & ,0 < t < \infty \\ 0 & o.w \end{cases} \qquad (2)$$

It helps to derive different moments of the Birnbaum-Saunders distribution the r^{th} moment of T is

$$E[T^r] = \beta^r \sum_{j=0}^{r} \binom{2r}{2j} \sum_{i=0}^{j} \binom{j}{i} \frac{(2r-2j+2i)!}{2^{r-j+i}(r-j+i)!} \left[\frac{\alpha}{2}\right]^{2r-2j+2i} \qquad r = 1,2 \ldots. \quad (3)$$

The Moments and measures for Birnbaum –Saunders distribution can be Expressed in explicit form equation (3) as follow:

$$E(T) = \beta\left(1+\frac{\alpha^2}{2}\right) \qquad (4)$$

$$V(T) = (\alpha\beta)^2\left(1+\frac{5}{4}\alpha^2\right) \qquad (5)$$

$$CS(T) = \frac{16\,\alpha^2(11\alpha^2+6)}{(5\alpha^2+4)^3} \qquad (6)$$

$$CK(T) = 3 + \frac{6\,\alpha^2(93\alpha^2+41)}{(5\alpha^2+4)^2} \qquad (7)$$

Where E(T) ,V(T) , CS(T) , CK(T) and CV(T) are the expected value ,variance ,coefficient of skewness, coefficient of kurtosis and coefficient of variation respectively.

If T is a Birnbaum-Saunders distribution, i ;e .BS($\alpha;\ \beta$), then $T^{-1}$is also α Birnbaum-Saunders with parameters (α , β^{-1}).

The above observation is very useful. Immediately we obtain

$$E(T^{-1}) = \beta^{-1}\left(1+\frac{1}{2}\alpha^2\right) \qquad (9)$$

$$V(T^{-1}) = \alpha^2\beta^{-2}\left(1+\tfrac{5}{4}\alpha^2\right) \qquad (10)$$

"A more general derivation was provided by Desmond (1985) based on a Biological model";[5]. Desmond (1985) also strengthened the physical justification for the use of this distribution by relaxing the assumption made by Birnbaum and Saunders (1969). "Desmond (1986) investigated the relationship between Birnbaum- Saunders distribution and the Inverse-Gaussian distribution";[9].

Some recent works on Birnbaum - Saunders distribution can be found in Barndorff – Nielsen (1986, 1991), Chang and Tang (1993, 1994), Johnson et al (1995), Rieck (1995, 1999), Reid (1996), Dupuis and Mills (1998), Fraser et al

(1999), Upadhyay (2000), Tsionas (2001), Wu et al (2005), Owen (2005) and Ng et al (2003, 2006).

"Padgett (1982) carried out Bayesian estimation of reliability function when both parameters of the B-S distribution are unknown";[17]. He suggested the use of a gamma prior in connection with reliability estimation. "Upadhyay (2000) successfully used Gibbs sampler to analyze the posterior surfaces on three parameter B-S distribution which at times are difficult when using non-sample based approaches";[22], while Tsionas (2001) gave a posterior analysis of a linear regression model with disturbances from a logarithmic B-S distribution. "Tsionas (2001) also noted that, posterior distributions cannot be easily analyzed using analytical techniques and thus proposed posterior simulation methods using Metropolis algorithm"; [21].

Although some Bayesian estimation has been carried out by Padgett (1982), Upadhyay (2000) and Tsionas (2001), it is clearly noted that Bayesian estimation for the parameters or a function of the parameters involves evaluation of a ratio of two intractable integrals. "In this article, we apply the Lindley (1980) Bayesian approximation method to evaluate the Bayes estimates of the parameters for the Birnbaum – Saunders distribution";[13] .This paper is concerned using the two-parameter Birnbaum–Saunders model. Furthermore, we will estimate the parameters of the mentioned model using four methods—Maximum likelihood estimator, Modified Moment estimator Lindley (1980) Bayesian approximation technique estimator, and shrinkage Bayesian estimator—depending upon the iterative numerical method (Newton–Raphson method), and then utilized these parameters. Finally the four proposed estimators were compared using the mean absolute percentage error with respect to recommend the best estimator.

2. Estimation Methods

In this section, we introduce four methods for estimation the parameters of the Birnbaum–Saunders distribution.

2.1 Maximum likelihood estimator method (ML)

The idea behind the maximum likelihood approach to fitting a statistical distribution to a data set is to find the parameters of the distribution that maximize the likelihood of having observed the data. "Assuming the data are independent of each other, the likelihood of the data is the product of the likelihoods of each datum"; [4] [5].

Thus, the likelihood function of two-parameter Birnbaum–Saunders is:

$$L = \prod_{i=1}^{n} f(t_i, \alpha, \beta)$$

$$L = \frac{\prod_{i=1}^{n} t_i^{\frac{-3}{2}}(t+\beta)}{2^n \alpha^n \beta^{\frac{n}{2}} (2\pi)^{\frac{n}{2}}} exp - \left(\frac{1}{2\alpha^2}\sum_{i=1}^{n}(\frac{t_i}{\beta} - 2 + \frac{\beta}{t_i})\right) \quad (11)$$

Taking the Natural for equation (11)logarithm for the above likelihood function, so we get the following:

$$\mathrm{Ln}\,L = -n\,\mathrm{Ln}\,2 - n\,\mathrm{Ln}\,\alpha - \frac{n}{2}\mathrm{Ln}\,\beta - \frac{n}{2}\mathrm{Ln}(2\pi) - \frac{3}{2}\sum_{i=1}^{n}\mathrm{Ln}t_i + \sum_{i=1}^{n}\mathrm{Ln}(t_i+\beta) - \left(\frac{1}{2\alpha^2}\sum_{i=1}^{n}\left(\frac{t_i}{\beta}\right) + \left(\frac{\beta}{t_i}\right) - 2\right) \quad (12)$$

The partial derivative for equation (12) with respect to unknown parameters α and β, respectively, are:

$$L_{10} = \frac{\partial \mathrm{Ln}\,L}{\partial \alpha} = \frac{-n}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{n}\left(\frac{t_i}{\beta} - 2 + \frac{\beta}{t_i}\right) \quad (13)$$

Equating equation (13) to zero to solve this equation: $L_{10} = 0$

$$\frac{-n}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{n}\left(\frac{t_i}{\beta} - 2 + \frac{\beta}{t_i}\right) = 0$$

Where $s = \frac{1}{n}\sum_{i=1}^{n} t_i$ is the arithmetic mean and $r = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{t_i}\right)^{-1}$ is the is the harmonic mean.

$$\hat{\alpha} = \left(\frac{s}{\hat{\beta}} + \frac{\hat{\beta}}{r} - 2\right)^{\frac{1}{2}} \quad (14)$$

Also, the partial derivative for log-likelihood w.r.t. β , is as follows:

$$L_{02} = \frac{\partial Ln\,L}{\partial \beta} = \frac{-n}{2\beta} + \sum_{i=1}^{n}\frac{1}{t_i+\beta} + \frac{1}{2\hat{\alpha}^2}\sum_{i=1}^{n}\left(\frac{t_i}{\beta^2} - \frac{1}{t_i}\right) \quad (15)$$

Equating equation (15) to zero to solve this equation: $L_{02} = 0$

$$\frac{-n}{2\hat{\beta}} + \sum_{i=1}^{n}\frac{1}{t_i+\hat{\beta}} + \frac{1}{2\hat{\alpha}^2}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}^2} - \frac{1}{t_i}\right) = 0 \quad (16)$$

From equations (14) and (16) we can write the formula as:

$$\hat{\beta}^2 - \hat{\beta}\left[\frac{2n}{\sum_{i=1}^{n} t_i^{-1}} + \frac{n}{\sum_{i=1}^{n}(t_i+\hat{\beta})^{-1}}\right] + \frac{n}{\sum_{i=1}^{n} t_i^{-1}}\left[\frac{1}{n}\sum_{i=1}^{n} t_i + \frac{n}{\sum_{i=1}^{n}(t_i+\hat{\beta})^{-1}}\right] = 0 \quad (17)$$

$$\hat{\beta}^2 - \hat{\beta}\left[2r + k(\hat{\beta})\right] + r\left[s + k(\hat{\beta})\right] = 0 \quad (18)$$

Where, $K(\hat{\beta}) = \left(\frac{1}{n}\sum_{i=1}^{n}(t_i+\hat{\beta})^{-1}\right)^{-1}$; $\beta \geq 0$

Since (18) is a non-linear equation in $\hat{\beta}$, we shall use the Newton-Raphson method to solve for $\hat{\beta}$ (see Ng *et al*, 2003). Suppose that $g(\beta) = 0$ where g is a function then

$$g(\beta) = \beta^2 - \beta[2r + k(\beta)] + r[s + k(\beta)] = 0 \quad (19)$$

If Newton–Raphson method is instead used to solve Eq. (20)we would require

$$g'(\beta) = 2\beta - 2r + (r-\beta)K'(\beta) - K(\beta). \quad (20)$$

$$K'(\beta) = [K(\beta)]^2\left[\frac{1}{n}\sum_{i=1}^{n}(t_i+\beta)^{-1}\right]^{-2} \qquad for \beta \geq 0$$

then the Newton iteration procedure

$$\beta_{n+1} = \beta_n - \frac{g(\beta)}{g'(\beta)} \qquad n = 1,2,\ldots\ldots \quad (21)$$

Condition stop when $|\beta_{i+1} - \beta_i| < \epsilon$ and $\epsilon = 0$

The two – functions $f_1(\alpha)$ and $f_2(\beta)$ are the first derivative of equation (12) with respect to unknown parameters α and β and respectively .

$$f_1(\hat{\alpha}) = L_{10} = \frac{-n}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}} - 2 + \frac{\hat{\beta}}{t_i}\right) \quad (22)$$

$$f_2(\hat{\beta}) = L_{01} = \frac{-n}{2\hat{\beta}} + \sum_{i=1}^{n}\frac{1}{t_i+\hat{\beta}} + \frac{1}{2\hat{\alpha}^2}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}^2} - \frac{1}{t_i}\right) \quad (23)$$

The Jacobean matrix J_{k_1} is the first derivative for eachfunction of L_{10} and L_{01} with respect to $\hat{\alpha}$ and $\hat{\beta}$ or it is the second derivative of the equation (12) to the two – parameters.

$$J_{k_1} = \begin{bmatrix} \frac{\partial f_1(\hat{\alpha})}{\partial\hat{\alpha}} & \frac{\partial f_1(\hat{\alpha})}{\partial\hat{\beta}} \\ \frac{\partial f_2(\hat{\beta})}{\partial\hat{\alpha}} & \frac{\partial f_2(\hat{\beta})}{\partial\hat{\beta}} \end{bmatrix} \quad (24)$$

$$\frac{\partial f_1(\alpha)}{\partial\alpha} = \frac{n}{\alpha^2} - \frac{3}{\alpha^4}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}} - 2 + \frac{\hat{\beta}}{t_i}\right) \quad (25)$$

$$\frac{\partial f_2(\hat{\beta})}{\partial\hat{\beta}} = \frac{n}{2\hat{\beta}^2} - \sum_{i=1}^{n}\frac{t_i}{(t_i+\hat{\beta})^2} - \frac{1}{\hat{\alpha}^2\hat{\beta}^3}\sum_{i=1}^{n}t_i \quad (26)$$

$$\frac{\partial f_1(\hat{\alpha})}{\partial\hat{\beta}} = -\frac{1}{\hat{\alpha}^3}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}^2} + \frac{1}{t_i}\right) \quad (27)$$

$$\frac{\partial f_2(\hat{\beta})}{\partial\hat{\alpha}} = -\frac{1}{\hat{\alpha}^3}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}^2} - \frac{1}{t_i}\right) \quad (28)$$

$$L_{10} = \frac{-n}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}} - 2 + \frac{\hat{\beta}}{t_i}\right) \quad (29)$$

$$L_{01} = \frac{-n}{2\hat{\beta}} + \sum_{i=1}^{n}\frac{1}{t_i+\hat{\beta}} + \frac{1}{2\hat{\alpha}^2}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}^2} - \frac{1}{t_i}\right) \quad (30)$$

$$L_{20} = \frac{n}{\alpha^2} - \frac{3}{\alpha^4}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}} - 2 + \frac{\hat{\beta}}{t_i}\right) \quad (31)$$

$$L_{02} = \frac{n}{2\hat{\beta}^2} - \sum_{i=1}^{n}\frac{t_i}{(t_i+\hat{\beta})^2} - \frac{1}{\hat{\alpha}^2\hat{\beta}^3}\sum_{i=1}^{n}t_i \quad (32)$$

$$L_{11} = -\frac{1}{\hat{\alpha}^3}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}^2} + \frac{1}{t_i}\right) \quad (33)$$

$$L_{21} = -\frac{3}{\hat{\alpha}^4}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}^2} - \frac{1}{t_i}\right) \quad (34)$$

$$L_{12} = \frac{2}{\alpha^3\hat{\beta}^2}\sum_{i=1}^{n}t_i \quad (35)$$

$$L_{03} = -\frac{n}{\beta^3} + \sum_{i=1}^{n}\frac{1}{(t_i+\beta)^3} + \frac{3}{\alpha^2\beta^4}\sum_{i=1}^{n}t_i \quad (36)$$

$$L_{30} = \frac{2n}{\hat{\alpha}^3} + \frac{12}{\hat{\alpha}^5}\sum_{i=1}^{n}\left(\frac{t_i}{\hat{\beta}} - 2 + \frac{\hat{\beta}}{t_i}\right) \quad (37)$$

The Jacobean matrix in maximum likelihood method estimator must be anon – singular symmetric matrix in this procedure because depending upon the first derivatives, so its inverse can be found.

$$\begin{bmatrix}\alpha_{K+1} \\ \beta_{k+1}\end{bmatrix} = \begin{bmatrix}\alpha_K \\ \beta_k\end{bmatrix} - J_{k_i}^{-1}\begin{bmatrix}f_1(\alpha) \\ f_2(\beta)\end{bmatrix} \quad i = 1,2 \ldots.. \quad (38)$$

The absolute value for the difference between the new founded values with the initial value is the error term, it must be a symbol by ϵ , which is a very small value and assumed.

Then, error term is formulated as:

$$\begin{bmatrix}\epsilon_{K+1}(\alpha) \\ \epsilon_{k+1}(\beta)\end{bmatrix} = \left\| \begin{bmatrix}\alpha_{K+1} \\ \beta_{k+1}\end{bmatrix} - \begin{bmatrix}\alpha_K \\ \beta_k\end{bmatrix} \right\|$$

where α_K and β_k are the initial values which are assumed.

The asymptotic distribution for MLE of α and β is normal with mean α and β and the covariance matrix is given by the inverse of the expectation of the Fisher information .i.e. for a sample of size n we have

$$\begin{pmatrix}\hat{\alpha} \\ \hat{\beta}\end{pmatrix} \overset{a}{\sim} N\left(\begin{pmatrix}\alpha \\ \beta\end{pmatrix}, \frac{-1}{n}\left(E\begin{pmatrix}\frac{\partial^2 lnL}{\partial\alpha^2} & \frac{\partial^2 lnL}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 lnL}{\partial\beta\,\partial\alpha} & \frac{\partial^2 lnL}{\partial\beta^2}\end{pmatrix}\right)^{-1}\right) \quad (39)$$

Can compute the covariance matrix as

$$\frac{-1}{n}\left(E\begin{pmatrix}\frac{\partial^2 lnL}{\partial\alpha^2} & \frac{\partial^2 lnL}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 lnL}{\partial\beta\,\partial\alpha} & \frac{\partial^2 lnL}{\partial\beta^2}\end{pmatrix}\right)^{-1} = \begin{bmatrix}\frac{\hat{\alpha}^2}{2n} & 0 \\ 0 & \frac{\hat{\beta}^2\ \hat{\alpha}^2}{n(1+\alpha(2\pi)^{-1/2}\,h(\alpha))}\end{bmatrix} \quad (40)$$

Where $h(\alpha) = \alpha\sqrt{\pi/2} - \pi e^{2/\alpha}[1 - \Phi(2/\alpha)]$, $\Phi(.)$ is normal C.D.F.

The asymptotic variance of $\hat{\alpha}$ is given as

$$\hat{v}ar(\hat{\alpha}) = \frac{\hat{\alpha}^2}{2n} \quad (41)$$

While that of $\hat{\beta}$ is given by

$$\hat{v}ar(\hat{\beta}) = \frac{\hat{\beta}^2\ \hat{\alpha}^2}{n(1+\alpha(2\pi)^{-1/2}\,h(\alpha))} \quad (42)$$

2.2 Modified Moment Estimators (MO)

For the usual moment estimators in a two-parameter case, the first and second population moments are equated with the corresponding sample moments. In this case, the sample mean and the sample variance can be equated to the right-hand sides of (4) and (5), respectively, and the corresponding moment estimators of α and βcan then be obtained as solutions of α and β to these equations. It can be easily seen from these equations that if the sample coefficient of variation is greater than $\sqrt{5}$, then the moment estimators do not exist. If the sample coefficient of variation is less than $\sqrt{5}$, the moment estimators exist; however, the moment estimator of β may not be unique." Instead of using (4) and (5), we propose to use (4) and (9) and equate them with the corresponding sample estimates to obtain the MMEs ";[14]. In this case, we have the following two moment equations:

$$s = \beta\left(1 + \frac{1}{2}\alpha^2\right) \quad (43)$$

$$r^{-1} = \beta^{-1}\left(1 + \frac{1}{2}\alpha^2\right) \quad (44)$$

Solving Esq. (43) And (44) for α and β , we obtain the MMEs for α and β denoted by $\hat{\alpha}_{MO}$ and $\hat{\beta}_{MO}$ as

$$\hat{\alpha}_{MO} = \left\{2\left[\left(\frac{s}{r}\right)^{1/2} - 1\right]\right\}^{1/2} \quad , \quad \hat{\beta}_{MO} = (sr)^{1/2}$$

2.3 Bayes estimators (BS)

We shall use a diffuse prior $h(\theta) = h(\alpha, \beta) \propto \frac{1}{\alpha\beta}$ (see Sinha, 1986 for the case of Inverse-Gaussian Distribution). It is noted as in Sinha (1986) that Bayes estimates of the parameters of the B-S (α , β) would require a derivation of the posterior distributions of α and β which is extremely difficult due to the complex nature of the joint posterior of (α , β). The marginal posterior p.d.fs of α and β are obtained in order to compute the corresponding posterior expectations as

$$E\left(\alpha / \underline{t}\right) = \int_{\Omega} \alpha\, h\left(\alpha, \beta / \underline{t}\right) d\alpha\, d\beta = \frac{\int_{\Omega} \alpha L\left(\underline{t} / \alpha, \beta\right) h(\alpha, \beta) d\alpha\, d\beta}{\int_{\Omega} L\left(\underline{t} / \alpha, \beta\right) h(\alpha, \beta)\, d\alpha\, d\beta} \quad (45)$$

And

$$E\left(\beta / \underline{t}\right) = \int_{\Omega} \beta\, h\left(\alpha, \beta / \underline{t}\right) d\alpha\, d\beta = \frac{\int_{\Omega} \beta L\left(\underline{t} / \alpha, \beta\right) h(\alpha, \beta)\, d\alpha\, d\beta}{\int_{\Omega} L\left(\underline{t} / \alpha, \beta\right) h(\alpha, \beta)\, d\alpha\, d\beta} \quad (46)$$

"Lindley (1980) suggested an asymptotic approximation to the ratio of the two integrals of the form"; [13]

$$\frac{\int_{\Omega} W(\theta) \exp L(\theta)\, d\theta}{\int_{\Omega} V(\theta) \exp L(\theta)\, d\theta} \quad (47)$$

where $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, L (θ) is the log-likelihood function, $W(\theta)$ and $V(\theta)$ are arbitrary functions of $\underline{\theta}$.

Let $W(\theta) = U(\theta)V(\theta)$ and $V(\theta)$ be the prior distribution of θ. From (3.3) we use the expectation of $U(\theta)$ given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ as

$$E\left[U(\theta) / \underline{t}\right] = \frac{\int_{\Omega} U(\theta)\, V(\theta) \exp L(\theta)\, d\theta}{\int_{\Omega} V(\theta) \exp L(\theta)\, d\theta} \quad (48)$$

which is the Bayes estimator of $U(\theta)$ under the squared error loss function.

For two – parameter case, Lindley's (1980) approximation[13] leads to:-

$$E\left[U(\theta) / \underline{t}\right] = U + ½ [U_{11}\sigma_{11} + U_{12}\sigma_{12} + U_{21}\sigma_{21} + U_{22}\sigma_{22}] + \rho_1 [U_1\sigma_{11} + U_2\sigma_{21}] + \rho_2 [U_2\sigma_{22} + U_1\sigma_{12}] + ½ \{ L_{30} (U_1\sigma^2_{11} + U_2\sigma_{11}\sigma_{12}) + L_{21} [3U_1\sigma_{11}\sigma_{12} + U_2 (\sigma_{11}\sigma_{22} + 2\sigma^2_{12})] + L_{12} [3U_2\sigma_{22}\sigma_{21} + U_1(\sigma_{22}\sigma_{11} + 2\sigma^2_{21})] L_{03} (U_1\sigma_{12}\sigma_{22} + U_2\sigma^2_{22}) \} \quad (49)$$

all evaluated at MLE $\left(\hat{\theta}_1, \hat{\theta}_2\right)$ and where

$\rho = \log V(\theta)$; $\underline{\theta} = (\theta_1, \theta_2)$; $\rho_j = \frac{\partial \rho}{\partial \theta_j}$; $L_{ij} = \frac{\partial^{i+j} L}{\partial \alpha^i \partial \beta^j}$; $U_i = \frac{\partial U}{\partial \theta_i}$; $U_{ij} = \frac{\partial^2 U}{\partial \theta_i\, \partial \theta_j}$ and

σ_{ij} = (i, j)th element in the inverse of matrix $\{- L_{ij}\}$ evaluated at $\hat{\underline{\theta}} = (\hat{\theta}_1, \hat{\theta}_2)$; i , j = 1,2 . (50)

Using Lindley's asymptotic expansion (49), we will obtain Bayes estimates of α and β from (45) and (46) respectively and compare them with the MLE's in (18) and (14). These estimates are not easy to obtain, need no special tables nor does expensive computer time unless one is conducting extensive simulation studies.

Given a random sample $\underline{t} = (t_1, t_2, \ldots, t_n)$ from the p.d.f (2), the log-likelihood equation becomes

The observed negative of the Fisher's information matrix is then given by

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \frac{1}{L_{20} L_{02} - L_{11} L_{11}} \begin{bmatrix} -L_{02} & L_{11} \\ L_{11} & -L_{20} \end{bmatrix} \quad (51)$$

For the diffuse prior

$$h(\theta) = h(\alpha, \beta) \propto \frac{1}{\alpha\beta} \; ;$$

We have

$\rho = \log h(\theta) = - \ln(\alpha\beta)$

So that

$$\rho_1 = \frac{\partial \rho}{\partial \alpha} = -\frac{1}{\alpha} \quad \text{and}$$

$$\rho_2 = \frac{\partial \rho}{\partial \beta} = -\frac{1}{\beta}$$

2.3.1 Bayes Inference on β

If we define U = β such that $U_1 = \frac{\partial \beta}{\partial \alpha} = 0$;

$U_2 = \frac{\partial \beta}{\partial \beta} = 1$ and $U_{ij} = 0$; i,j = 1,2

then, Bayes estimate of β is given as:

$$\beta^* = E\left[\beta / \underline{t}\right]$$

$$= \beta - \left[\frac{\sigma_{21}}{\alpha} + \frac{\sigma_{22}}{\beta} \right] + \frac{1}{2} \left\{ L_{30}\sigma_{11}\sigma_{12} + L_{21}\left[\sigma_{11}\sigma_{22} + 2\sigma_{12}^{2} \right] + 3L_{12}\sigma_{21}\sigma_{22} + L_{03}\sigma_{22}^{2} \right\} \quad (52)$$

all evaluated at MLE $(\hat{\alpha}, \hat{\beta})$.

2.3.2 Bayes Inference on α

If we let U = α such that

$$U_1 = \frac{\partial \alpha}{\partial \alpha} = 1 \quad ; \quad U_2 = \frac{\partial \alpha}{\partial \beta} = 0$$

And

$U_{ij} = 0$; i,j = 1,2

then, Bayes estimate of α is given as:

$$\alpha^* = E\left[\alpha / \underline{t} \right]$$

$$= \alpha - \left[\frac{\sigma_{11}}{\alpha} + \frac{\sigma_{12}}{\beta} \right] + \frac{1}{2} \left\{ L_{30}\sigma_{11}^{2} + 3L_{21}\sigma_{11}\sigma_{12} + L_{12}\left(\sigma_{11}\sigma_{22} + 2\sigma_{21}^{2} \right) + L_{03}\sigma_{12}\sigma_{22} \right\} \quad (53)$$

All evaluated at MLE $(\hat{\alpha}, \hat{\beta})$.

2.3 Shrinkage Bayesian (SB):

"The Shrinkage Bayesian estimator combines between the Shrinkage and Bayesian methods"; [1]. So , we will use the Bayesian estimator instead of MLE estimator in shrinkage method as follows.

$$\left. \begin{aligned} \hat{\alpha}_{SB} &= \hat{\alpha}_{SB} K + (1-K)\alpha_{\circ} \\ \hat{\beta}_{SB} &= \hat{\beta}_{SB} K + (1-K)\beta_{\circ} \end{aligned} \right\} \quad where\ 0 \le K \le 1 \quad (54)$$

Where k equals $\hat{V}ar(\hat{\alpha}_{ml}) = \frac{\hat{\alpha}_{ml}{}^{2}}{2n}$ for α and

$$\hat{V}ar(\hat{\beta}_{ml}) = \frac{\hat{\beta}_{ml}{}^{2}\ \hat{\alpha}_{ml}{}^{2}}{n\left(1 + \hat{\alpha}_{ml}\,(2\pi)^{-\frac{1}{2}}\, h(\hat{\alpha}_{ml}) \right)} \quad \text{for} \beta.$$

3. Simulation Results

There are many methods of simulation (especially after the rapid development that took place in the use of electronic computers), which provides the time, effort ,cost and achieve analytical solutions. Simulation is the imitation of the operation of a real-world process or system over time. The act of simulating something first requires a model to be developed; this model represents the key characteristics, behaviors of the selected physical , abstract system ,or process. The model represents the system itself, while the simulation represents the operation of the system over time. Computer simulations have become a useful part of mathematical modeling of many natural systems

in sciences .So the simulation is a type of sampling techniques. A simulation study was carried out to compare the performance of the Maximum likelihood estimator, Modified moment, Bayesian estimates, Shrinkage Bayesian estimates of the model parameters of the B-S (α, β) distribution. If T has a B-S (α, β) distribution then the monotone transformation

$$X = \frac{1}{2}\left\{ \left(\frac{T}{\beta} \right)^{1/2} - \left(\frac{\beta}{T} \right)^{1/2} \right\}$$

has a normal distribution with mean zero and variance $\frac{1}{4}\alpha^2$. A random sample from B-S (α, β) distribution was generated using this relationship by applying Box-Muller (1958) transformation. For our study, different sample sizes and different parameter values were used. At first, we took sample sizes as n = 25, 50, 100, the shape parameter as α = 0.5, 1 while β =0.5, 1 .The process of simulation strategy is explained the numerical results in the Tables (1-6) as below.

Table 1: Estimate parameters based on simulations when α=1 and β=1

n	α_{ML}	α_{MO}	α_{BS}	α_{SB}	β_{ML}	β_{MO}	β_{BS}	β_{SB}
25	0.9638	0.9638	1.0120	1.0010	1.0207	1.0201	1.0207	1.0025
50	0.9877	0.9877	1.0138	1.0003	1.0056	1.0059	1.0056	1.0006
100	0.9889	0.9889	1.0019	1.0001	1.0028	1.0028	1.0028	1.0001

Table 2: Estimate parameters based on simulations when α=0.5 and β=1

n	α_{ML}	α_{MO}	α_{BS}	α_{SB}	β_{ML}	β_{MO}	β_{BS}	β_{SB}
25	0.4867	0.4867	0.4386	0.4998	1.0044	1.0044	1.0044	1.0002
50	0.4903	0.4903	0.4674	0.4999	0.9996	0.9996	0.9996	1.0000
100	0.4965	0.4965	0.4857	0.5000	1.0021	1.0021	1.0021	1.0000

Table 3: Estimate parameters based on simulations when α=1 and β=0.5

n	α_{ML}	α_{MO}	α_{BS}	α_{SB}	β_{ML}	β_{MO}	β_{BS}	β_{SB}
25	0.9622	0.9621	1.0999	1.0028	0.5082	0.5081	0.5082	0.5003
50	0.9819	0.9818	1.0526	1.0007	0.5021	0.5020	0.5021	0.5001
100	0.9920	0.9919	1.0273	1.0002	0.5042	0.5042	0.5042	0.5000

Table 4: Means absolute percentage (MAPE) of estimates based on simulations (α=1 and β =1.0)

n	α_{ML}	α_{MO}	α_{BS}	α_{SB}	β_{ML}	β_{MO}	β_{BS}	β_{SB}
25	0.1137	0.1136	0.1267	0.0025	0.1413	0.1411	0.1413	0.0049
50	0.0783	0.0783	0.0841	0.0009	0.1018	0.1019	0.1018	0.0017
100	0.0585	0.0585	0.0600	0.0003	0.0676	0.0677	0.0676	0.0005

Table 5: Means absolute percentage (MAPE) of estimates based on simulations (α=0.5 and β =1)

n	α_{ML}	α_{MO}	α_{BS}	α_{SB}	β_{ML}	β_{MO}	β_{BS}	β_{SB}
25	0.1128	0.1128	0.1778	0.0008	0.0755	0.0755	0.0755	0.0007
50	0.0818	0.0818	0.1061	0.0002	0.0528	0.0528	0.0528	0.0002
100	0.0559	0.0559	0.0639	0.0001	0.0392	0.0392	0.0392	0.0001

Table 6: Means absolute percentage (MAPE) of estimates based on simulations (α=1 and β =0.5)

n	α_{ML}	α_{MO}	α_{BS}	α_{SB}	β_{ML}	β_{MO}	β_{BS}	β_{SB}
25	0.1164	0.1163	0.1590	0.0035	0.1395	0.1402	0.1395	0.0012
50	0.0827	0.0827	0.0976	0.0011	0.0979	0.0981	0.0979	0.0004
100	0.0560	0.0560	0.0624	0.0003	0.0704	0.0706	0.0704	0.0001

4. Conclusions

As mentioned above, we have used four estimation methods which are; Maximum likelihood, Modified Moment, Bayesian and Shrinkage Bayesian estimators. In simulation study, we compare the results and we find:

1. The means absolute percentage error

$$\left(\text{MAPE} = \frac{1}{1000}\sum_{1}^{1000}\left|\frac{\hat{\theta}_i - \theta}{\theta}\right|\right)$$ Variant from one sample to another.

2. For all models (α=1 and β =1.0; α=0.5 and β =1; α=1 and β =0.5) for estimating the fatigue life distribution parameters α and β, and for all sample sizes (n=25, 50,100), the Shrinkage Bayesian estimator is the best among all proposed estimators in the sense of mean absolute percentage error (MAPE), and then Maximum Likelihood estimator, Modified Moment estimator and Bayesian estimator is the last one.

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