Mathematical Behaviour of a System with Failures due to Machinery Defects and Random Shocks

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Abstract: Many engineering systems have been analyzed in the field of reliability theory with the assumption that all the units of the system are of similar type. In most situations, there exists some two unit standby systems in which both the units are dissimilar with different costs and operating conditions. Keeping this view, the present paper analyze a two unit redundant system in which both the units are dissimilar. Here the units can fail due to machinery defects as well as due to random shocks. The reliability characteristics of interest are obtained using regenerative point technique with Markov renewal process.

Keywords: Transition probabilities, mean sojourn time, mean time to system failure, availability analysis, busy period analysis and expected number of visits by repair facility

1. Introduction

Agarwal, Manju and Kumar, A (1981), Eric Chatelat (2005), Dhillon (1982), Goyel and Agnihotri (1981) and Al-Ali (1990) analyzed many engineering systems in the field of reliability with the assumption that all the units of the system are of similar type. There may exists two unit standby systems in which both the units are dissimilar with different costs and operating conditions. Keeping this view in the present paper, we analyze two unit redundant system in which both the units are dissimilar. The units here can fail due to machinery defects as well as due to random shocks. The reliability characteristics of interest using regenerative point technique with Markov renewal process are obtained such as Transition and Steady state transition probabilities, Mean Sojourn time, MTSF, Point wise and steady state such as Transition and Steady state transition probabilities, Mean Sojourn time, MTSF, Point wise and steady state availability of the system, expected Busy period of the repairman, end expected number of visits by repairman in (o, t).

2. Model Description and Assumptions

1) The system consists of only two non-identical units in which first is operative and the second unit is kept as cold standby.
2) First unit can sustain almost two shocks i.e. if the unit does not fail in the first shock then it will definitely fail in the second shock. It can also fail directly due to machinery defects.
3) Second unit can fail due to machinery defects as well as due to random shocks and it cannot sustain more than one random shock.
4) A single repair facility is considered in the system.
5) Second unit is repairable if it is failed due to machinery defects otherwise in case of random shocks send it for replacement.
6) The priority in repair and replacement is given to the second unit over first unit.
7) All the failure time distributions are assumed to be negative exponential while the distribution of repair and replacement are arbitrary.

3. Notation and Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
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<tbody>
<tr>
<td>N₀₀</td>
<td>Normal unit as operative</td>
</tr>
<tr>
<td>N₁₁</td>
<td>Normal unit as cold standby</td>
</tr>
<tr>
<td>N₀₁</td>
<td>Normal unit as operative after observing first random shock</td>
</tr>
<tr>
<td>F₁</td>
<td>Failed unit under repair</td>
</tr>
<tr>
<td>F₀</td>
<td>Failed unit waiting for repair</td>
</tr>
<tr>
<td>F₀₁</td>
<td>Failed unit under replacement</td>
</tr>
<tr>
<td>α</td>
<td>Constant rate of occurring first random shock to the first unit</td>
</tr>
<tr>
<td>β</td>
<td>Constant rate of occurring second random shock to the first unit</td>
</tr>
<tr>
<td>γ</td>
<td>Constant failure rate of first unit failed due to machinery defects</td>
</tr>
<tr>
<td>δ</td>
<td>Constant failure rate of second unit failed due to random shock</td>
</tr>
<tr>
<td>λ</td>
<td>Constant failure rate of second unit failed due to machinery defects</td>
</tr>
<tr>
<td>f(.), F(.)</td>
<td>pdf and cdf of time to repair of first unit</td>
</tr>
<tr>
<td>g(.), G(.)</td>
<td>pdf and cdf of time to replacement of second unit failed due to random shock</td>
</tr>
<tr>
<td>h(.), H(.)</td>
<td>pdf and cdf of time to repair of second unit failed due to machinery defects</td>
</tr>
</tbody>
</table>

The possible states of the system are here under using the notations and symbols above:

**Up States:** S₀ = (N₀₀, N₁₁) S₁ = (N₀₁, N₁) S₂ = (F₁, N₀₀)

**Down States:** S₁ = (F₀, F₀₁) S₂ = (F₁, F₁)

The transitions between various states are shown below:

[Diagram showing the transitions between states]

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4. Transition Probabilities

Let \( T_0 (= 0), T_1, T_2, \ldots \) be the epochs at which enters the states \( S_i \in E \). Let \( X_0 \) denotes the state entered at epoch \( T_{n+1} \) i.e. just after the transition of \( T_n \). Then \( \{T_n, X_n\} \) constitutes a Markov-renewal process with state space \( E \) and

\[
Q_k(t) = \Pr \{ X_{n+1} = S_k | X_n = S_k, T_n \leq t | X_0 = S_i \}
\]

(1.1)

is semi Markov over \( E \). The stochastic matrix of embedded Markov chain is

\[
P = p_{ik} = \lim_{t \to \infty} Q_k(t) = Q(\infty)
\]

(1.2)

By simple probabilistic consideration, the non-zero elements of \( Q_k(t) \) are:

\[
Q_{01}(t) = \int_{0}^{t} e^{-(\alpha+\gamma)u} du = \frac{\alpha}{\alpha+\gamma} [1 - e^{-(\alpha+\gamma)t}]
\]

\[
Q_{02}(t) = \int_{0}^{t} e^{-(\alpha+\gamma)u} du = \frac{\gamma}{\alpha+\gamma} [1 - e^{-(\alpha+\gamma)t}]
\]

\[
Q_{12}(t) = \int_{0}^{t} e^{-\beta u} du = [1 - e^{-\beta t}]
\]

\[
Q_{20}(t) = \int_{0}^{t} e^{-(\delta+\gamma)u} f(u)du
\]

\[
Q_{23}(t) = \int_{0}^{t} \delta e^{-(\delta+\gamma)u} F(u)du = \frac{\delta}{\delta+\gamma} [1 - e^{-(\delta+\gamma)t}]
\]

\[
Q_{24}(t) = \int_{0}^{t} \lambda e^{-(\delta+\gamma)u} F(u)du = \frac{\lambda}{\delta+\gamma} [1 - e^{-(\delta+\gamma)t}]
\]

\[
Q_{22}(t) = \int_{0}^{t} g(u)du
\]

(1.3-1.7)

The steady state transition \( p_i \) can be obtain by taking limit as \( t \to \infty \)

i.e. \( p_{ik} = \lim_{t \to \infty} Q_k(t) \)

(1.4)

Thus,

\[
p_{01} = \frac{\alpha}{\alpha+\gamma}, \quad p_{02} = \frac{\gamma}{\alpha+\gamma}, \quad p_{12} = 1, \quad p_{20} = f\ast(\delta + \lambda)
\]

\[
p_{23} = \frac{\delta}{\delta+\gamma} [1 - f\ast(\delta + \lambda)]
\]

and

\[
p_{24} = \frac{\lambda}{\delta+\gamma} [1 - f\ast(\delta + \lambda)], \quad p_{32} = 1 \text{ and } p_{42} = 1
\]

(1.4.1-7)

The above probabilities establish the following relations:

\[
p_{01} + p_{02} = 1 \quad p_{12} = p_{13} = p_{14}
\]

\[
p_{20} + p_{23} + p_{24} = 1
\]

(1.5)

5. Mean Sojourn Times

The mean time taken by the system in a particular state \( S_i \) before transiting to any other state is known as mean sojourn time and is defined by

\[
\mu_i = \int_{0}^{\infty} P\{T > t\} dt
\]

(2.1)

where \( T \) is time of stay in state \( S_i \) by the system.

To calculate mean sojourn time \( \mu_i \) in state \( S_i \), we assume that so long as the system is in state \( S_i \), it will not transit to any other state. Therefore,

\[
\mu_0 = \int_{0}^{\infty} e^{-(\alpha+\gamma)t} dt = \frac{1}{\alpha+\gamma}, \quad \mu_1 = \int_{0}^{\infty} e^{-\beta t} dt = \frac{1}{\beta}, \quad \mu_2 = \int_{0}^{\infty} e^{-(\delta+\gamma)t} F(t)dt = \frac{1}{\delta+\gamma} [1 - f\ast(\delta + \lambda)]
\]

\[
\mu_3 = \int_{0}^{\infty} G(t)dt = \int_{0}^{\infty} t.g(t)dt \quad \text{and} \quad \mu_4 = \int_{0}^{\infty} H(t)dt = \int_{0}^{\infty} t.h(t)dt
\]

(2.1.1-5)

For the contribution to mean sojourn time in state \( S_i \in E \) and non-regenerative state occurs, before transiting to \( S_i \in E \), i.e.

\[
m_{ij} = \int tq_{ij}(t)dt = -q_{ij}^\ast(0)
\]

(2.2)

Therefore,

\[
m_{01} = \int_{0}^{\infty} \alpha t e^{-\alpha(\lambda+\gamma)} dt = \frac{\alpha}{(\alpha+\gamma)^2}
\]

\[
m_{02} = \int_{0}^{\infty} \gamma t e^{-\alpha(\lambda+\gamma)} dt = \frac{\gamma}{(\alpha+\gamma)^2}
\]

\[
m_{12} = \int_{0}^{\infty} \beta t e^{-\beta t} dt = \frac{1}{\beta}
\]

\[
m_{20} = \int_{0}^{\infty} t e^{(\delta+\gamma)t} f(t)dt
\]

\[
m_{23} = \delta \int_{0}^{\infty} t e^{(\delta+\gamma)t} F(t)dt
\]

\[
m_{24} = \lambda \int_{0}^{\infty} t e^{(\delta+\gamma)t} F(t)dt
\]

\[
m_{32} = \int_{0}^{\infty} t h(t)dt
\]

(2.2.1-8)

Hence,

\[
m_{01} + m_{02} = 1, \quad m_{12} = \frac{1}{\beta}, \quad m_{20} + m_{23} + m_{24} = 1
\]

\[
m_{32} = \int_{0}^{\infty} t g(t)dt = \mu_3 \quad \text{and} \quad m_{42} = \int_{0}^{\infty} t h(t)dt = \mu_4
\]

(2.3.1-5)

6. Mean Time to System Failure (MISF)

The mean time to system failure (MTSF) can be obtained by \( E(T) \) given below by using Laplace Stieltjes transform of the relations for the distribution function \( F_\lambda(t) \) of the time to system failure with starting time \( S_0 \)

\[
E(T) = \frac{d}{dt} F_\lambda(t) \big|_{t=0} = \frac{d}{dt} \left[ \frac{1}{\beta} \right] = \frac{1}{\beta}
\]

(3.1)

where \( N_1 = \mu_0 + \mu_1 p_{01} + \mu_2 \) and

\[
D_1 = 1 - p_{20}
\]

(3.2)

7. Availability Analysis

System availability is defined as

\[
A_0(t) = A_1(t) \quad \text{[Starting from state } S_0 \text{, the system is available at epoch } t \text{ without passing through any regenerative state]}
\]

\[
M_0(t) = M_1(t) \quad \text{[Starting from up state } S_1 \text{, the system remains up till epoch } t \text{ without passing through any regenerative state]}
\]

Hence, obtaining \( A_0(t) \) by using elementary probability argument, we get

\[
A_0(t) = M_0(t) + q_{01} A_1(t) + q_{02} A_2(t)
\]

\[
A_1(t) = M_1(t) + q_{12} A_2(t)
\]

\[
A_2(t) = M_2(t) + q_{20} A_0(t) + q_{23} A_3(t)
\]

\[
+ q_{24} A_4(t)
\]

\[
A_3(t) = q_{32} A_2(t)
\]

\[
A_4(t) = q_{42} A_2(t)
\]

(4.1.1-5)
where $M_0(t) = e^{-(a+\gamma)t}, M_1(t) = e^{-\beta t}$ and $M_2(t) = e^{-(\delta+\gamma)t}$. 

Taking Laplace transform of the equations (4.1.1-5) and solving for point wise availability by omitting the arguments "s" for brevity, the steady state functioning availability of the system, when the system starts operation from the state $S_i$, can be obtained as:

$$A_0(\infty) = \lim_{t \to \infty} A_0(t) = \lim_{s \to 0} A_0(s) = \frac{N_2(0)}{D_0(0)} = \frac{N_2}{D_2}$$

(4.2)

where $N_2 = (\mu_0 + \mu_1)p_{20} + \mu_2$ and $D_2 = (\mu_0 + \mu_1)p_{20} + \mu_2 + \mu_3p_{23} + \mu_4p_{42}$

(4.3)

### 8. Busy Period Analysis

(a) Let $W_3(t)$ be the probability that the system is under repair by repair facility in the state $S_i \in E$ at time $t$ without transiting to any regenerative state. Therefore,

$$W_3(t) = e^{-(t+\delta)\alpha}F(t)$$

(5.1)

Let $B_4(t)$ be the probability that the system is under repair at time $t$. We obtain the following recursive relations among $B_i(t)$:

$$B_0(t) = q_{01}(t)B_1(t) + q_{02}(t)B_2(t)$$

$$B_1(t) = q_{12}(t)B_2(t)$$

$$B_2(t) = W_2(t) + q_{20}(t)B_0(t) + q_{23}(t)B_3(t) + q_{24}(t)B_4(t)$$

$$B_3(t) = q_{32}(t)B_1(t)$$

$$B_4(t) = W_2(t) + q_{42}(t)B_2(t)$$

(5.1.1-5)

Taking Laplace transform of the equations (5.1.1-5) and solving the equations by omitting the argument for brevity we get the fraction of time for which the repair facility is busy in repair as

$$B_0(t) = \lim_{s \to \infty} B_0(s) = \lim_{s \to \infty} B_0(s) = N_4(0)/D_4(0) = N_4/D_4$$

(5.2)

where $N_4 = \mu_0 + p_{24}\mu_4$ and $D_4$ is same as in (4.3).

(b) In case if the system is under replacement in state $S_i \in E$ at time $t$ without transiting to any regenerative state, the fraction of time for which the repair facility is busy in replacement can be obtained as

$$R_0(t) = \lim_{s \to \infty} R_0(s) = \lim_{s \to \infty} R_0(s) = N_4(0)/D_4(0) = N_4/D_4$$

(5.3)

where $N_4 = \mu_3p_{23}$ and $D_4$ is same as in (4.3).

### 9. Expected Number of Visits by Repair Facility

Let $V_i(t)$ be the expected number of visits by the repair facility in the repair facility in $0, t$ given that the system initially started from regenerative state $S_i$ at $t = 0$. The following recurrence relations among $V_i(t)$’s can be obtained as:

$$V_0(t) = Q_{01}(t)V_1(t) + Q_{02}(t)[1 + V_2(t)]$$

$$V_1(t) = Q_{12}(t)[1 + V_2(t)]$$

$$V_2(t) = Q_{20}(t)V_0(t) + Q_{23}(t)V_3(t) + Q_{24}(t)V_4(t)$$

$$V_3(t) = Q_{32}(t)V_2(t)$$

$$V_4(t) = Q_{42}(t)V_2(t)$$

(6.1.1-5)

Using Laplace Stieltjes transform of the above equations and omitting the argument "s" for brevity, we can get the number of visits per unit of time when the system starts after entrance into state $S_i$ as:

$$V_0 = \lim_{t \to \infty} V_0(t)/t = \lim_{s \to 0} sV_0(s) = N_5/D_5$$

(6.2)

Where $N_5 = p_{20}$ and $D_5$ is same as in (4.3)

With the help of this study we concluded that the performance of the manufacturing system can be improved by improving the procedures, proper training of employees and proper maintenance of the system. The results derived in this paper are valuable in a study of improving the reliability of the systems and additionally they can be extensively used in many engineering disciplines.

**References**