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Suzuki Type Unique Common Tripled Fixed Point Theorem for Four Maps under Ψ - Φ Contractive Condition in Partial Metric Spaces

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Abstract: ***In this paper, we obtain a Suzuki type unique common tripled fixed point theorem for four maps in partial metric spaces.***

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1. Introduction

The notion of a partial metric space was introduced by Matthews [9] as a part of the study of denotational semantics of data ow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation and domain theory in computer science.

Matthews [9] and Romaguera [11] and Altun et al. [2] proved some fixed point theorems in partial metric spaces for a single map. For more works on fixed, common fixed point theorems in partial metric spaces, we refer [1-12]). The aim of this paper is to study Suzuki type unique common tripled fixed point theorem for four maps satisfying a ψ - ϕ contractive condition in partial metric spaces.

First we give the following theorem of Suzuki [15].

Theorem 1.1. (See [15]): Let (X, d) be a complete metric space and let T be a mapping on X. Define a non - increasing function θ from [0, 1) onto $(\frac{1}{2},1]$ by

$$\theta(r)=\begin{cases} 1 & if \quad 0\le r\le \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & if \quad \frac{\sqrt{5}-1}{2}\le r\le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & if \quad \frac{1}{\sqrt{2}}\le r<1. \end{cases}$$

Assume that there exists r ∈ [0, 1) such that

$\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le rd(x,y)$

for all $x,y\in X$.

Then there exists a unique fixed point z of T.

Moreover $\lim_{n\to\infty} T^n x = z$ for all $x\in X$.

First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.4. (See [1,9]) A partial metric on a nonempty set X is a function $p: X\times X\to R^+$ such that for all $x,y,z\in X$:

$(p_1)\, x=y \Leftrightarrow p(x,x)=p(x,y)=p(y,y),$

$(p_2)\, p(x,x)\le p(x,y),\ p(y,y)\le p(x,y),$

$(p_3)\, p(x,y)=p(y,x),$

$(p_4)\, p(x,y)\le p(x,z)+p(z,y)-p(z,z).$

The pair (X,P) is called a partial metric space (PMS).

If p is a partial metric on X, then the function $p^s: X\times X\to R^+$ given by

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y), \quad (1.1)$$

is a metric on X.

Example 1.5. (See e.g. [9]) Consider X = [0, ∞) with $p(x,y)=\max\{x,y\}$.Then (X, p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $p^s(x,y)=|x-y|$.

Example 1.6. Let X = $\{[a,b]: a,b\in R, a\le b\}$ and define p([a,b],[c,d]) = max{b,d} – min{a,c}. Then (X,p) is a partial metric space.

We now state some basic topological notions (such as convergence, completeness, continuity) on partial metric spaces (see e.g. [1, 2, 7, 8, 9].)

Definition 1.7.

(i) A sequence $\{x_n\}$ in the PMS (X, p) converges to the limit x if and only if

$p(x,x)=\lim_{n\to\infty} p(x,x_n).$

(ii) A sequence $\{x_n\}$ in the PMS (X, p) is called a Cauchy sequence if

$\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(iii) A PMS (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_m, x_n)$.

The following lemma is one of the basic results in PMS([1, 2, 7, 8, 9]).

Lemma 1.8.

(i) A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(ii) A PMS (X, p) is complete if and only if the metric space (X, p^s) is complete.

Moreover $\lim_{n\to\infty} p^s(x, x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_m, x_n)$.

Next, we give two simple lemmas which will be used in the proof of our main result. For the proofs we refer to [1].

Lemma 1.9.
Assume $x_n \to z$ as n $\to \infty$ in a PMS (X, p) such that $p(z,z) = 0$. Then $p(z,y) = \lim_{n\to\infty} p(x_n, y)$ for every $y \in X$.

Lemma 1.10. Let (X, p) be a PMS. Then
(A) If p(x, y) = 0 then x = y,
(B) If x $\neq$ y, then p(x, y) > 0.

Remark 1.11. If x = y, p(x, y) may not be 0.

Definition 1.12. Let Ψ be the set of all altering distance functions $\psi : [0,\infty) \to [0,\infty)$
such that
(i) ψ is continuous and non - decreasing.
(ii) ψ (t) = 0 if and only if t = 0.

Definition 1.13. Let Φ be the set of all functions $\phi : [0,\infty) \to [0,\infty)$ such that
(i) ϕ is continuous .
(ii) ϕ (t) = 0 if and only if t = 0.

Defnition 1.14. [14] An element (x, y,z) $\in$ X $\times$ X $\times$X is called
(i) a tripled coincident point of F : X $\times$ X $\times$X $\to$ X and f : X $\to$X if fx = F(x, y, z), fy = F(y, z, x) and fz = F(z, x, y),
(ii) a common tripled fixed point of F : X $\times$X $\times$X $\to$ X and f : X $\to$X if
x = fx = F(x, y, z), y = fy = F(y, z, x) and z = fz = F(z, x, y).

Defnition 1.15. [14] The mappings F : X $\times$X $\times$X $\to$ X and f : X $\to$X are called w - compatible if f(F(x, y,z)) = F(fx, fy,fz), f(F(y, z,x)) = F(fy, fz, fx) and f(F(z, x,y)) = F(fz,fx,fy) whenever fx = F(x, y, z), fy = F(y, z, x) and fz = F(z, x, y).

Now we prove our main result.

2. Main Result

Theorem 2.1. Let (X, p) be a partial metric space and let S, T: X $\times$X $\times$X $\to$ X and f, g : X $\to$ X be mappings satisfying

(2.1.1) $$\frac{1}{2}\min\{p(fx,S(x,y,z)), p(gu,T(u,v,w))\} \le \max\{p(fx,gu),\quad p(fy,gv),\quad p(fz,gw)\}$$

or

$$\frac{1}{2}\min\{p(fy,S(y,z,x)), p(gv,T(v,w,u))\} \le \max\{p(fx,gu),\quad p(fy,gv),\quad p(fz,gw)\}$$

or

$$\frac{1}{2}\min\{p(fz,S(z,x,y)), p(gw,T(w,u,v))\} \le \max\{p(fx,gu),\quad p(fy,gv),\quad p(fz,gw)\}$$

implies that

$$\psi(p(S(x,y,z),T(u,v,w))) \le \psi(M(x,y,z,u,v,w)) - \phi(M(x,y,z,u,v,w)),$$

for all x, y , z ,u , v, w in X, where $\psi \in \Psi$ and $\phi \in \Phi$ are functions and

$$M(x,y,z,u,v,w) = \max\left\{\begin{array}{c} p(fx,gu),\quad p(fy,gv),\quad p(fz,gw), \\ p(fx,S(x,y,z)),\quad p(fy,S(y,z,x)), \\ p(fz,S(z,x,y)),\quad p(gu,T(u,v,w)), \\ p(gv,T(v,w,u)),\quad p(gw,T(w,u,v)), \\ \frac{1}{2}[p(fx,T(u,v,w)) + p(gu,S(x,y,z))], \\ \frac{1}{2}[p(fy,T(v,w,u)) + p(gv,S(y,z,x))], \\ \frac{1}{2}[p(fz,T(w,u,v)) + p(gw,S(z,x,y))] \end{array}\right\},$$

(2.1.2) $S(X \times X \times X) \subseteq g(X)$, $T(X \times X \times X) \subseteq f(X)$,
(2.1.3) either f(X) or g(X) is a complete subspaces of X,
(2.1.4) the pairs (f, S) and (g, T) are weakly compatible.
Then S, T, f and g have a unique common tripled fixed point of the form $(\alpha,\alpha,\alpha) \in X \times X \times X$.

Proof: Let $x_0, y_0, z_0 \in X$ be arbitrary points in X. From (2.1.2), there exist sequences of $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ in X such that

$S(x_{2n}, y_{2n}, z_{2n}) = gx_{2n+1} = u_{2n}$,
$S(y_{2n}, z_{2n}, x_{2n}) = gy_{2n+1} = v_{2n}$,
$S(z_{2n}, x_{2n}, y_{2n}) = gz_{2n+1} = w_{2n}$,
$T(x_{2n+1}, y_{2n+1}, z_{2n+1}) = fx_{2n+2} = u_{2n+1}$,
$T(y_{2n+1}, z_{2n+1}, x_{2n+1}) = fy_{2n+2} = v_{2n+1}$,
$T(z_{2n+1}, x_{2n+1}, y_{2n+1}) = fz_{2n+2} = w_{2n+1}$, n = 0, 1, 2, · · ·

For simplification we denote

$$R_n = \max\{p(u_n,u_{n+1}),\quad p(v_n,v_{n+1}),\quad p(w_n,w_{n+1})\}.$$

Clearly

$$\frac{1}{2}\min\left\{\begin{array}{c} p(fx_{2n},S(x_{2n},y_{2n},z_{2n})), \\ p(gx_{2n+1},T(x_{2n+1},y_{2n+1},z_{2n+1})) \end{array}\right\} \le \max\{p(fx_{2n},gx_{2n+1}),\quad p(fy_{2n},gy_{2n+1}),\quad p(fz_{2n},gz_{2n+1})\}.$$

From (2.1.1), we

$$\psi(p(S(x_{2n},y_{2n},z_{2n}),T(x_{2n+1},y_{2n+1},z_{2n+1}))) \le \psi(M(x_{2n},y_{2n},z_{2n},x_{2n+1},y_{2n+1},z_{2n+1})) - \phi(M(x_{2n},y_{2n},z_{2n},x_{2n+1},y_{2n+1},z_{2n+1})).$$

$$M(x_{2n},y_{2n},z_{2n},x_{2n+1},y_{2n+1},z_{2n+1}) = \max\left\{\begin{array}{c} p(u_{2n-1},u_{2n}),\quad p(v_{2n-1},v_{2n}),\quad p(w_{2n-1},w_{2n}),\quad p(u_{2n-1},u_{2n}), \\ p(v_{2n-1},v_{2n}),\quad p(w_{2n-1},w_{2n}),\quad p(u_{2n+1},u_{2n}),\quad p(v_{2n+1},v_{2n}), \\ p(w_{2n+1},w_{2n}),\quad \frac{1}{2}[p(u_{2n-1},u_{2n+1})+p(u_{2n},u_{2n})], \\ \frac{1}{2}[p(v_{2n-1},v_{2n+1})+p(v_{2n},v_{2n})],\quad \frac{1}{2}[p(w_{2n-1},w_{2n+1})+p(w_{2n},w_{2n})] \end{array}\right\}$$

But $\frac{1}{2}[p(u_{2n-1},u_{2n+1})+p(u_{2n},u_{2n})] \le \frac{1}{2}[p(u_{2n-1},u_{2n})+p(u_{2n},u_{2n+1})]$, from (p_4)

$$\le \max\{p(u_{2n-1},u_{2n}),\ p(u_{2n},u_{2n+1})\}.$$

Similarly,

$$\frac{1}{2}[p(v_{2n-1},v_{2n+1})+p(v_{2n},v_{2n})] \le \max\{p(v_{2n-1},v_{2n}),\ p(v_{2n},v_{2n+1})\}.$$

and

$$\frac{1}{2}[p(w_{2n-1},w_{2n+1})+p(w_{2n},w_{2n})] \le \max\{p(w_{2n-1},w_{2n}),\ p(w_{2n},w_{2n+1})\}.$$

Hence

$$M(x_{2n},y_{2n},z_{2n},x_{2n+1},y_{2n+1},z_{2n+1}) = \max\left\{\begin{array}{c} p(u_{2n-1},u_{2n}),\quad p(v_{2n-1},v_{2n}),\quad p(w_{2n-1},w_{2n}), \\ p(u_{2n},u_{2n+1}),\quad p(v_{2n},v_{2n+1}),\quad p(w_{2n},w_{2n+1}) \end{array}\right\}$$
$$= \max\{R_{2n-1},R_{2n}\}.$$

Thus

$$\psi(p(u_{2n},u_{2n+1})) \le \psi(\max\{R_{2n-1},R_{2n}\}) - \phi(\max\{R_{2n-1},R_{2n}\}).$$

Similarly,

$$\psi(p(v_{2n},v_{2n+1})) \le \psi(\max\{R_{2n-1},R_{2n}\}) - \phi(\max\{R_{2n-1},R_{2n}\})$$

and

$$\psi(p(w_{2n},w_{2n+1})) \le \psi(\max\{R_{2n-1},R_{2n}\}) - \phi(\max\{R_{2n-1},R_{2n}\}).$$

Now,

$$\psi(R_{2n}) = \psi(\max\{p(u_{2n},u_{2n+1}), \quad p(v_{2n},v_{2n+1}), \quad p(w_{2n},w_{2n+1})\})$$
$$= \max\{\psi(p(u_{2n},u_{2n+1})), \quad \psi(p(v_{2n},v_{2n+1})), \quad \psi(p(w_{2n},w_{2n+1}))\}$$
$$\leq \psi(\max\{R_{2n-1},R_{2n}\}) - \phi(\max\{R_{2n-1},R_{2n}\}).$$

We have the following two cases.

Case (a) : If R_{2n} is maximum in the right hand side, we obtain

$$\psi(R_{2n}) \leq \psi(R_{2n}) - \phi(R_{2n}).$$

It follows that $\phi(R_{2n}) = 0$ so that $R_{2n} = 0$:

Thus

$$\max\{p(u_{2n},u_{2n+1}), \quad p(v_{2n},v_{2n+1}), \quad p(w_{2n},w_{2n+1})\} = 0.$$

$$\Rightarrow p(u_{2n},u_{2n+1}) = 0, \quad p(v_{2n},v_{2n+1}) = 0 \quad and \quad p(w_{2n},w_{2n+1}) = 0.$$

$$Thus \; u_{2n} = u_{2n+1}, \quad v_{2n} = v_{2n+1} \text{ and } w_{2n} = w_{2n+1}.$$

Now we claim that $u_{2n+2} = u_{2n+1}, \quad v_{2n+2} = v_{2n+1} \; and \quad w_{2n+2} = w_{2n+1}$.

Clearly

$$\frac{1}{2}\min\left\{\begin{matrix} p(fx_{2n+2},S(x_{2n+2},y_{2n+2},z_{2n+2})), \\ p(gx_{2n+1},T(x_{2n+1},y_{2n+1},z_{2n+1})) \end{matrix}\right\} \leq \max\{p(fx_{2n+2},gx_{2n+1}), \quad p(fy_{2n+2},gy_{2n+1}), \quad p(fz_{2n+2},gz_{2n+1})\}.$$

From (2.1.1),we get

$$\psi(p(S(x_{2n+2},y_{2n+2},z_{2n+2}),T(x_{2n+1},y_{2n+1},z_{2n+1}))) \leq \psi(M(x_{2n+2},y_{2n+2},z_{2n+2},x_{2n+1},y_{2n+1},z_{2n+1})) - \phi(M(x_{2n+2},y_{2n+2},z_{2n+2},x_{2n+1},y_{2n+1},z_{2n+1})).$$

$$M(x_{2n+2},y_{2n+2},z_{2n+2},x_{2n+1},y_{2n+1},z_{2n+1}) =$$
$$\max\left\{\begin{matrix} p(u_{2n+1},u_{2n}), \quad p(v_{2n+1},v_{2n}), \quad p(w_{2n+1},w_{2n}), \quad p(u_{2n+1},u_{2n+2}), \\ p(v_{2n+1},v_{2n+2}), \quad p(w_{2n+1},w_{2n+2}), \quad p(u_{2n+1},u_{2n}), \quad p(v_{2n+1},v_{2n}), \\ p(w_{2n+1},w_{2n}), \quad \frac{1}{2}[p(u_{2n+1},u_{2n+1}) + p(u_{2n},u_{2n+2})], \\ \frac{1}{2}[p(v_{2n+1},v_{2n+1}) + p(v_{2n},v_{2n+2})], \quad \frac{1}{2}[p(w_{2n+1},w_{2n+1}) + p(w_{2n},w_{2n+2})] \end{matrix}\right\}.$$

But

$$\frac{1}{2}[p(u_{2n+2},u_{2n}) + p(u_{2n+1},u_{2n+1})] = \frac{1}{2}[p(u_{2n+2},u_{2n+1}) + p(u_{2n+1},u_{2n+1})]$$
$$\leq p(u_{2n+2},u_{2n+1}), \; from(p_2).$$

Similarly,

$$\frac{1}{2}[p(v_{2n+2},v_{2n}) + p(v_{2n+1},v_{2n+1})] \leq p(v_{2n+2},v_{2n+1})$$

and

$$\frac{1}{2}[p(w_{2n+2},w_{2n}) + p(w_{2n+1},w_{2n+1})] \leq p(w_{2n+2},w_{2n+1}).$$

Hence

$$M(x_{2n+2},y_{2n+2},z_{2n+2},x_{2n+1},y_{2n+1},z_{2n+1}) = R_{2n+1}, \; from(p_2).$$

Thus

$$\psi(p(u_{2n+2},u_{2n+1})) \leq \psi(R_{2n+1}) - \phi(R_{2n+1}).$$

Similarly,

$$\psi(p(v_{2n+2},v_{2n+1})) \leq \psi(R_{2n+1}) - \phi(R_{2n+1})$$

and

$$\psi(p(w_{2n+2},w_{2n+1})) \leq \psi(R_{2n+1}) - \phi(R_{2n+1}).$$

Now,

$$\psi(R_{2n+1}) = \psi(\max\{p(u_{2n+2},u_{2n+1}), \quad p(v_{2n+2},v_{2n+1}), \quad p(w_{2n+2},w_{2n+1})\})$$
$$= \max\{\psi(p(u_{2n+2},u_{2n+1})), \quad \psi(p(v_{2n+2},v_{2n+1})), \quad \psi(p(w_{2n+2},w_{2n+1}))\}$$
$$\leq \psi(R_{2n+1}) - \phi(R_{2n+1}).$$

It follows that $\phi(R_{2n+1}) \leq 0.$ So that $R_{2n+1} = 0.$

Hence $u_{2n+2} = u_{2n+1}, v_{2n+2} = v_{2n+1}$ and $w_{2n+2} = w_{2n+1}$.

Continuing in this way, we can conclude that $u_n = u_{n+k}$, $v_n = v_{n+k}$ and $w_n = w_{n+k}$ for all $k > 0$.

Thus, $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are Cauchy sequences.

Case(b): If R_{2n-1} is maximum, then

$$\psi(R_{2n}) \leq \psi(R_{2n-1}) - \phi(R_{2n-1}) \leq \psi(R_{2n-1}). \quad (1)$$

Since ψ is monotone increasing, we have $R_{2n} \leq R_{2n-1}$.

Similarly $R_{2n+1} \leq R_{2n}$.

Continuing in this way we can conclude that $\{R_n\}$ is non - increasing sequence of non - negative real numbers and must converges to a real number $r \geq 0$ say.

Suppose $r > 0$.

Letting $n \to \infty$ in (1), we get

$$\psi(r) \leq \psi(r) - \phi(r) < \psi(r),$$

a contradiction.

Hence $r = 0$. Thus

$$\lim_{n\to\infty} \max\{p(u_n,u_{n+1}), \quad p(v_n,v_{n+1}), \quad p(w_n,w_{n+1})\} = 0.$$

$$\lim_{n\to\infty} p(u_n,u_{n+1}) = \lim_{n\to\infty} p(v_n,v_{n+1}) = \lim_{n\to\infty} p(w_n,w_{n+1}) = 0. \quad (2)$$

Hence from (p_2), we get

$$\lim_{n\to\infty} p(u_n,u_n) = \lim_{n\to\infty} p(v_n,v_n) = \lim_{n\to\infty} p(w_n,w_n) = 0. \quad (3)$$

From (2) and (3), and by definition of p^s, we get

$$\lim_{n\to\infty} p^s(u_n,u_{n+1}) = 0. \quad (4)$$

$$\lim_{n\to\infty} p^s(v_n,v_{n+1}) = 0. \quad (5)$$

and

$$\lim_{n\to\infty} p^s(w_n,w_{n+1}) = 0. \quad (6)$$

Now we prove that $\{u_{2n}\}$,$\{v_{2n}\}$ and $\{w_{2n}\}$ is a Cauchy sequence in (X, p^s). On contrary suppose that $\{u_{2n}\}$ or $\{v_{2n}\}$ or $\{w_{2n}\}$ is not Cauchy.

There exist an $\varepsilon > 0$ and monotone increasing sequences of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k > k$,

$$\max\{p^s(u_{2m_k},u_{2n_k}), \quad p^s(v_{2m_k},v_{2n_k}), \quad p^s(w_{2m_k},w_{2n_k})\} \geq \varepsilon \quad (7)$$

and

$$\max\{p^s(u_{2m_k},u_{2n_{k-2}}), \quad p^s(v_{2m_k},v_{2n_{k-2}}), \quad p^s(w_{2m_k},w_{2n_{k-2}})\} < \varepsilon. \quad (8)$$

From (7) and (8), we obtain

$$\varepsilon \leq \max\{p^s(u_{2m_k},u_{2n_k}), \quad p^s(v_{2m_k},v_{2n_k}), \quad p^s(w_{2m_k},w_{2n_k})\}$$
$$\leq \max\{p^s(u_{2m_k},u_{2n_{k-2}}), \quad p^s(v_{2m_k},v_{2n_{k-2}}), \quad p^s(w_{2m_k},w_{2n_{k-2}})\}$$
$$+ \max\{p^s(u_{2n_{k-2}},u_{2n_{k-1}}), \quad p^s(v_{2n_{k-2}},v_{2n_{k-1}}), \quad p^s(w_{2n_{k-2}},w_{2n_{k-1}})\}$$
$$+ \max\{p^s(u_{2n_{k-1}},u_{2n_k}), \quad p^s(v_{2n_{k-1}},v_{2n_k}), \quad p^s(w_{2n_{k-1}},w_{2n_k})\}$$
$$< \varepsilon + \max\{p^s(u_{2n_{k-2}},u_{2n_{k-1}}), \quad p^s(v_{2n_{k-2}},v_{2n_{k-1}}), \quad p^s(w_{2n_{k-2}},w_{2n_{k-1}})\}$$
$$+ \max\{p^s(u_{2n_{k-1}},u_{2n_k}), \quad p^s(v_{2n_{k-1}},v_{2n_k}), \quad p^s(w_{2n_{k-1}},w_{2n_k})\}.$$

Letting $k \to \infty$ and then using (4), (5) and (6) ,we get

$$\lim_{k\to\infty} \max\{p^s(u_{2m_k},u_{2n_k}), \quad p^s(v_{2m_k},v_{2n_k}), \quad p^s(w_{2m_k},w_{2n_k})\} = \varepsilon. \quad (9)$$

Hence from definition of p^s, we have

$$\lim_{k\to\infty} \max\left\{\begin{array}{c} 2p(u_{2m_k},u_{2n_k}) - p(u_{2m_k},u_{2m_k}) - p(u_{2n_k},u_{2n_k}), \\ 2p(v_{2m_k},v_{2n_k}) - p(v_{2m_k},v_{2m_k}) - p(v_{2n_k},v_{2n_k}), \\ 2p(w_{2m_k},w_{2n_k}) - p(w_{2m_k},w_{2m_k}) - p(w_{2n_k},w_{2n_k}) \end{array}\right\} = \varepsilon.$$

By using (3) , we have

$$\lim_{k\to\infty} \max\left\{p(u_{2m_k}, u_{2n_k}), \quad p(v_{2m_k}, v_{2n_k}), \quad p(w_{2m_k}, w_{2n_k})\right\} = \frac{\varepsilon}{2}. \tag{10}$$

From (7), we have

$$\begin{aligned}
\varepsilon &\le \max\left\{p^s(u_{2m_k}, u_{2n_k}), \quad p^s(v_{2m_k}, v_{2n_k}), \quad p^s(w_{2m_k}, w_{2n_k})\right\} \\
&\le \max\left\{p^s(u_{2m_k}, u_{2m_{k-1}}), \quad p^s(v_{2m_k}, v_{2m_{k-1}}), \quad p^s(w_{2m_k}, w_{2m_{k-1}})\right\} \\
&\qquad + \max\left\{p^s(u_{2m_{k-1}}, u_{2n_k}), \quad p^s(v_{2m_{k-1}}, v_{2n_k}), \quad p^s(w_{2m_{k-1}}, w_{2n_k})\right\} \qquad (11) \\
&\le \max\left\{p^s(u_{2m_k}, u_{2m_{k-1}}), \quad p^s(v_{2m_k}, v_{2m_{k-1}}), \quad p^s(w_{2m_k}, w_{2m_{k-1}})\right\} \\
&\qquad + \max\left\{p^s(u_{2m_{k-1}}, u_{2m_k}), \quad p^s(v_{2m_{k-1}}, v_{2m_k}), \quad p^s(w_{2m_{k-1}}, w_{2m_k})\right\} \\
&\qquad + \max\left\{p^s(u_{2m_k}, u_{2n_k}), \quad p^s(v_{2m_k}, v_{2n_k}), \quad p^s(w_{2m_k}, w_{2n_k})\right\} \\
&= 2\max\left\{p^s(u_{2m_k}, u_{2m_{k-1}}), \quad p^s(v_{2m_k}, v_{2m_{k-1}}), \quad p^s(w_{2m_k}, w_{2m_{k-1}})\right\} \\
&\qquad + \max\left\{p^s(u_{2m_k}, u_{2n_k}), \quad p^s(v_{2m_k}, v_{2n_k}), \quad p^s(w_{2m_k}, w_{2n_k})\right\}.
\end{aligned}$$

Letting k $\longrightarrow \infty$,using (4),(5),(6),(9) and (11), we have

$$\lim_{k\to\infty} \max\left\{p^s(u_{2m_{k-1}}, u_{2n_k}), \quad p^s(v_{2m_{k-1}}, v_{2n_k}), \quad p^s(w_{2m_{k-1}}, w_{2n_k})\right\} = \varepsilon. \tag{12}$$

Hence, we have

$$\lim_{k\to\infty} \max\left\{p(u_{2m_{k-1}}, u_{2n_k}), \quad p(v_{2m_{k-1}}, v_{2n_k}), \quad p(w_{2m_{k-1}}, w_{2n_k})\right\} = \frac{\varepsilon}{2}. \tag{13}$$

Again from (7), we have

$$\begin{aligned}
\varepsilon &\le \max\left\{p^s(u_{2m_k}, u_{2n_k}), \quad p^s(v_{2m_k}, v_{2n_k}), \quad p^s(w_{2m_k}, w_{2n_k})\right\} \\
&\le \max\left\{p^s(u_{2m_k}, u_{2m_{k-1}}), \quad p^s(v_{2m_k}, v_{2m_{k-1}}), \quad p^s(w_{2m_k}, w_{2m_{k-1}})\right\} \\
&\qquad + \max\left\{p^s(u_{2m_{k-1}}, u_{2n_{k+1}}), \quad p^s(v_{2m_{k-1}}, v_{2n_{k+1}}), \quad p^s(w_{2m_{k-1}}, w_{2n_{k+1}})\right\} \\
&\qquad + \max\left\{p^s(u_{2n_{k+1}}, u_{2n_k}), \quad p^s(v_{2n_{k+1}}, v_{2n_k}), \quad p^s(w_{2n_{k+1}}, w_{2n_k})\right\} \qquad (14) \\
&\le \max\left\{p^s(u_{2m_k}, u_{2m_{k-1}}), \quad p^s(v_{2m_k}, v_{2m_{k-1}}), \quad p^s(w_{2m_k}, w_{2m_{k-1}})\right\} \\
&\qquad + \max\left\{p^s(u_{2m_{k-1}}, u_{2m_k}), \quad p^s(v_{2m_{k-1}}, v_{2m_k}), \quad p^s(w_{2m_{k-1}}, w_{2m_k})\right\} \\
&\qquad + \max\left\{p^s(u_{2m_k}, u_{2n_k}), \quad p^s(v_{2m_k}, v_{2n_k}), \quad p^s(w_{2m_k}, w_{2n_k})\right\} \\
&\qquad + \max\left\{p^s(u_{2n_{k+1}}, u_{2n_k}), \quad p^s(v_{2n_{k+1}}, v_{2n_k}), \quad p^s(w_{2n_{k+1}}, w_{2n_k})\right\} \\
&\qquad + \max\left\{p^s(u_{2n_{k+1}}, u_{2n_k}), \quad p^s(v_{2n_{k+1}}, v_{2n_k}), \quad p^s(w_{2n_{k+1}}, w_{2n_k})\right\} \\
&= 2\max\left\{p^s(u_{2m_k}, u_{2m_{k-1}}), \quad p^s(v_{2m_k}, v_{2m_{k-1}}), \quad p^s(w_{2m_k}, w_{2m_{k-1}})\right\} \\
&\qquad + \max\left\{p^s(u_{2m_k}, u_{2n_k}), \quad p^s(v_{2m_k}, v_{2n_k}), \quad p^s(w_{2m_k}, w_{2n_k})\right\} \\
&\qquad + 2\max\left\{p^s(u_{2n_{k+1}}, u_{2n_k}), \quad p^s(v_{2n_{k+1}}, v_{2n_k}), \quad p^s(w_{2n_{k+1}}, w_{2n_k})\right\}.
\end{aligned}$$

Letting k $\longrightarrow \infty$ and then using (4), (5), (6), (9) and (14), we have

$$\lim_{k\to\infty} \max\left\{p^s(u_{2m_{k-1}}, u_{2n_{k+1}}), \quad p^s(v_{2m_{k-1}}, v_{2n_{k+1}}), \quad p^s(w_{2m_{k-1}}, w_{2n_{k+1}})\right\} = \varepsilon. \tag{15}$$

Hence, we have

$$\lim_{k\to\infty} \max\left\{p(u_{2m_{k-1}}, u_{2n_{k+1}}), \quad p(v_{2m_{k-1}}, v_{2n_{k+1}}), \quad p(w_{2m_{k-1}}, w_{2n_{k+1}})\right\} = \frac{\varepsilon}{2}. \tag{16}$$

Again from (7), we have

$$\varepsilon \le \max\{p^s(u_{2m_k}, u_{2n_k}),\quad p^s(v_{2m_k}, v_{2n_k}),\quad p^s(w_{2m_k}, w_{2n_k})\}$$
$$\le \max\{p^s(u_{2m_k}, u_{2n_{k+1}}),\quad p^s(v_{2m_k}, v_{2n_{k+1}}),\quad p^s(w_{2m_k}, w_{2n_{k+1}})\}$$
$$+ \max\{p^s(u_{2n_{k+1}}, u_{2n_k}),\quad p^s(v_{2n_{k+1}}, v_{2n_k}),\quad p^s(w_{2n_{k+1}}, w_{2n_k})\}$$

Letting k $\to \infty$, using (4),(5) and (6), we have

$$\varepsilon \le \lim_{k\to\infty} \max\{p^s(u_{2m_k}, u_{2n_{k+1}}),\quad p^s(v_{2m_k}, v_{2n_{k+1}}),\quad p^s(w_{2m_k}, w_{2n_{k+1}})\} + 0$$

$$\le \lim_{k\to\infty} \max\left\{\begin{array}{c} 2p(u_{2m_k}, u_{2n_{k+1}}) - p(u_{2m_k}, u_{2m_k}) - p(u_{2n_{k+1}}, u_{2n_{k+1}}), \\ 2p(v_{2m_k}, v_{2n_{k+1}}) - p(v_{2m_k}, v_{2m_k}) - p(v_{2n_{k+1}}, v_{2n_{k+1}}), \\ 2p(w_{2m_k}, w_{2n_{k+1}}) - p(w_{2m_k}, w_{2m_k}) - p(w_{2n_{k+1}}, w_{2n_{k+1}}) \end{array}\right\}$$
$$\le 2\lim_{k\to\infty} \max\{p(u_{2m_k}, u_{2n_{k+1}}),\quad p(v_{2m_k}, v_{2n_{k+1}}),\quad p(w_{2m_k}, w_{2n_{k+1}})\},\ from (3).$$

Thus,

$$\frac{\varepsilon}{2} \le \lim_{k\to\infty} \max\{p(u_{2m_k}, u_{2n_{k+1}}),\quad p(v_{2m_k}, v_{2n_{k+1}}),\quad p(w_{2m_k}, w_{2n_{k+1}})\}.$$

By the properties of ψ,

$$\psi\left(\frac{\varepsilon}{2}\right) \le \psi\left(\lim_{k\to\infty} \max\{p(u_{2m_k}, u_{2n_{k+1}}),\quad p(v_{2m_k}, v_{2n_{k+1}}),\quad p(w_{2m_k}, w_{2n_{k+1}})\}\right)$$
$$= \lim_{k\to\infty} \max\{\psi(p(u_{2m_k}, u_{2n_{k+1}})),\quad \psi(p(v_{2m_k}, v_{2n_{k+1}})),\quad \psi(p(w_{2m_k}, w_{2n_{k+1}}))\}. \qquad (17)$$

Now , we show that

$$\frac{1}{2}\min\{p(u_{2m_{k-1}}, u_{2m_k}), p(u_{2n_k}, u_{2n_{k+1}})\} \le \max\left\{\begin{array}{l} p(u_{2m_{k-1}}, u_{2n_k}), \\ p(v_{2m_{k-1}}, v_{2n_k}), \\ p(w_{2m_{k-1}}, w_{2n_k}) \end{array}\right\}.$$

On contrary suppose that

$$\max\left\{\begin{array}{l} p(u_{2m_{k-1}}, u_{2n_k}), \\ p(v_{2m_{k-1}}, v_{2n_k}), \\ p(w_{2m_{k-1}}, w_{2n_k}) \end{array}\right\} < \frac{1}{2}\min\{p(u_{2m_{k-1}}, u_{2m_k}), p(u_{2n_k}, u_{2n_{k+1}})\}.$$

as k $\to \infty$, in above we get $\varepsilon < 1$.
It is a contradiction.
Hence

$$\frac{1}{2}\min\{p(u_{2m_{k-1}}, u_{2m_k}), p(u_{2n_k}, u_{2n_{k+1}})\} \le \max\{p(u_{2m_{k-1}}, u_{2n_k}),\quad p(v_{2m_{k-1}}, v_{2n_k}),\quad p(w_{2m_{k-1}}, w_{2n_k})\}.$$

From (2.1.1), we have

$$\psi(p(u_{2m_k},u_{2n_{k+1}}))=\psi(p(S(x_{2m_k},y_{2m_k},z_{2m_k}),T(x_{2n_{k+1}},y_{2n_{k+1}},z_{2n_{k+1}})))$$

$$\psi(M(x_{2m_k},y_{2m_k},z_{2m_k},x_{2n_{k+1}},y_{2n_{k+1}},z_{2n_{k+1}}))-\phi(M(x_{2m_k},y_{2m_k},z_{2m_k},x_{2n_{k+1}},y_{2n_{k+1}},z_{2n_{k+1}}))$$

$$\leq\psi\left(\max\left\{\begin{array}{c} p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2n_k}),\quad p(u_{2m_{k-1}},u_{2m_k}),\\ p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2m_k}),\quad p(u_{2n_{k+1}},u_{2n_k}),\quad p(v_{2n_{k+1}},v_{2n_k}),\\ p(w_{2n-1},w_{2n}),\quad p(u_{2m_{k-1}},u_{2n_{k+1}}),\quad p(v_{2m_{k-1}},v_{2n_{k+1}}),\quad p(w_{2m_{k-1}},w_{2n_{k+1}}),\\ p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2n_k}),\quad p(w_{2m_{k-1}},w_{2n_k}) \end{array}\right\}\right)$$

$$-\phi\left(\max\left\{\begin{array}{c} p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2n_k}),\quad p(w_{2m_{k-1}},w_{2n_k}),\quad p(u_{2m_{k-1}},u_{2m_k}),\\ p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2m_k}),\quad p(u_{2n_{k+1}},u_{2n_k}),\quad p(v_{2n_{k+1}},v_{2n_k}),\\ p(w_{2n-1},w_{2n}),\quad \frac{1}{2}\left[p(u_{2m_{k-1}},u_{2n_{k+1}})+p(u_{2m_{k-1}},u_{2n_k})\right],\\ \frac{1}{2}\left[p(v_{2m_{k-1}},v_{2n_{k+1}})+p(v_{2m_{k-1}},v_{2n_k})\right],\quad \frac{1}{2}\left[p(w_{2m_{k-1}},w_{2n_{k+1}})+p(w_{2m_{k-1}},w_{2n_k})\right] \end{array}\right\}\right)$$

since ψ is increasing.

Similarly, we have

$$\psi(p(v_{2m_k},v_{2n_{k+1}}))=\psi(p(S(x_{2m_k},y_{2m_k},z_{2m_k}),T(x_{2n_{k+1}},y_{2n_{k+1}},z_{2n_{k+1}})))$$

$$\leq\psi\left(\max\left\{\begin{array}{c} p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2n_k}),\quad p(u_{2m_{k-1}},u_{2m_k}),\\ p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2m_k}),\quad p(u_{2n_{k+1}},u_{2n_k}),\quad p(v_{2n_{k+1}},v_{2n_k}),\\ p(w_{2n-1},w_{2n}),\quad p(u_{2m_{k-1}},u_{2n_{k+1}}),\quad p(v_{2m_{k-1}},v_{2n_{k+1}}),\quad p(w_{2m_{k-1}},w_{2n_{k+1}}),\\ p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2n_k}),\quad p(w_{2m_{k-1}},w_{2n_k}) \end{array}\right\}\right)$$

$$-\phi\left(\max\left\{\begin{array}{c} p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2n_k}),\quad p(w_{2m_{k-1}},w_{2n_k}),\quad p(u_{2m_{k-1}},u_{2m_k}),\\ p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2m_k}),\quad p(u_{2n_{k+1}},u_{2n_k}),\quad p(v_{2n_{k+1}},v_{2n_k}),\\ p(w_{2n-1},w_{2n}),\quad \frac{1}{2}\left[p(u_{2m_{k-1}},u_{2n_{k+1}})+p(u_{2m_{k-1}},u_{2n_k})\right],\\ \frac{1}{2}\left[p(v_{2m_{k-1}},v_{2n_{k+1}})+p(v_{2m_{k-1}},v_{2n_k})\right],\quad \frac{1}{2}\left[p(w_{2m_{k-1}},w_{2n_{k+1}})+p(w_{2m_{k-1}},w_{2n_k})\right] \end{array}\right\}\right)$$

and

$$\psi(p(w_{2m_k},w_{2n_{k+1}}))=\psi(p(S(x_{2m_k},y_{2m_k},z_{2m_k}),T(x_{2n_{k+1}},y_{2n_{k+1}},z_{2n_{k+1}})))$$

$$\leq\psi\left(\max\left\{\begin{array}{c} p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2n_k}),\quad p(u_{2m_{k-1}},u_{2m_k}),\\ p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2m_k}),\quad p(u_{2n_{k+1}},u_{2n_k}),\quad p(v_{2n_{k+1}},v_{2n_k}),\\ p(w_{2n-1},w_{2n}),\quad p(u_{2m_{k-1}},u_{2n_{k+1}}),\quad p(v_{2m_{k-1}},v_{2n_{k+1}}),\quad p(w_{2m_{k-1}},w_{2n_{k+1}}),\\ p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2n_k}),\quad p(w_{2m_{k-1}},w_{2n_k}) \end{array}\right\}\right)$$

$$-\phi\left(\max\left\{\begin{array}{c} p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2n_k}),\quad p(w_{2m_{k-1}},w_{2n_k}),\quad p(u_{2m_{k-1}},u_{2m_k}),\\ p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2m_k}),\quad p(u_{2n_{k+1}},u_{2n_k}),\quad p(v_{2n_{k+1}},v_{2n_k}),\\ p(w_{2n-1},w_{2n}),\quad \frac{1}{2}\left[p(u_{2m_{k-1}},u_{2n_{k+1}})+p(u_{2m_{k-1}},u_{2n_k})\right],\\ \frac{1}{2}\left[p(v_{2m_{k-1}},v_{2n_{k+1}})+p(v_{2m_{k-1}},v_{2n_k})\right],\quad \frac{1}{2}\left[p(w_{2m_{k-1}},w_{2n_{k+1}})+p(w_{2m_{k-1}},w_{2n_k})\right]\end{array}\right\}\right)$$

Hence from (17), we have

$$\psi\left(\frac{\varepsilon}{2}\right)\leq\lim_{k\to\infty}\psi\left(\max\left\{\begin{array}{c} p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2n_k}),\quad p(u_{2m_{k-1}},u_{2m_k}),\\ p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2m_k}),\quad p(u_{2n_{k+1}},u_{2n_k}),\quad p(v_{2n_{k+1}},v_{2n_k}),\\ p(w_{2n-1},w_{2n}),\quad p(u_{2m_{k-1}},u_{2n_{k+1}}),\quad p(v_{2m_{k-1}},v_{2n_{k+1}}),\quad p(w_{2m_{k-1}},w_{2n_{k+1}}),\\ p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2n_k}),\quad p(w_{2m_{k-1}},w_{2n_k})\end{array}\right\}\right)$$

$$-\lim_{k\to\infty}\phi\left(\max\left\{\begin{array}{c} p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2n_k}),\quad p(w_{2m_{k-1}},w_{2n_k}),\quad p(u_{2m_{k-1}},u_{2m_k}),\\ p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2m_k}),\quad p(u_{2n_{k+1}},u_{2n_k}),\quad p(v_{2n_{k+1}},v_{2n_k}),\\ p(w_{2n-1},w_{2n}),\quad \frac{1}{2}\left[p(u_{2m_{k-1}},u_{2n_{k+1}})+p(u_{2m_{k-1}},u_{2n_k})\right],\\ \frac{1}{2}\left[p(v_{2m_{k-1}},v_{2n_{k+1}})+p(v_{2m_{k-1}},v_{2n_k})\right],\quad \frac{1}{2}\left[p(w_{2m_{k-1}},w_{2n_{k+1}})+p(w_{2m_{k-1}},w_{2n_k})\right]\end{array}\right\}\right)$$

$$=\psi\left(\frac{\varepsilon}{2}\right)-\phi(t),$$

$$t=\lim_{k\to\infty}\max\left\{\begin{array}{c} p(u_{2m_{k-1}},u_{2n_k}),\quad p(v_{2m_{k-1}},v_{2n_k}),\quad p(w_{2m_{k-1}},w_{2n_k}),\quad p(u_{2m_{k-1}},u_{2m_k}),\\ p(v_{2m_{k-1}},v_{2m_k}),\quad p(w_{2m_{k-1}},w_{2m_k}),\quad p(u_{2n_{k+1}},u_{2n_k}),\quad p(v_{2n_{k+1}},v_{2n_k}),\\ p(w_{2n-1},w_{2n}),\quad \frac{1}{2}\left[p(u_{2m_{k-1}},u_{2n_{k+1}})+p(u_{2m_{k-1}},u_{2n_k})\right],\\ \frac{1}{2}\left[p(v_{2m_{k-1}},v_{2n_{k+1}})+p(v_{2m_{k-1}},v_{2n_k})\right],\quad \frac{1}{2}\left[p(w_{2m_{k-1}},w_{2n_{k+1}})+p(w_{2m_{k-1}},w_{2n_k})\right]\end{array}\right\}>0$$

$$<\psi\left(\frac{\varepsilon}{2}\right),\quad \text{since}\quad \phi(t)>0\quad \textit{for}\quad t>0.$$

Is a contradiction.
Hence $\{u_{2n}\}$,$\{v_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in the metric space (X, p^s).
Letting n,m $\to\infty$ in

$$\left|p^s(u_{2n+1},u_{2m+1})-p^s(u_{2n},u_{2m})\right|\leq p^s(u_{2n+1},u_{2n})+p^s(u_{2m},u_{2m+1})$$

we have

$$\lim_{n,m\to\infty} p^s(u_{2n+1},u_{2m+1})=0.$$

Similarly we have

$$\lim_{n,m\to\infty} p^s(v_{2n+1},v_{2m+1})=0$$

and

$$\lim_{n,m\to\infty} p^s(w_{2n+1},w_{2m+1})=0.$$

Thus $\{u_{2n+1}\}$,$\{v_{2n+1}\}$ and $\{w_{2n+1}\}$ are Cauchy sequences in the metric space (X, p^s).
Hence $\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are Cauchy sequences in the metric space (X, p^s).
Hence, we have $\lim_{n,m\to\infty} p^s(u_n,u_m)=\lim_{n,m\to\infty} p^s(v_n,v_m)=\lim_{n,m\to\infty} p^s(w_n,w_m)=0.$

Now, from the definition of p^s and from (3), we obtain

$$\lim_{n,m\to\infty} p(u_n, u_m) = 0. \quad (18)$$

$$\lim_{n,m\to\infty} p(v_n, v_m) = 0. \quad (19)$$

and

$$\lim_{n,m\to\infty} p(w_n, w_m) = 0. \quad (20)$$

Suppose g(X) is complete.
Since $u_{2n} = S(x_{2n}, y_{2n}, z_{2n}) = gx_{2n+1}$, it follows $\{u_{2n}\} \subseteq g(X)$, $\{v_{2n}\} \subseteq g(X)$ and $\{w_{2n}\} \subseteq g(X)$ are Cauchy sequence in the complete metric space $(g(X), p^s)$, it follows that $\{u_{2n}\}, \{v_{2n}\}$ and $\{w_{2n}\}$ are convergent in $(g(X), p^s)$.

Thus $$\lim_{n\to\infty} p^s(u_{2n}, \alpha) = 0,$$

$$\lim_{n\to\infty} p^s(v_{2n}, \beta) = 0,$$

and

$$\lim_{n\to\infty} p^s(w_{2n}, \gamma) = 0, \text{ for some } \alpha, \beta \text{ and } \gamma \in g(X).$$

Since $\alpha, \beta, \gamma \in g(X)$, there exist s, t, r $\in$ X such that $\alpha = gs$, $\beta = gt$ and $\gamma = gr$.
Since $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are Cauchy sequences in X and $\{u_{2n}\} \to \alpha$, $\{v_{2n}\} \to \beta$ and $\{w_{2n}\} \to \gamma$, it follows that $\{u_{2n+1}\} \to \alpha$, $\{v_{2n+1}\} \to \beta$ and $\{w_{2n+1}\} \to \gamma$.
From Lemma 1:5 (b), we have

$$p(\alpha,\alpha) = \lim_{n\to\infty} p(u_{2n+1},\alpha) = \lim_{n\to\infty} p(u_{2n},\alpha) = \lim_{n,m\to\infty} p(u_n,u_m). \quad (21)$$

$$p(\beta,\beta) = \lim_{n\to\infty} p(v_{2n+1},\beta) = \lim_{n\to\infty} p(v_{2n},\beta) = \lim_{n,m\to\infty} p(v_n,v_m) \quad (22)$$

and

$$p(\gamma,\gamma) = \lim_{n\to\infty} p(w_{2n+1},\gamma) = \lim_{n\to\infty} p(w_{2n},\gamma) = \lim_{n,m\to\infty} p(w_n,w_m). \quad (23)$$

From (23),(22), (21), (20), (19) and (18) we obtain

$$p(\alpha,\alpha) = \lim_{n\to\infty} p(u_{2n+1},\alpha) = \lim_{n\to\infty} p(u_{2n},\alpha) = 0. \quad (24)$$

$$p(\beta,\beta) = \lim_{n\to\infty} p(v_{2n+1},\beta) = \lim_{n\to\infty} p(v_{2n},\beta) = 0 \quad (25)$$

and

$$p(\gamma,\gamma) = \lim_{n\to\infty} p(w_{2n+1},\gamma) = \lim_{n\to\infty} p(w_{2n},\gamma) = 0. \quad (26)$$

Now we claim that, for each n $\geq$1, atleast one of the following assertions holds.

$$\frac{1}{2} p(u_{2n-1},u_{2n}) \leq \max\{p(u_{2n-1},\alpha), \quad p(v_{2n-1},\beta), \quad p(w_{2n-1},\gamma)\}$$

or

$$\frac{1}{2} p(u_{2n+1},u_{2n}) \leq \max\{p(u_{2n},\alpha), \quad p(v_{2n},\beta), \quad p(w_{2n},\gamma)\}$$

Suppose to the contrary that

$$\frac{1}{2} p(u_{2n-1},u_{2n}) > \max\{p(u_{2n-1},\alpha), \quad p(v_{2n-1},\beta), \quad p(w_{2n-1},\gamma)\}$$

and

$$\frac{1}{2} p(u_{2n+1},u_{2n}) > \max\{p(u_{2n},\alpha), \quad p(v_{2n},\beta), \quad p(w_{2n},\gamma)\}$$

$$p(u_{2n-1},u_{2n}) \leq p(u_{2n-1},\alpha) + p(\alpha,u_{2n}) - p(\alpha,\alpha)$$
$$\leq \frac{1}{2}[p(u_{2n-1},u_{2n}) + p(u_{2n},u_{2n+1})]$$
$$\leq p(u_{2n-1},u_{2n})$$

.

is a contradiction .
Hence claim holds.
Sub case(a) :
Claim : Show that $\alpha = T(s,t,r),\ \beta = T(t,r,s)\ and\ \gamma = T(r,s,t).$

Suppose $\frac{1}{2}p(u_{2n-1},u_{2n}) \leq \max\{p(u_{2n-1},\alpha),\ p(v_{2n-1},\beta),\ p(w_{2n-1},\gamma)\}$

.

Suppose $\alpha \neq T(s,t,r)$or $\beta \neq T(t,r,s)\ or\ \gamma \neq T(r,s,t).$

From (2.1.1), we get

$$\psi(p(S(x_{2n},y_{2n},z_{2n}),T(s,t,r)))$$
$$\leq \psi\left(\max\left\{\begin{array}{c} p(fx_{2n},\alpha),\quad p(fy_{2n},\beta),\quad p(fz_{2n},\gamma), \\ p(fx_{2n},S(x_{2n},y_{2n},z_{2n})),\quad p(fy_{2n},S(y_{2n},z_{2n},x_{2n})), \\ p(fz_{2n},S(z_{2n},x_{2n},y_{2n})),\quad p(\alpha,T(s,t,r)), \\ p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t)), \\ p(fx_{2n},T(s,t,r)),\quad p(fy_{2n},T(t,r,s)),\quad p(fz_{2n},T(r,s,t)), \\ p(\alpha,S(x_{2n},y_{2n},z_{2n})),\quad p(\beta,S(y_{2n},z_{2n},x_{2n})),\quad p(\gamma,S(z_{2n},x_{2n},y_{2n})) \end{array}\right\}\right)$$
$$-\phi\left(\text{mx}\left\{\begin{array}{c} p(fx_{2n},\alpha),\quad p(fy_{2n},\beta),\quad p(fz_{2n},\gamma), \\ p(fx_{2n},S(x_{2n},y_{2n},z_{2n})),\quad p(fy_{2n},S(y_{2n},z_{2n},x_{2n})), \\ p(fz_{2n},S(z_{2n},x_{2n},y_{2n})),\quad p(\alpha,T(s,t,r)), \\ p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t)), \\ \frac{1}{2}[p(fx_{2n},T(s,t,r)) + p(\alpha,S(x_{2n},y_{2n},z_{2n}))], \\ \frac{1}{2}[p(fy_{2n},T(t,r,s)) + p(\beta,S(y_{2n},z_{2n},x_{2n}))], \\ \frac{1}{2}[p(fz_{2n},T(r,s,t)) + p(\gamma,S(z_{2n},x_{2n},y_{2n}))] \end{array}\right\}\right).$$

Letting n $\to \infty$.

$$\psi(p(\alpha,T(s,t,r))) \leq \psi(\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\})$$
$$-\phi(\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\}).$$

Similarly ,

$$\psi(p(\beta,T(t,r,s))) \leq \psi(\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\})$$
$$-\phi(\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\})$$

and

$$\psi(p(\gamma,T(r,s,t))) \leq \psi(\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\})$$
$$-\phi(\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\})$$

Now,

$$\begin{aligned}&\psi(\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\})\\&=\max\{\psi(p(\alpha,T(s,t,r))),\quad \psi(p(\beta,T(t,r,s))),\quad \psi(p(\gamma,T(r,s,t)))\}\\&\le\psi(\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\})\\&\qquad-\phi(\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\}).\end{aligned}$$

It follows that

$$\max\{p(\alpha,T(s,t,r)),\quad p(\beta,T(t,r,s)),\quad p(\gamma,T(r,s,t))\}=0.$$

Therefore $T(s,t,r)=\alpha=gs, T(t,r,s)=\beta=gt \text{ and } T(r,s,t)=\gamma=gr.$

Since (g, T) is weakly compatible, we have

$$T(\alpha,\beta,\gamma)=g\alpha, T(\beta,\gamma,\alpha)=g\beta \text{ and } T(\gamma,\alpha,\beta)=g\gamma.$$

We claim that

$$T(\alpha,\beta,\gamma)=\alpha, T(\beta,\gamma,\alpha)=\beta \text{ and } T(\gamma,\alpha,\beta)=\gamma.$$

Let us suppose that

$$T(\alpha,\beta,\gamma)\ne\alpha \text{ or } T(\beta,\gamma,\alpha)\ne\beta \text{ or } T(\gamma,\alpha,\beta)\ne\gamma.$$

Since $\frac{1}{2}\min\{p(fx_{2n},S(x_{2n},y_{2n},z_{2n})), p(g\alpha,T(\alpha,\beta,\gamma))\}\le\max\{p(fx_{2n},g\alpha),\quad p(fy_{2n},g\beta),\quad p(fz_{2n},g\gamma)\}.$

From (2.1.1), we get

$$\psi(p(S(x_{2n},y_{2n},z_{2n}),T(\alpha,\beta,\gamma)))$$

$$\le\psi\left(\max\left\{\begin{matrix}p(fx_{2n},g\alpha),\quad p(fy_{2n},g\beta),\quad p(fz_{2n},g\gamma),\\ p(fx_{2n},S(x_{2n},y_{2n},z_{2n})),\quad p(fy_{2n},S(y_{2n},z_{2n},x_{2n})),\\ p(fz_{2n},S(z_{2n},x_{2n},y_{2n})),\quad p(g\alpha,T(\alpha,\beta,\gamma)),\\ p(g\beta,T(\beta,\gamma,\alpha)),\quad p(g\gamma,T(\gamma,\alpha,\beta)),\\ p(fx_{2n},T(\alpha,\beta,\gamma)),\quad p(fy_{2n},T(\beta,\gamma,\alpha)),\quad p(fz_{2n},T(\gamma,\alpha,\beta)),\\ p(g\alpha,S(x_{2n},y_{2n},z_{2n})),\quad p(g\beta,S(y_{2n},z_{2n},x_{2n})),\quad p(g\gamma,S(z_{2n},x_{2n},y_{2n}))\end{matrix}\right\}\right)$$

$$-\phi\left(\max\left\{\begin{matrix}p(fx_{2n},g\alpha),\quad p(fy_{2n},g\beta),\quad p(fz_{2n},g\gamma),\quad p(fx_{2n},S(x_{2n},y_{2n},z_{2n})),\\ p(fy_{2n},S(y_{2n},z_{2n},x_{2n})),\quad p(fz_{2n},S(z_{2n},x_{2n},y_{2n})),\quad p(g\alpha,T(\alpha,\beta,\gamma)),\quad p(g\beta,T(\beta,\gamma,\alpha)),\\ p(g\gamma,T(\gamma,\alpha,\beta)),\quad \frac{1}{2}[p(fx_{2n},T(\alpha,\beta,\gamma))+p(g\alpha,S(x_{2n},y_{2n},z_{2n}))],\\ \frac{1}{2}[p(fy_{2n},T(\beta,\gamma,\alpha))+p(g\beta,S(y_{2n},z_{2n},x_{2n}))],\quad \frac{1}{2}[p(fz_{2n},T(\gamma,\alpha,\beta))+p(g\gamma,S(z_{2n},x_{2n},y_{2n}))]\end{matrix}\right\}\right)$$

Letting n $\to\infty$.

$$\begin{aligned}\psi(p(\alpha,T(\alpha,\beta,\gamma)))&\le\psi(\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\})\\&\quad-\phi(\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\}).\end{aligned}$$

Similarly,

$$\begin{aligned}\psi(p(\beta,T(\beta,\gamma,\alpha)))&\le\psi(\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\})\\&\quad-\phi(\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\})\end{aligned}$$

and

$$\begin{aligned}\psi(p(\gamma,T(\gamma,\alpha,\beta)))&\le\psi(\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\})\\&\quad-\phi(\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\})\end{aligned}$$

Now,

$$\psi(\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\})$$
$$=\max\{\psi(p(\alpha,T(\alpha,\beta,\gamma))),\quad \psi(p(\beta,T(\beta,\gamma,\alpha))),\quad \psi(p(\gamma,T(\gamma,\alpha,\beta)))\}$$
$$\leq \psi(\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\})$$
$$-\phi(\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\}).$$

It follows that.

$$\max\{p(\alpha,T(\alpha,\beta,\gamma)),\quad p(\beta,T(\beta,\gamma,\alpha)),\quad p(\gamma,T(\gamma,\alpha,\beta))\}=0.$$

Hence

$$T(\alpha,\beta,\gamma)=g\alpha=\alpha,\quad T(\beta,\gamma,\alpha)=g\beta=\beta \text{ and } T(\gamma,\alpha,\beta)=g\gamma=\gamma. \qquad (27)$$

Since T(X × X× X) $\subseteq$ f(X), then there exist a,b,c $\in$ X such that

$$T(\alpha,\beta,\gamma)=fa=\alpha,\quad T(\beta,\gamma,\alpha)=fb=\beta \text{ and } T(\gamma,\alpha,\beta)=fc=\gamma.$$

Claim: S(a, b,c) = fa = α , S(b, c,a) = fb = β and S(c, a, b) = fc = γ .

Suppose that S(a, b,c) $\neq \alpha$ or S(b, c,a) $\neq \beta$ or S(c, a, b) $\neq \gamma$.

Clearly,

$$\frac{1}{2}\min\{p(fa,S(a,b,c)),p(g\alpha,T(\alpha,\beta,\gamma))\}\leq \max\{p(fa,g\alpha),\quad p(fb,g\beta),\quad p(fc,g\gamma)\}.$$

From (2.1.1), we get

$$\psi(p(S(a,b,c),fa))=\psi(p(S(a,b,c),T(\alpha,\beta,\gamma))$$
$$\leq \psi(M(a,b,c,\alpha,\beta,\gamma))$$
$$-\phi(M(a,b,c,\alpha,\beta,\gamma)).$$

$$M(a,b,c,\alpha,\beta,\gamma)=\max\left\{\begin{array}{c} p(\alpha,\alpha),\quad p(\beta,\beta),\quad p(\gamma,\gamma),\quad p(\alpha,S(a,b,c)),\\ p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b)),\quad p(\alpha,\alpha),\quad p(\beta,\beta),\\ p(\gamma,\gamma),\quad \frac{1}{2}[p(\alpha,\alpha)+p(\alpha,S(a,b,c))],\\ \frac{1}{2}[p(\beta,\beta)+p(\beta,S(b,c,a))],\quad \frac{1}{2}[p(\gamma,\gamma)+p(\gamma,S(c,a,b))] \end{array}\right\}$$
$$=\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\}, \mathit{from}(24),(25)\mathit{and}(26).$$

Hence

$$\psi(p(S(a,b,c),\alpha))\leq \psi(\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\})$$
$$-\phi(\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\}).$$

Similarly,

$$\psi(p(S(b,c,a),\beta))\leq \psi(\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\})$$
$$-\phi(\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\})$$

and

$$\psi(p(S(c,a,b),\gamma))\leq \psi(\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\})$$
$$-\phi(\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\}).$$

Now,

$$\psi(\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\})$$
$$=\max\{\psi(p(\alpha,S(a,b,c))),\quad \psi(p(\beta,S(b,c,a))),\quad \psi(p(\gamma,S(c,a,b)))\}$$
$$\leq \psi(\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\})$$
$$-\phi(\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\}).$$

It follows that

$$\max\{p(\alpha,S(a,b,c)),\quad p(\beta,S(b,c,a)),\quad p(\gamma,S(c,a,b))\}=0.$$

Hence $S(a,b,c)=fa=\alpha,\quad S(b,c,a)=fb=\beta,\quad and\quad S(c,a,b)=fc=\gamma.$

Since (S,f) is weakly compatible.

Thus $S(\alpha,\beta,\gamma)=f\alpha,\quad S(\beta,\gamma,\alpha)=f\beta,\quad and\quad S(\gamma,\alpha,\beta)=f\gamma.$

Suppose $S(\alpha,\beta,\gamma)\neq\alpha$ or $S(\beta,\gamma,\alpha)\neq\beta$ or $S(\gamma,\alpha,\beta)\neq\gamma$.
Clearly,

$$\frac{1}{2}\min\{p(f\alpha,S(\alpha,\beta,\gamma)),p(gs,T(s,t,r))\}\leq\max\{p(f\alpha,gs),\quad p(f\beta,gt),\quad p(f\gamma,gr)\}.$$

From (2.1.1), we get

$$\begin{aligned}\psi(p(f\alpha,\alpha)) &= \psi(p(S(\alpha,\beta,\gamma),T(s,t,r))\\ &\leq \psi(M(\alpha,\beta,\gamma,s,t,r))\\ &\quad -\phi(M(\alpha,\beta,\gamma,s,t,r)).\end{aligned}$$

$$M(\alpha,\beta,\gamma,s,t,r)$$

$$=\max\left\{\begin{array}{c} p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma),\quad p(f\alpha,f\alpha),\\ p(f\beta,f\beta),\quad p(f\gamma,f\gamma),\quad p(\alpha,\alpha),\quad p(\beta,\beta),\\ p(\gamma,\gamma),\quad \frac{1}{2}[p(f\alpha,\alpha)+p(f\alpha,\alpha)],\\ \frac{1}{2}[p(f\beta,\beta)+p(f\beta,\beta)],\quad \frac{1}{2}[p(f\gamma,\gamma)+p(f\gamma,\gamma)]\end{array}\right\}$$

$$=\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\},\ from (24), (25) and (26).$$

Hence

$$\begin{aligned}\psi(p(f\alpha,\alpha)) &\leq \psi(\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\})\\ &\quad -\phi(\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\}).\end{aligned}$$

Similarly,

$$\begin{aligned}\psi(p(f\beta,\beta)) &\leq \psi(\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\})\\ &\quad -\phi(\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\})\end{aligned}$$

and

$$\begin{aligned}\psi(p(f\gamma,\gamma)) &\leq \psi(\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\})\\ &\quad -\phi(\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\}).\end{aligned}$$

Now,

$$\begin{aligned}\psi(\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\}) &= \max\{\psi(p(f\alpha,\alpha)),\quad \psi(p(f\beta,\beta)),\quad \psi(p(f\gamma,\gamma))\}\\ &\leq \psi(\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\})\\ &\quad -\phi(\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\}).\end{aligned}$$

It follows that.

$$\max\{p(f\alpha,\alpha),\quad p(f\beta,\beta),\quad p(f\gamma,\gamma)\}=0.$$

Therefore

$$S(\alpha,\beta,\gamma)=f\alpha=\alpha,\quad S(\beta,\gamma,\alpha)=f\beta=\beta\quad and\quad S(\gamma,\alpha,\beta)=f\gamma=\gamma. \qquad (28)$$

From (27) and (28) , (α,β,γ) is common tripled fixed point of S, T, f and g. Suppose (α',β',γ') is another common tripled fixed point of S, T, f and g.
Clearly

$$\frac{1}{2}\min\{p(f\alpha,S(\alpha,\beta,\gamma)),p(g\alpha',T(\alpha',\beta',\gamma'))\}\leq\max\left\{\begin{array}{c}p(f\alpha,g\alpha'),\\ p(f\beta,g\beta'),\\ p(f\gamma,g\gamma')\end{array}\right\}.$$

From (2.1.1), we get

$$\psi(p(\alpha,\alpha'))=\psi(p(S(\alpha,\beta,\gamma),T(\alpha',\beta',\gamma'))$$
$$\leq\psi(M(\alpha,\beta,\gamma,\alpha',\beta',\gamma'))$$
$$-\phi(M(\alpha,\beta,\gamma,\alpha',\beta',\gamma')).$$

$$M(\alpha,\beta,\gamma,\alpha',\beta',\gamma')=\max\left\{\begin{array}{c} p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma'),\quad p(\alpha,\alpha), \\ p(\beta,\beta),\quad p(\gamma,\gamma),\quad p(\alpha',\alpha'),\quad p(\beta',\beta'), \\ p(\gamma',\gamma'),\quad \frac{1}{2}[p(\alpha,\alpha')+p(\alpha',\alpha)], \\ \frac{1}{2}[p(\beta,\beta')+p(\beta',\beta)],\quad \frac{1}{2}[p(\gamma,\gamma')+p(\gamma',\gamma)] \end{array}\right\}$$
$$=\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\}, from (24),(25),(26) and (p_2).$$

Hence

$$\psi(p(\alpha,\alpha'))\leq\psi(\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\})$$
$$-\phi(\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\}).$$

Similarly,

$$\psi(p(\beta,\beta'))\leq\psi(\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\})$$
$$-\phi(\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\})$$

and

$$\psi(p(\gamma,\gamma'))\leq\psi(\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\})$$
$$-\phi(\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\}).$$

Now,

$$\psi(\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\})=\max\{\psi(p(\alpha,\alpha')),\quad \psi(p(\beta,\beta')),\quad \psi(p(\gamma,\gamma'))\}$$
$$\leq\psi(\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\})$$
$$-\phi(\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\}).$$

It follows that.

$$\max\{p(\alpha,\alpha'),\quad p(\beta,\beta'),\quad p(\gamma,\gamma')\}=0.$$

Therefore

$$\alpha=\alpha',\quad \beta=\beta'\quad and\quad \gamma=\gamma'.$$

Therefore (α,β,γ) is a unique common tripled fixed point of S, T, f and g.

Now we claim that $\alpha=\beta,\quad \beta=\gamma\quad and\quad \gamma=\alpha.$

Suppose $\alpha\neq\beta$ or $\beta\neq\gamma$ or $\gamma\neq\alpha.$

Clearly

$$\frac{1}{2}\min\{p(f\alpha,S(\alpha,\beta,\gamma)),p(g\beta,T(\beta,\gamma,\alpha))\}\leq\max\left\{\begin{array}{c} p(f\alpha,g\beta), \\ p(f\beta,g\gamma), \\ p(f\gamma,g\alpha) \end{array}\right\}.$$

From (2.1.1), we get

$$\psi(p(\alpha,\beta))=\psi(p(S(\alpha,\beta,\gamma),T(\beta,\gamma,\alpha))$$
$$\leq\psi(M(\alpha,\beta,\gamma,\beta,\gamma,\alpha))$$
$$-\phi(M(\alpha,\beta,\gamma,\beta,\gamma,\alpha)).$$

$$M(\alpha,\beta,\gamma,\beta,\gamma,\alpha) = \max \left\{ \begin{array}{c} p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha), \quad p(\alpha,\alpha), \\ p(\beta,\beta), \quad p(\gamma,\gamma), \quad p(\beta,\beta), \quad p(\gamma,\gamma), \\ p(\alpha,\alpha), \quad \frac{1}{2}[p(\alpha,\beta)+p(\beta,\alpha)], \\ \frac{1}{2}[p(\beta,\gamma)+p(\gamma,\beta)], \quad \frac{1}{2}[p(\gamma,\alpha)+p(\alpha,\gamma)] \end{array} \right\}$$

$$= \max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\}.$$

$$\psi(p(\alpha,\beta)) \le \psi(\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\}) - \phi(\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\}).$$

Similarly,

$$\psi(p(\beta,\gamma)) \le \psi(\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\}) - \phi(\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\})$$

and

$$\psi(p(\gamma,\alpha)) \le \psi(\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\}) - \phi(\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\}).$$

Now,

$$\psi(\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\}) = \max\{\psi(p(\alpha,\beta)), \quad \psi(p(\beta,\gamma)), \quad \psi(p(\gamma,\alpha))\}$$
$$\le \psi(\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\}) - \phi(\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\}).$$

It follows that,

$$\max\{p(\alpha,\beta), \quad p(\beta,\gamma), \quad p(\gamma,\alpha)\} = 0.$$

Hence $\alpha=\beta, \beta=\gamma$ and $\gamma=\alpha$.

Therefore (α,α,α) is a unique common tripled fixed point of S, T, f and g. Sub case (b): There exist a unique common tripled fixed point of S, T, f and g when $\frac{1}{2}p(u_{2n+1},u_{2n}) \le \max\{p(u_{2n},\alpha), \quad p(v_{2n},\beta), \quad p(w_{2n},\gamma)\}$ holds.

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