International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

MSGDTM for Solution of Frictional Order Dengue Disease Model

Rashid Jan[1], Asif Jan[2]

[1]School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, PR China
Email: rashid_ash2000@yahoo.com

[2]Department of Microbiology, Abasyn University, Peshawar 25000, KPK, Pakistan
Email: asifjan2117@gmail.com

Abstract: ***Multi-step generalized differential transform method (MSGDTM) is one of the most proficient and effective method, which provides better and improved approximate solution for a system than other numerical and analytic methods for frictional derivatives and it's performance and reliability is superior than other methods. In this research paper we will employ Multi-Step Generalized Differential Transform Method (MSGDTM) to find the approximate solution of the frictional order Host-Vector Dengue disease model and the non-negativity of the solutions of frictional order Host-Vector model will be presented.***

Keywords: Dengue fever, Host-Vector model, Caputo derivative, IVP, Approximate solution, MSGDTM.

1. Introduction

Many authors formulated, investigated and analyzed mathematical models in Physical sciences, finance, economics, engineering and particular in life science (Mathematical-Biology) using the conventional integer order system of differential equations and established some important results in the past several years [1-12]. On the other hand due to the effective nature of fractional derivatives and integrals, many epidemiological models and other models in engineering and science have successful being originated and analyzed [13-21].

To be more specific, Fractional calculus has been accustomed to model physical and engineering processes, which are found to be best described by fractional differential equations. It is worth noting that the standard mathematical models of integer-order derivatives, including nonlinear models, which fails to work sufficiently in many cases. In new era, fractional calculus has played a very significant part in various fields such as chemistry, mechanics, economics, electricity, control theory, image and signal processing [22, 23, 24], and particularly in mathematical-biology.

The most important purpose of this paper is to exploit the multi-step generalized differential transform method to approximate the numerical solution of the frictional order Host-Vector model for Dengue disease.

2. Description of Model

A Host-Vector dengue disease transmission model was developed by Esteva and Vargas in [25], they supposed that a recover individual from the disease will not be re-infected by the disease. They also assume that the host population H is constant with death and birth rate μh. Where S_h, I_h, R_h are susceptible, infective, and recover individuals in the host population and S_v, I_v are susceptible, Infective in the vector population V. Their model is given as follows:

$$\begin{aligned}\frac{dS_h}{dt} &= \mu_h H - \frac{\beta_h b}{H} S_h I_v - \mu_h S_h\\ \frac{dI_h}{dt} &= \frac{\beta_h b}{H} S_h I_v - (\mu_h+\gamma_h) I_h\\ \frac{dR_h}{dt} &= \gamma_h I_h - \mu_h R_h\\ \frac{dS_v}{dt} &= \Lambda - \frac{\beta_v b}{H} S_v I_h - \mu_v S_v\\ \frac{dI_v}{dt} &= \frac{\beta_v b}{H} S_v I_h - \mu_v I_v,\end{aligned} \tag{1}$$

where β_h,β_v are the transmission probability from vector to host and host to vector. γ_h represent the recovery rate in the host population and b is the biting rate of the vector. Further-more equation (1) can be reduced to three dimension dynamics with the condition $S_h + I_h + R_h = H$ and $S_v + I_v = \frac{\Lambda}{\mu_v}$.

$$\begin{aligned}\frac{dS_h}{dt} &= \mu_h H - \frac{\beta_h b}{H} S_h I_v - \mu_h S_h\\ \frac{dI_h}{dt} &= \frac{\beta_h b}{H} S_h I_v - (\mu_h+\gamma_h) I_h\\ \frac{dI_v}{dt} &= \frac{\beta_v b}{H} S_v I_h - \mu_v I_v.\end{aligned} \tag{2}$$

To normalize (2), we set
$S = \frac{S_h}{H}, Y = \frac{I_h}{H}, Z = \frac{I_v}{\Lambda/\mu_v}$
and get

$$\begin{aligned}\frac{d}{dt}S &= \mu(1-S) - \rho S Z\\ \frac{d}{dt}Y &= \rho S Z - \eta Y\\ \frac{d}{dt}Z &= \lambda(1-Z)Y - \delta Z,\end{aligned} \tag{3}$$

where

$$\rho = \frac{b\beta_h \Lambda}{\mu_v H}, \eta = \gamma_h + \mu_h, \lambda = b\beta_v, \delta = \mu_v, \mu = \mu_h.$$

3. Fractional Calculus

Fractional calculus is a governing, dominant and attractive tool for mathematical modeling. It has been applied in many areas of research such as science, economics, finance and engineering. Fractional calculus contains several interesting and attractive definitions of fractional derivatives [19, 21], but here the famous Caputo derivatives is used due its advantage on initial value problems. Some important definitions related to frictional calculus are given below:

Definition 3.1 A function g(x) for positive x is said to be in the space $\mathbf{G}_\alpha$, (where α belongs to R) if it is expressed in the form $g(x) = x^p g_1(x)$ for some p>α where $g_1(x)$ is continuous in [0, ∞), and g(x) be in the space $\mathbf{G}_\alpha^m$ *if* $g^m \in \mathbf{G}_\infty, m \in N$.

Definition 3.2 *The Riemann-Liouville integral operator is defined as:*

$$(J_u^\alpha g)(x) = \frac{1}{\Gamma(\alpha)} \int_u^x \frac{g(x)}{(x-t)^{1-\alpha}} dt, x > u \quad (4)$$

$$(J_u^0 g)(x) = g(x), \quad (5)$$

where α is the order of operator and u ≥ 0, properties of this operator are given in [21, 22, 26]. Here, we just required the following:

For g ∈ $\mathbf{G}_\alpha$, α, β are positive, u is non-negative, v ∈ R, γ > −1, we get

$$(J_u^\alpha J_u^\beta g)(x) = (J_u^\beta J_u^\alpha g)(x) = (J_u^{\alpha+\beta} g)(x), \quad (6)$$

$$J_u^\alpha(x^\gamma) = \frac{x^{\alpha+\gamma}}{\Gamma(\alpha)} \mathbf{B}_{\frac{x-u}{x}}(\alpha, \gamma+1), \quad (7)$$

where $\mathbf{B}_\tau(\alpha, \gamma+1)$ *represents incomplete beta function and is defined as:*

$$\mathbf{B}_\tau(\alpha, \gamma+1) = \int_0^\tau \frac{(1-t)^\gamma}{t^{1-\alpha}} dt, \quad (8)$$

$$J_u^\alpha e^{vx} = e^{uv}(x-u)^\alpha \sum_{k=0}^{\infty} \frac{[v(x-u)]^k}{\Gamma(\alpha+k+1)}. \quad (9)$$

In the real world application Riemann-Liouville has some disadvantages with frictional order derivatives, that's why here we use the Caputo frictional derivative D_u^α .

Definition 3.3 *For a function g(x) the Caputo Fractional derivative is defined by*

$$D_u^\alpha g(x) = \frac{1}{\Gamma(p-\alpha)} \int_u^x \frac{g^p(t)}{(x-t)^{\alpha+1-p}} dt,$$

where α is the order of Caputo frictional derivative with the condition p − 1 < α < p, p ∈ N, x ≥ u, g ∈ $\mathbf{G}_{-1}^m$.

Many authors investigated Caputo frictional derivative for $p-1 < \alpha < p$, $g(x) \in \mathbf{G}_\alpha^m$ and α ≥−1; we have

$$(J_n^\alpha D_u^\alpha g)(x) = J^p D^p g(x) = g(x) - \sum_{k=0}^{p-1} g^k(u) \frac{(x-u)^k}{k!} \quad (10)$$

4. Multi-step Generalized Differential Transform Method

It is known that the generalized differential transform method (GDTM) is applied to find the approximate solutions for nonlinear problems which gives accurate approximate solution for small step of time but for large time it has been proved that the approximate solution obtained by GDTM are not accurate and applicable [29, 30, 31, 32]. Multi-step generalized differential transform method is modified form GDTM, which present precise and accurate approximate solutions of the model over a longer time frame compared to the standard GDTM see [33, 34, 35, 36, 37, 38, 39].

For more explanation we supposed the following IVP for systems of the fractional differential equations:

$$D_*^{\alpha_1} w_1(t) = g_1(t, w_1, w_2, ..., w_n)$$
$$D_*^{\alpha_2} w_2(t) = g_1(t, w_1, w_2, ..., w_n) \quad (11)$$
$$\vdots$$
$$D_*^{\alpha_n} w_n(t) = g_1(t, w_1, w_2, ..., w_n)$$

with the initial conditions

$$w_i(t_0) = c_i, \quad i = 1, 2, 3, ..., n, \quad (12)$$

where $D_*^{\alpha_1} w_1(t), D_*^{\alpha_2} w_2(t), D_*^{\alpha_3} w_3(t), ..., D_*^{\alpha_n} w_n(t)$ are the Caputo fractional derivative of order $\alpha_1, \alpha_2, ..., \alpha_n$, with the condition that $0 < \alpha_1, \alpha_2, ..., \alpha_n \le 1$. Assume $[t_0, T]$ be the interval over which we desire to get the approximate solution of the IVP (11, 12). In concrete applications of the generalized differential transform method the K_{th}-order approximate solution of the IVP (11, 12) can be expressed by the finite series

$$w_i(t) = \sum_{i=0}^{K} W_i(k)(t-t_0)^{k\alpha_i}, t \in [t_0, T], \quad (13)$$

where Wi(k) satisfied the recurrence relation

$$\frac{\Gamma((k+1)\alpha_i + 1)}{\Gamma(k\alpha_i + 1)} W_i(k+1) = G_i(k, W_1, W_2, ..., W_n), \quad (14)$$

$W_i(0) = c_i$, and $G_i(k, W_1, W_2, ..., W_n)$ is the differential transform of function $g_i(t, w_1, w_2, ..., w_n)$ for i =1, 2,..., n. The fundamental step of GDTM can be found in [31, 32, 38, 40].

Assume that the interval $[t_0, T]$ is divided into M subintervals $[t_{m-1}, t_m]$, m = 1, 2, 3, ... , M of equal step size $h = (T - t_0)/M$ by using the nodes $t_m = t_0 + mh$. The procedure and main steps of the MSGDTM are given below:

Initially, we apply the generalized differential transform method to the IVP (11, 12) over the interval $[t_0, t_1]$, we get the approximate solution $w_{i,1}(t), t \in [t_0, t_1]$ using the initial condition $w_i(t_0)= c_i$, for i =1, 2, ..., n. For m ≥ 2 and at each subinterval $[t_{m-1},t_m]$, we use the initial condition $w_{i,m}(t_{m-1}) = w_{i,m-1}(t_{m-1})$ and apply the GDTM to the IVP (11, 12) over the interval $[t_{m-1},t_m]$. With the repetition of this process a sequence of approximate solutions $w_{i,m}(t)$,m =1, 2, ..., M, for i =1, 2, ..., n is generated. At last, the MSGDTM assumes the following solution

$$w_i(t) = \begin{cases} w_{i,1}(t), & t \in [t_0, t_1] \\ w_{i,2}(t), & t \in [t_1, t_2], \\ . & . \\ . & . \\ . & . \\ w_{i,M}(t), & t \in [t_{M-1}, t_M] \end{cases}$$

MSGDTM, the new algorithm, which is obtained from GDTM is simple for computational performance for all values of h. The solution obtained by MSGDTM converges for wide range of time.

5. MSGDTM Algorithm for Solution of Frictional Order Host-Vector Dengue Fever Model

To show the effectiveness of this method, we consider the fractional order Host-Vector model of epidemic. As the approximate solutions of MSGDTM are accurate and better than other numerical methods, that's why we want to get a better and accurate result of frictional order Host-Vector model by using MSGDTM.

Now we established the fractional order dengue disease model of the system described by (15). For which we replace the integer order derivatives by the fractional order derivatives, as follows

$$\begin{aligned} D^{\alpha_1}S &= \mu(1-S) - \rho SZ, \\ D^{\alpha_2}Y &= \rho SZ - \eta Y, \\ D^{\alpha_3}Z &= \lambda(1-Z)Y - \delta Z, \end{aligned} \tag{15}$$

where (S, Y, Z) are the state variables μ, ρ, η, λ, δ are non-negative constant and α_1, α_2, α_3 are parameters describing the order of the frictional time derivatives in the Caputo sense. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to find and get different responses. Applying the multi-step generalized differential transform method to system (15), we get

$$\begin{aligned} S(k+1) &= \frac{\Gamma(\alpha_1 k+1)}{\Gamma(\alpha_1(k+1)+1)}[\mu(1-S(k)) - \rho\sum_{l=0}^{k} S(l)Z(k-l)], \\ Y(k+1) &= \frac{\Gamma(\alpha_2 k+1)}{\Gamma(\alpha_2(k+1)+1)}[\rho\sum_{l=0}^{k} S(l)Z(k-l) - \eta Y(k)], \\ Z(k+1) &= \frac{\Gamma(\alpha_3 k+1)}{\Gamma(\alpha_3(k+1)+1)}[\lambda Y(k) - \lambda\sum_{l=0}^{k} Z(k)Y(k-l) - \delta Z(k)], \end{aligned} \tag{16}$$

where S(k),Y (k) and Z(k) are the differential transforms of S(t),Y (t) and Z(t) respectively. S (0) = c_1, Y (0) = c_2 and Z(0) = c_3 are the differential transform of the initial conditions. In view of the differential inverse transform, the differential transform series solution for the system (15) can be obtained as

$$\begin{aligned} s(t) &= \sum_{n=0}^{N} S(n)t^{\alpha_1 n}, \\ y(t) &= \sum_{n=0}^{N} Y(n)t^{\alpha_2 n}, \\ z(t) &= \sum_{n=0}^{N} Z(n)t^{\alpha_3 n}. \end{aligned} \tag{17}$$

According to the MSGDTM, the series solution for the system (15) is

$$s(t) = \begin{cases} \sum_{n=0}^{K} S_1(n)t^{\alpha_1 n}, & t \in [0, t_1], \\ \sum_{n=0}^{K} S_2(n)(t-t_1)^{\alpha_1 n}, & t \in [t_1, t_2], \\ . & . \\ . & . \\ . & . \\ \sum_{n=0}^{K} S_M(n)(t-t_{M-1})^{\alpha_1 n}, & t \in [t_{M-1}, t_M], \end{cases} \tag{18}$$

$$y(t) = \begin{cases} \sum_{n=0}^{K} Y_1(n)t^{\alpha_2 n}, & t \in [0, t_1], \\ \sum_{n=0}^{K} Y_2(n)(t-t_1)^{\alpha_2 n}, & t \in [t_1, t_2], \\ . & . \\ . & . \\ . & . \\ \sum_{n=0}^{K} Y_M(n)(t-t_{M-1})^{\alpha_2 n}, & t \in [t_{M-1}, t_M], \end{cases} \tag{19}$$

$$z(t)=\begin{cases} \sum_{n=0}^{K} Z_1(n)t^{\alpha_3 n}, & t\in[0,t_1],\\ \sum_{n=0}^{K} Z_2(n)(t-t_1)^{\alpha_3 n}, & t\in[t_1,t_2],\\ . & .\\ . & .\\ . & .\\ \sum_{n=0}^{K} Z_M(n)(t-t_{M-1})^{\alpha_3 n}, & t\in[t_{M-1},t_M], \end{cases} \quad (20)$$

where $S_i(n)$, $Y_i(n)$ and $Z_i(n)$ for $i = 1, 2, ..., M$ satisfies the following recurrence relations

$$\begin{aligned} S(k+1) &= \frac{\Gamma(\alpha_1 k+1)}{\Gamma(\alpha_1(k+1)+1)}[\mu(1-S(k))-\rho\sum_{l=0}^{k} S(l)Z(k-l)],\\ Y(k+1) &= \frac{\Gamma(\alpha_2 k+1)}{\Gamma(\alpha_2(k+1)+1)}[\rho\sum_{l=0}^{k} S(l)Z(k-l)-\eta Y(k)],\\ Z(k+1) &= \frac{\Gamma(\alpha_3 k+1)}{\Gamma(\alpha_3(k+1)+1)}[\lambda Y(k)-\lambda\sum_{l=0}^{k} Z(k)Y(k-l)-\delta Z(k)], \end{aligned} \quad (21)$$

such that

$$S_i(0)=s_i(t_{i-1})=s_{i-1}(t_{i-1}), Y_i(0)=y_i(t_{i-1})=y_{i-1}(t_{i-1})$$

And

$$Z_i(0)=z_i(t_{i-1})=z_{i-1}(t_{i-1})$$

At last, we start with $S_0(0) = c_1$, $Y_0(0) = c_2$ and $Z_0(0) = c_3$ and using the recurrence relation given in (21), we get the multi-step solution given in (18)-(20).

6. Non-negative Solution

Assume $R^3 = \{Z \in R_3 : Z \geq 0\}$, where $Z = (S, Y, Z)^T$. To show the non-negative solution of the model we will apply the following lemma presented in [27].

Lemma 6.1 [27], Generalized Mean Value Theorem: *Let $g(x) \in C[c, d]$* and $D^\alpha g(x) \in C[c, d]$ for $0 \leq \alpha \leq 1$, then we have

$$g(x)-g(c)+\frac{1}{\Gamma(\alpha)}D^\alpha g(\xi)(x-c)^\alpha,$$

with the condition $c \leq \xi \leq x$, for all $x \in [c, d]$.

Remark 3.1: Assume that $g(x) \in C[c, d]$ and $D^\alpha g(x)$ belongs to $C[c, d]$, for $0 \leq \alpha \leq 1$. It follows form lemma 3.1 that $g(x)$ is non-decreasing if $D^\alpha g(x) \geq 0$, for all $x \in [c, d]$ and $g(x)$ non increasing if $D^\alpha g(x) \leq 0$ for all $x \in [c, d]$.

Theorem 6.1: *A unique solution of the fractional order initial value problem (15) exists and it remains in R^3_+.*

Proof 6.1 *Existence and uniqueness of the solution of model problem (15) in $(0, \infty)$ follows by the use of theorem 3.1 and remark 3.2 in [28]. The domain R^3_+ is positively invariant for the model problem, because*

$$D^\alpha S|_{S=0}=\mu\geq 0,$$
$$D^\alpha Y|_{Y=0}=0,$$
$$D^\alpha Z|_{Z=0}=\gamma\geq 0,$$

on each hyper-plane bounding the non negative orthant, the vector field points into R^3_+.

7. Conclusion

Multi-Step Generalized Differential Transform Method is a simple method for computing the solution of epidemic models and other non-linear problems, it has been proved that this method is more reliable and effective than other method to find the approximate solution of problems. In this research paper, a frictional order Host-Vector dengue disease model is formulated and MSGDTM is used to find the approximate solution of the model. The approximate solutions obtained by MSGDTM are valid for long time and highly accurate.

References

[1] Xue Y, Yuan X, Liu M. Global stability of a multi-group SEI model, Applied Mathematics and Computation. 2014; 226:51-60.
[2] Magal P, Ruan S. Susceptible-infectious-recovered models revisited: From the individual level to the population level. Mathematical, Mathematical Biosciences. 2014; 250:26-40.
[3] Hethcote HW, van den Driessche. Two SIS epidemiologic models with delays, J. Math. Biol. 2000; 40:3-26.
[4] J. Dushoff J, Huang W, Castillo-Chavez C. Backwards bifurcations and catastrophe in simple models of fatal

diseases, Journal of Mathematical Biology. 1998; 36:227-248.

[5] Castillo-Chavez C, Thieme HR. Asymptotically autonomous epidemic models, Mathematical Population Dynamics: Analysis of Heterogeneity. 1995; 1:33-50.

[6] Chitnis N, Cushing JM, Hyman JM. Bifurcation analysis of a mathematical model for malaria transmission, SIAM Journal on Applied Mathematics. 2006; 67:24-45.

[7] Okosun KO, Makinde OD. Optimal control analysis of malaria in the presence of non-linear incidence rate, Appl. Comput. Math. 2013; 12:20-32.

[8] Koella JC, Anita R. Epidemiological models for the spread of anti-malarial resistance, Malaria Journal. 2003; 2:3.

[9] Mukandavire Z, Gumel AB, Garira W, Tchuenche JM. Mathematical analysis of a model for HIV-malaria co-infection, Math Biosci Eng. 2009; 6:333-362.

[10] Brauer F, Sanchez DA. Constant rate population harvesting: Equilibrium and stability, Theoretical Population Biology. 1975; 8(1):12-30.

[11] Feng Z, Velasco-Hernandez JX. Competitive exclusion in a vector-host model for the dengue fever, Journal of Mathematical Biology. 1997; 35(5):523-544.

[12] Hethcote HW. Note on determining the limiting susceptible population in an epidemic model., Mathematical Biosciences. 1970; 9:161163.

[13] Ahmed A, El-Sayed AMA, El-Saka HA. Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, Journal of Mathematical Analysis and Applications. 2007; 325:542-553.

[14] Arafa AAM, Rida SZ, Khalil M. Solution of fractional order model of childhood diseases with constant vaccination strategy, Mathematical Sciences Letters. 2012; 1:17-23.

[15] Javidi M, Ahmad B. A study of fractional-order cholera mode, Appl: Math. Inf. Sci. 2014; 8:2195-2206.

[16] Rocco A, West BJ. Fractional calculus and the evolution of fractional phenomena, Physica A Statistical Mechanics and Its Applications. 999; 265(3-4):535-546.

[17] Ding Y, Ye H. A fractional-order differential equation model of HIV infection of CD4+T¬cells, Mathematical and Computer Modeling. 2009; 50(3-4):386-392.

[18] Ahmed E, El-Saka HA. On fractional order models for hepatitis C, Nonlinear Biomedical Physics. 2010;4:1-3.

[19] Petras I. Fractional-order nonlinear systems: Modeling, analysis and simulation, Springer-Verlag; 2011.

[20] Petras I, Magin IR. Simulation of drug uptake in a two compartmental fractional model for a biological system, Common Nonlinear Sci Numer Simul. 2011; 16(12):4588-4595.

[21] Podlubny I. Fractional differential equations, Academic Press, New York; 1999.

[22] Miller, S. and Ross, B. An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, USA 1993.

[23] Ertrk, V. S., Odibat, Z. and Momani, S. An approximate solution of a fractional order differential equation model of human T-cell lymphotropic virus I (HTLV-I) infection of CD4+ T-cells, Computers and Mathematics with Applications, 2011; 62, 992-1002.

[24] Lin W. Global existence theory and chaos control of fractional differential equations, JMAA, 2007; 332, 709-726.

[25] Esteva L and Vargas C. Analysis of a Dengue Fever Transmission Model, Journal of Bioscience 150(1998); 131-151.

[26] Caputo M. Linear Models of Dissipation whose Q is almost Frequency Independent-II, Geophysical Journal of the Royal Astronomical Society, 1967; 13(5), 529539.

[27] Odibat ZM, Shawagfeh NT. Generalized Taylor's formula, Applied Mathematics and Computation. 2007; 186(1):286-293.

[28] Lin W. Global existence theory and chaos control of fractional differential equations, Journal of Mathematical Analysis and Applications. 2007; 332(1):709-726.

[29] Ertrk V, Momani S, and Odibat Z. Application of generalized differential transform method to multi-order fractional differential equations, Communications in Nonlinear Science and Numerical Simulation, 2008;13, 16421654.

[30] Momani S and Odibat Z. A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylors formula. Journal of Comput. and Applied Mathematics, 2008;220, 8595.

[31] Odibat Z, Momani S and Ertrk V.S. Generalized differential transform method: Application to differential equations of fractional order, Appl. Math. Comput. 2008; 197, 467477.

[32] Odibat Z, Momani S. A generalized differential transform method for linear partial differential equations of fractional order, Applied Mathematics Letters, 2008;21, 194199.

[33] Ertrk V S, Odibat Z and Momani S. An approximate solution of a fractional order differential equation model of human T-cell lymphotropic virus I (HTLV-I) infection of CD4+ T-cells, Computers and Mathematics with Applications, 2011;62, 992-1002.

[34] Freihat A and Momani S. Adaptation of differential transform method for the numeric-analytic solution of fraction-order Rssler chaotic and hyperchaotic systems, Abstract and Applied Analysis, 2012; doi:10.1155/2012/934219.

[35] Freihat A and Momani S. Application of multi-step generalized differential transform method for the solutions of the fractional-order Chuas system. Discrete Dynamics in Nature and Society, Discrete Dynamics in Nature and Society, 2012;doi: 10.1155/2012/427393.

[36] Freihat A and AL-Smadi M. A new reliable algorithm using the generalized differential transform method for the numeric-analytic solution of fractional-order Liu chaotic and hyperchaotic systems, Pensee Journal, 2013; Vol 75, No. 9, 263276.

[37] Momani S, Freihat A and AL-Smadi M. Analytical study of fractional-order multiple chaotic FitzHugh-Nagumo neurons model using multi-step generalized differential transform method, Abstract and Applied Analysis, Vol. 2014;Article ID 276279, 10 pages.

[38] Odibat Z, Bertelle C, Aziz-Alaoui M. A and Duchamp G. A multi-step differential transform method and application to non-chaotic or chaotic systems, Computers and Mathematics with Applications, 2010;59(4), 1462-1472.

[39] Zeb A, Zaman G, Chohan M. I, Momani S and Ertrk V. S. Analytic Numeric Solution for SIRC Epidemic Model in Fractional Order, Asian Journal of Mathematics and Applications, Volume 2013;Article ID ama0058.

[40] Olsen L. F, Truty G. L, and Schaffer W. M. Oscillations and chaos in epidemics: a nonlinear dynamic study of six childhood diseases in Copenhagen, Denmark. Theoretical Population Biol.1988; 33, 344370.