Suzuki Type Unique Common Tripled Fixed Point Theorem for Four Maps under $\Psi$ - $\Phi$ Contractive Condition in Partial Metric Spaces

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Abstract: In this paper, we obtain a Suzuki type unique common tripled fixed point theorem for four maps in partial metric spaces.

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1. Introduction

The notion of a partial metric space was introduced by Matthews [9] as a part of the study of denotational semantics of data ow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation and domain theory in computer science.

Matthews [9] and Romaguera [11] and Altun et al. [2] proved some fixed point theorems in partial metric spaces for a single map. For more works on fixed, common fixed point theorems in partial metric spaces, we refer [1-12]. The aim of this paper is to study Suzuki type unique common tripled fixed point theorem for four maps satisfying a $\Psi$ - $\Phi$ contractive condition in partial metric spaces.

First we give the following theorem of Suzuki [15].

Theorem 1.1. (See [15]): Let $(X, d)$ be a complete metric space and let $T$ be a mapping on $X$. Define a non-increasing function $\theta$ from $[0, 1]$ onto $(\frac{1}{2}, 1]$ by

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5} - 1}{2}, \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5} - 1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r) d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$.

Then there exists a unique fixed point $z$ of $T$.

Moreover $\lim_{n \to \infty} T^n x = z$ for all $x \in X$.

First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.4. (See [1,9]) A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

$$(p_1) x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$$

$$(p_2) p(x, x) \leq p(x, y), p(y, y) \leq p(x, y),$$

$$(p_3) p(x, y) = p(y, x),$$

$$(p_4) p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

The pair $(X, p)$ is called a partial metric space (PMS).

If $p$ is a partial metric on $X$, then the function $p^r : X \times X \to \mathbb{R}^+$ given by

$$p^r(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

(1.1) is a metric on $X$.

Example 1.5. (See e.g. [9]) Consider $X = [0, \infty)$ with $p(x, y) = \max\{x, y\}$. Then $(X, p)$ is a partial metric space.

It is clear that $p$ is not a (usual) metric. Note that in this case $p^r(x, y) = |x - y|$.

Example 1.6. Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and define

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}.$$ Then $(X, p)$ is a partial metric space.

We now state some basic topological notions (such as convergence, completeness, continuity) on partial metric spaces (see e.g. [1, 2, 7, 8, 9]).

Definition 1.7.

(i) A sequence $\{x_n\}$ in the PMS $(X, p)$ converges to the limit $x$ if and only if

$$p(x, x_n) = \lim_{n \to \infty} p(x, x_n).$$

(ii) A sequence $\{x_n\}$ in the PMS $(X, p)$ is called a Cauchy sequence if

$$\lim_{n,m \to \infty} p(x_n, x_m)$$

exists and is finite.

(iii) A PMS $(X, p)$ is called complete if every Cauchy
sequence \( \{ x_n \} \) in \( X \) converges with respect to \( \tau_\rho \), to a point \( x \in X \) such that \( p(x, x) = \lim_{n,m \to \infty} p(x_m, x_n) \).

The following lemma is one of the basic results in PMS([1, 2, 7, 8, 9]).

**Lemma 1.8.**

Moreover \( \lim_{n \to \infty} p^*(x, x_n) = 0 \iff \lim_{n \to \infty} p(x, x_n) = \lim_{n \to \infty} p(x_m, x_n) \).

Next, we give two simple lemmas which will be used in the proof of our main result. For the proofs we refer to [1].

**Lemma 1.9.** Assume \( x_n \to z \) as \( n \to \infty \) in a PMS \( (X, p) \) such that \( p(z, z) = 0 \). Then \( p(z, y) = \lim_{n \to \infty} p(x_n, y) \) for every \( y \in X \).

**Lemma 1.10.** Let \((X, p)\) be a PMS. Then

(A) If \( p(x, y) = 0 \) then \( x = y \).

(B) If \( x \neq y \), then \( p(x, y) > 0 \).

**Remark 1.11.** If \( x = y \), \( p(x, y) \) may not be 0.

**Definition 1.12.** Let \( \Psi : [0, \infty) \to [0, \infty) \) be the set of all altering distance functions

such that

(i) \( \Psi \) is continuous and non-decreasing.

(ii) \( \Psi (t) = 0 \) if and only if \( t = 0 \).

**Definition 1.13.** Let \( \Phi \) be the set of all functions

\( \phi : [0, \infty) \to [0, \infty) \) such that

(i) \( \phi \) is continuous.

(ii) \( \phi (t) = 0 \) if and only if \( t = 0 \).

\[ (2.1.1) \quad \frac{1}{2} \min \{ p(fx, S(x, y, z)), p(gu, T(u, v, w)) \} \leq \max \{ p(fx, gu), p(fy, gv), p(fz, gw) \} \]

or

\[ \frac{1}{2} \min \{ p(fy, S(y, z, x)), p(gv, T(v, w, u)) \} \leq \max \{ p(fx, gu), p(fy, gv), p(fz, gw) \} \]

or

\[ \frac{1}{2} \min \{ p(fz, S(z, x, y)), p(gw, T(w, u, v)) \} \leq \max \{ p(fx, gu), p(fy, gv), p(fz, gw) \} \]

implies that

\( \psi(p(S(x, y, z), T(u, v, w))) \leq \psi(M(x, y, z, u, v, w)) - \phi(M(x, y, z, u, v, w)) \),

for all \( x, y, z, u, v, w \) in \( X \), where \( \psi \in \Psi \) and \( \phi \in \Phi \) are functions and

\[
M(x, y, z, u, v, w) = \max \left\{ \frac{1}{2} \left[ p(fx, T(x, y, z)) + p(gu, S(x, y, z)) \right], \frac{1}{2} \left[ p(fy, T(y, z, x)) + p(gv, S(y, z, x)) \right], \frac{1}{2} \left[ p(fz, T(z, x, y)) + p(gw, S(z, x, y)) \right] \right\}.
\]

**2. Main Result**

**Theorem 2.1.** Let \((X, p)\) be a partial metric space and let \( S, T : X \times X \times X \to X \) and \( f, g : X \to X \) be mappings satisfying

(i) \( \psi(p(S(x, y, z), T(u, v, w))) \leq \psi(M(x, y, z, u, v, w)) - \phi(M(x, y, z, u, v, w)) \),

for all \( x, y, z, u, v, w \) in \( X \), where \( \psi \in \Psi \) and \( \phi \in \Phi \) are functions and

\[
M(x, y, z, u, v, w) = \max \left\{ \frac{1}{2} \left[ p(fx, T(x, y, z)) + p(gu, S(x, y, z)) \right], \frac{1}{2} \left[ p(fy, T(y, z, x)) + p(gv, S(y, z, x)) \right], \frac{1}{2} \left[ p(fz, T(z, x, y)) + p(gw, S(z, x, y)) \right] \right\}.
\]
(2.1.2) \( S(X \times X \times X) \subseteq g(X) \), \( T(X \times X \times X) \subseteq f(X) \).
(2.1.3) either \( f(X) \) or \( g(X) \) is a complete subspace of \( X \),
(2.1.4) the pairs \( (f, S) \) and \( (g, T) \) are weakly compatible.
Then \( S, T, f \) and \( g \) have a unique common tripled fixed point
of the form \((a, a, a) \in X \times X \times X \).

Proof: Let \( x_0, y_0, z_0 \in X \) be arbitrary points in \( X \). From
(2.1.2), there exist sequences of \( \{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}, \{v_n\}, \{w_n\} \)
in \( X \) such that

For simplification we denote
\[
R_n = \max \{ p(u_n, u_{n+1}), \quad p(v_n, v_{n+1}), \quad p(w_n, w_{n+1}) \}.
\]

Clearly
\[
\frac{1}{2} \min \left\{ p(fx_{2n}, S(x_{2n}, y_{2n}, z_{2n})), \quad p(gx_{2n+1}, T(x_{2n+1}, y_{2n+1}, z_{2n+1})) \right\} \leq \max \left\{ p(fx_{2n}, gx_{2n+1}), \quad p(fy_{2n}, gy_{2n+1}), \quad p(fz_{2n}, gz_{2n+1}) \right\}.
\]

From (2.1.1), we
\[
\psi(p(S(x_{2n}, y_{2n}, z_{2n}), T(x_{2n+1}, y_{2n+1}, z_{2n+1})) \leq \psi(M(x_{2n}, y_{2n}, z_{2n}, x_{2n+1}, y_{2n+1}, z_{2n+1})) - \phi(M(x_{2n}, y_{2n}, z_{2n}, x_{2n+1}, y_{2n+1}, z_{2n+1})).
\]

Thus
\[
\psi(p(u_{2n-1}, u_{2n+1})) \leq \psi(\max \{R_{2n-1}, R_{2n}\}) - \phi(\max \{R_{2n-1}, R_{2n}\}).
\]

Similarly,
\[
\psi(p(v_{2n-1}, v_{2n+1})) \leq \psi(\max \{R_{2n-1}, R_{2n}\}) - \phi(\max \{R_{2n-1}, R_{2n}\}),
\]

and
\[
\psi(p(w_{2n-1}, w_{2n+1})) \leq \psi(\max \{R_{2n-1}, R_{2n}\}) - \phi(\max \{R_{2n-1}, R_{2n}\}).
\]

Hence
\[
M(x_{2n}, y_{2n}, z_{2n}, x_{2n+1}, y_{2n+1}, z_{2n+1}) = \max \left\{ p(u_{2n-1}, u_{2n}), \quad p(v_{2n-1}, v_{2n}), \quad p(w_{2n-1}, w_{2n}), \quad p(u_{2n-1}, u_{2n+1}), \quad p(v_{2n-1}, v_{2n+1}), \quad p(w_{2n-1}, w_{2n+1}) \right\}
\]

Thus
\[
\psi(p(u_{2n-1}, u_{2n+1})) \leq \psi(\max \{R_{2n-1}, R_{2n}\}) - \phi(\max \{R_{2n-1}, R_{2n}\}).
\]

Similarly,
\[
\psi(p(v_{2n-1}, v_{2n+1})) \leq \psi(\max \{R_{2n-1}, R_{2n}\}) - \phi(\max \{R_{2n-1}, R_{2n}\}),
\]

and
\[
\psi(p(w_{2n-1}, w_{2n+1})) \leq \psi(\max \{R_{2n-1}, R_{2n}\}) - \phi(\max \{R_{2n-1}, R_{2n}\}).
\]

Now,
\[ \psi(R_{2n}) = \psi(\max\{p(u_{2n}, u_{2n+1}),\ p(v_{2n}, v_{2n+1}),\ p(w_{2n}, w_{2n+1})\}) \]
\[ = \max\{\psi(p(u_{2n}, u_{2n+1})),\ \psi(p(v_{2n}, v_{2n+1})),\ \psi(p(w_{2n}, w_{2n+1}))\} \]
\[ \leq \psi(\max\{R_{2n-1}, R_{2n}\}) - \phi(\max\{R_{2n-1}, R_{2n}\}). \]

We have the following two cases.

Case (a) : If \( R_{2n} \) is maximum in the right hand side, we obtain
\[ \psi(R_{2n}) \leq \psi(R_{2n}) - \phi(R_{2n}). \]

It follows that \( \phi(R_{2n}) = 0 \) so that \( R_{2n} = 0 \):

Thus
\[ \max\{p(u_{2n}, u_{2n+1}),\ p(v_{2n}, v_{2n+1}),\ p(w_{2n}, w_{2n+1})\} = 0. \]
\[ \Rightarrow p(u_{2n}, u_{2n+1}) = 0, \quad p(v_{2n}, v_{2n+1}) = 0 \quad \text{and} \quad p(w_{2n}, w_{2n+1}) = 0. \]

Thus \( u_{2n} = u_{2n+1}, \ v_{2n} = v_{2n+1} \) and \( w_{2n} = w_{2n+1} \).

Now we claim that \( u_{2n+2} = u_{2n+1}, \ v_{2n+2} = v_{2n+1} \) and \( w_{2n+2} = w_{2n+1} \).

Clearly
\[ \frac{1}{2} \min\left\{ p(f x_{2n+2}, S(x_{2n+2}, y_{2n+2}, z_{2n+2})), \ p(g x_{2n+1}, T(x_{2n+1}, y_{2n+1}, z_{2n+1})) \right\} \leq \max\{p(f x_{2n+2}, g x_{2n+1}), \ p(f y_{2n+2}, g y_{2n+1}), \ p(f z_{2n+2}, g z_{2n+1})\}. \]

From (2.1.1), we get
\[ \psi(p(S(x_{2n+2}, y_{2n+2}, z_{2n+2}), T(x_{2n+1}, y_{2n+1}, z_{2n+1}))) \leq \psi(M(x_{2n+2}, y_{2n+2}, z_{2n+2}, x_{2n+1}, y_{2n+1}, z_{2n+1})) \]
\[ - \phi(M(x_{2n+2}, y_{2n+2}, z_{2n+2}, x_{2n+1}, y_{2n+1}, z_{2n+1})). \]

\[ M(x_{2n+2}, y_{2n+2}, z_{2n+2}, x_{2n+1}, y_{2n+1}, z_{2n+1}) = \]
\[ \max \left\{\begin{array}{c}
p(u_{2n+1}, u_{2n+2}), \ p(v_{2n+1}, v_{2n+2}), \ p(w_{2n+1}, w_{2n+2}), \ p(u_{2n+1}, u_{2n+2}), \\
p(v_{2n+1}, v_{2n+2}), \ p(w_{2n+1}, w_{2n+2}), \ p(u_{2n+1}, u_{2n+2}), \ p(v_{2n+1}, v_{2n+2}), \\
\frac{1}{2} [p(v_{2n+1}, v_{2n+2}) + p(v_{2n+1}, v_{2n+2})], \ \frac{1}{2} [p(w_{2n+1}, w_{2n+2}) + p(w_{2n+1}, w_{2n+2})]
\end{array} \right\}. \]

But
\[ \frac{1}{2} [p(u_{2n+2}, u_{2n+1}) + p(u_{2n+1}, u_{2n+2})] = \frac{1}{2} [p(u_{2n+2}, u_{2n+1}) + p(u_{2n+1}, u_{2n+2})] \]
\[ \leq p(u_{2n+2}, u_{2n+1}), \text{ from } (p_2). \]

Similarly,
\[ \frac{1}{2} [p(v_{2n+2}, v_{2n+1}) + p(v_{2n+1}, v_{2n+2})] \leq p(v_{2n+2}, v_{2n+1}) \]
and
\[ \frac{1}{2} [p(w_{2n+2}, w_{2n+1}) + p(w_{2n+1}, w_{2n+2})] \leq p(w_{2n+2}, w_{2n+1}). \]

Hence
\[ M(x_{2n+2}, y_{2n+2}, z_{2n+2}, x_{2n+1}, y_{2n+1}, z_{2n+1}) = R_{2n+1}, \text{ from } (p_2). \]

Thus
\[ \psi(p(u_{2n+2}, u_{2n+1})) \leq \psi(R_{2n+1}) - \phi(R_{2n+1}). \]

Similarly,
\[ \psi(p(w_{2n+2}, w_{2n+1})) \leq \psi(R_{2n+1}) - \phi(R_{2n+1}). \]

Now,
\[ \psi(R_{2n+1}) = \psi(\max\{p(u_{2n+2}, u_{2n+1}), \ p(v_{2n+2}, v_{2n+1}), \ p(w_{2n+2}, w_{2n+1})\}) \]
\[ = \max\{\psi(p(u_{2n+2}, u_{2n+1})), \ \psi(p(v_{2n+2}, v_{2n+1})), \ \psi(p(w_{2n+2}, w_{2n+1}))\} \]
\[ \leq \psi(R_{2n+1}) - \phi(R_{2n+1}). \]

It follows that \( \phi(R_{2n+1}) \leq 0. \) So that \( R_{2n+1} = 0. \)

Hence \( u_{2n+2} = u_{2n+1}, v_{2n+2} = v_{2n+1} \) and \( w_{2n+2} = w_{2n+1}. \)

Continuing in this way, we can conclude that \( u_n = u_{n+1}, v_n = v_{n+1} \) and \( w_n = w_{n+1} \) for all \( k > 0. \)

Thus, \( \{u_n\}, \{v_n\} \) and \( \{w_n\} \) are Cauchy sequences.

Case (b): If \( R_{2n+1} \) is maximum, then
\[ \psi(R_{2n+1}) \leq \psi(R_{2n+1}) - \phi(R_{2n+1}) \]
\[ \leq \psi(R_{2n+1}). \tag{1} \]

Since \( \psi \) is monotone increasing, we have \( R_{2n} \leq R_{2n+1}. \)

Similarly \( R_{2n+1} \leq R_{2n}. \)

Continuing in this way we can conclude that \( \{R_n\} \) is non - increasing sequence of non - negative real numbers and must converges to a real number \( r \geq 0 \) say.

Suppose \( r > 0. \)

Letting \( n \to \infty \) in (1), we get
\[ \psi(r) \leq \psi(r) - \phi(r) < \psi(r), \]
a contradiction.

Hence \( r = 0. \) Thus
\[ \lim_{n \to \infty} \max\{p(u_n, u_{n+1}), \ p(v_n, v_{n+1}), \ p(w_n, w_{n+1})\} = 0. \tag{2} \]

(2)

Hence from (2), we get
\[ \lim_{n \to \infty} p(u_n, u_{n+1}) = \lim_{n \to \infty} p(v_n, v_{n+1}) = \lim_{n \to \infty} p(w_n, w_{n+1}) = 0. \tag{3} \]

From (2) and (3), and by definition of \( p^s, \) we get
\[ \epsilon \leq \max\{p^s(u_{2n}, u_{2n+1}), \ p^s(v_{2n}, v_{2n+1}), \ p^s(w_{2n}, w_{2n+1})\} \]
\[ \leq \max\{p^s(u_{2n}, u_{2n+1}), \ p^s(v_{2n}, v_{2n+1}), \ p^s(w_{2n}, w_{2n+1})\} \]
\[ + \max\{p^s(u_{2n-1}, u_{2n}), \ p^s(v_{2n-1}, v_{2n}), \ p^s(w_{2n-1}, w_{2n})\} \]
\[ + \max\{p^s(u_{2n-1}, u_{2n}), \ p^s(v_{2n-1}, v_{2n}), \ p^s(w_{2n-1}, w_{2n})\} \]
\[ < \epsilon + \max\{p^s(u_{2n-1}, u_{2n-2}), \ p^s(v_{2n-1}, v_{2n-2}), \ p^s(w_{2n-1}, w_{2n-2})\} \]
\[ + \max\{p^s(u_{2n-1}, u_{2n-2}), \ p^s(v_{2n-1}, v_{2n-2}), \ p^s(w_{2n-1}, w_{2n-2})\} \]
\[ \text{Letting } k \to \infty \text{ and then using (4), (5) and (6), we get} \]
\[ \lim_{k \to \infty} \max\{p^s(u_{2n-1}, u_{2n}), \ p^s(v_{2n-1}, v_{2n}), \ p^s(w_{2n-1}, w_{2n})\} = \epsilon. \tag{9} \]

Hence from definition of \( p^s, \) we have
\[ \lim_{k \to \infty} \max\{2p(u_{2n}, u_{2n+1}) - p(u_{2n}, u_{2n+1}) - p(u_{2n}, u_{2n}), \ 2p(v_{2n}, v_{2n+1}) - p(v_{2n}, v_{2n+1}) - p(v_{2n}, v_{2n}), \ 2p(w_{2n}, w_{2n+1}) - p(w_{2n}, w_{2n+1}) - p(w_{2n}, w_{2n})\} = \epsilon. \]

By using (3), we have
\[
\lim_{k \to \infty} \max \left\{ p(u_{2m_k}, u_{2n_k}), \ p(v_{2m_k}, v_{2n_k}), \ p(w_{2m_k}, w_{2n_k}) \right\} = \frac{\varepsilon}{2}. \tag{10}
\]

From (7), we have
\[
\varepsilon \leq \max \left\{ p^\ast(u_{2m_k}, u_{2n_k}), \ p^\ast(v_{2m_k}, v_{2n_k}), \ p^\ast(w_{2m_k}, w_{2n_k}) \right\}
\]
\[
\leq \max \left\{ p^\ast(u_{2m_k}, u_{2m_k-1}), \ p^\ast(v_{2m_k}, v_{2m_k-1}), \ p^\ast(w_{2m_k}, w_{2m_k-1}) \right\}
\]
\[
+ \max \left\{ p^\ast(u_{2m_k-1}, u_{2n_k}), \ p^\ast(v_{2m_k-1}, v_{2n_k}), \ p^\ast(w_{2m_k-1}, w_{2n_k}) \right\} \tag{11}
\]
\[
\leq \max \left\{ p^\ast(u_{2m_k-1}, u_{2m_k-1}), \ p^\ast(v_{2m_k-1}, v_{2m_k-1}), \ p^\ast(w_{2m_k-1}, w_{2m_k-1}) \right\}
\]
\[
+ \max \left\{ p^\ast(u_{2m_k-1}, u_{2n_k}), \ p^\ast(v_{2m_k-1}, v_{2n_k}), \ p^\ast(w_{2m_k-1}, w_{2n_k}) \right\} \tag{12}
\]
\[
= 2 \max \left\{ p^\ast(u_{2m_k-1}, u_{2n_k}), \ p^\ast(v_{2m_k-1}, v_{2n_k}), \ p^\ast(w_{2m_k-1}, w_{2n_k}) \right\}
\]

Letting \( k \to \infty \), using (4), (5), (6), (9) and (11), we have
\[
\lim_{k \to \infty} \max \left\{ p^\ast(u_{2m_k-1}, u_{2n_k}), \ p^\ast(v_{2m_k-1}, v_{2n_k}), \ p^\ast(w_{2m_k-1}, w_{2n_k}) \right\} = \varepsilon. \tag{13}
\]

Hence, we have
\[
\lim_{k \to \infty} \max \left\{ p(u_{2m_k-1}, u_{2n_k}), \ p(v_{2m_k-1}, v_{2n_k}), \ p(w_{2m_k-1}, w_{2n_k}) \right\} = \frac{\varepsilon}{2}. \tag{14}
\]

Again from (7), we have
\[
\varepsilon \leq \max \left\{ p^\ast(u_{2m_k-1}, u_{2n_k}), \ p^\ast(v_{2m_k-1}, v_{2n_k}), \ p^\ast(w_{2m_k-1}, w_{2n_k}) \right\}
\]
\[
\leq \max \left\{ p^\ast(u_{2m_k-1}, u_{2m_k-1}), \ p^\ast(v_{2m_k-1}, v_{2m_k-1}), \ p^\ast(w_{2m_k-1}, w_{2m_k-1}) \right\}
\]
\[
+ \max \left\{ p^\ast(u_{2m_k-1}, u_{2n_k}), \ p^\ast(v_{2m_k-1}, v_{2n_k}), \ p^\ast(w_{2m_k-1}, w_{2n_k}) \right\} \tag{15}
\]
\[
\leq \max \left\{ p^\ast(u_{2m_k-1}, u_{2m_k-1}), \ p^\ast(v_{2m_k-1}, v_{2m_k-1}), \ p^\ast(w_{2m_k-1}, w_{2m_k-1}) \right\}
\]
\[
+ \max \left\{ p^\ast(u_{2m_k-1}, u_{2n_k}), \ p^\ast(v_{2m_k-1}, v_{2n_k}), \ p^\ast(w_{2m_k-1}, w_{2n_k}) \right\} \tag{16}
\]
\[
= 2 \max \left\{ p^\ast(u_{2m_k-1}, u_{2n_k}), \ p^\ast(v_{2m_k-1}, v_{2n_k}), \ p^\ast(w_{2m_k-1}, w_{2n_k}) \right\}
\]

Letting \( k \to \infty \) and then using (4), (5), (6), (9) and (16), we have
\[
\lim_{k \to \infty} \max \left\{ p^\ast(u_{2m_k-1}, u_{2n_k}), \ p^\ast(v_{2m_k-1}, v_{2n_k}), \ p^\ast(w_{2m_k-1}, w_{2n_k}) \right\} = \varepsilon. \tag{17}
\]

Hence, we have
\[
\lim_{k \to \infty} \max \left\{ p(u_{2m_k-1}, u_{2n_k}), \ p(v_{2m_k-1}, v_{2n_k}), \ p(w_{2m_k-1}, w_{2n_k}) \right\} = \frac{\varepsilon}{2}. \tag{18}
\]

Again from (7), we have
\[ \varepsilon \leq \max \left\{ p^* (u_{2m_i}, u_{2n_i}), \ p^* (v_{2m_i}, v_{2n_i}), \ p^* (w_{2m_i}, w_{2n_i}) \right\} \]

\[ \leq \max \left\{ p^* (u_{2m_i}, u_{2n_{i+1}}), \ p^* (v_{2m_i}, v_{2n_{i+1}}), \ p^* (w_{2m_i}, w_{2n_{i+1}}) \right\} \]

\[ + \max \left\{ p^* (u_{2n_{i+1}}, u_{2n_i}), \ p^* (v_{2n_{i+1}}, v_{2n_i}), \ p^* (w_{2n_{i+1}}, w_{2n_i}) \right\} \]

Letting \( k \rightarrow \infty \), using (4),(5) and (6), we have

\[ \varepsilon \leq \lim_{k \rightarrow \infty} \max \left\{ p^* (u_{2m_i}, u_{2n_{i+1}}), \ p^* (v_{2m_i}, v_{2n_{i+1}}), \ p^* (w_{2m_i}, w_{2n_{i+1}}) \right\} + 0 \]

\[ \leq \lim_{k \rightarrow \infty} \max \left\{ 2p(u_{2m_i}, u_{2n_{i+1}}) - p(u_{2m_i}, u_{2m_i}) - p(u_{2n_{i+1}}, u_{2n_{i+1}}), \right\} \]

\[ \leq 2 \lim_{k \rightarrow \infty} \max \left\{ p(u_{2m_i}, u_{2n_{i+1}}), \ p(v_{2m_i}, v_{2n_{i+1}}), \ p(w_{2m_i}, w_{2n_{i+1}}) \right\}, \text{from (3)}. \]

Thus,

\[ \frac{\varepsilon}{2} \leq \lim_{k \rightarrow \infty} \max \left\{ p(u_{2m_i}, u_{2n_{i+1}}), \ p(v_{2m_i}, v_{2n_{i+1}}), \ p(w_{2m_i}, w_{2n_{i+1}}) \right\} \]

By the properties of \( \psi \),

\[ \psi \left( \frac{\varepsilon}{2} \right) \leq \psi \left( \lim_{k \rightarrow \infty} \max \left\{ p(u_{2m_i}, u_{2n_{i+1}}), \ p(v_{2m_i}, v_{2n_{i+1}}), \ p(w_{2m_i}, w_{2n_{i+1}}) \right\} \right) \]

\[ = \lim_{k \rightarrow \infty} \max \left\{ \psi(p(u_{2m_i}, u_{2n_{i+1}})), \ \psi(p(v_{2m_i}, v_{2n_{i+1}})), \ \psi(p(w_{2m_i}, w_{2n_{i+1}})) \right\} ; \quad (17) \]

Now, we show that

\[ \frac{1}{2} \min \left\{ p(u_{2m_{i+1}}, u_{2m_i}), p(u_{2n_i}, u_{2n_{i+1}}) \right\} \leq \max \left\{ p(u_{2m_{i+1}}, u_{2n_i}), p(u_{2n_i}, u_{2n_{i+1}}) \right\} \]

On contrary suppose that

\[ \max \left\{ p(v_{2m_i}, v_{2n_i}), p(v_{2n_i}, v_{2n_{i+1}}) \right\} < \frac{1}{2} \min \left\{ p(u_{2m_{i+1}}, u_{2m_i}), p(u_{2m_i}, u_{2n_i}) \right\} \]

as \( k \rightarrow \infty \), in above we get \( \varepsilon < 1 \).

It is a contradiction.

Hence

\[ \frac{1}{2} \min \left\{ p(u_{2m_{i+1}}, u_{2m_i}), p(u_{2n_i}, u_{2n_{i+1}}) \right\} \leq \max \left\{ p(u_{2m_{i+1}}, u_{2n_i}), p(v_{2m_{i+1}}, v_{2n_i}), p(w_{2m_{i+1}}, w_{2n_i}) \right\} \]
From (2.1.1), we have

$$\psi(p(u_{2n_1}, u_{2n_1})) = \psi(p(S(x_{2m_1}, y_{2m_1}, z_{2m_1}), T(x_{2n_1}, y_{2n_1}, z_{2n_1})))$$

$$\psi(M(x_{2m_1}, y_{2m_1}, z_{2m_1}, x_{2n_1}, y_{2n_1}, z_{2n_1})) - \phi(M(x_{2m_1}, y_{2m_1}, z_{2m_1}, x_{2n_1}, y_{2n_1}, z_{2n_1}))$$

$$\leq \psi_{\max}
\begin{align*}
& p(u_{2n_1}, u_{2n_1}), \ p(v_{2n_1}, v_{2n_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2n_1}, u_{2n_1}), \\
& p(v_{2n_1}, v_{2n_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2n_1}, u_{2n_1}), \ p(v_{2n_1}, v_{2n_1}), \\
& p(w_{2n_1}, w_{2n_1}), \ p(u_{2n_1}, u_{2n_1}), \ p(v_{2n_1}, v_{2n_1}), \ p(w_{2m_1}, w_{2m_1})
\end{align*}$$

$$- \phi_{\max}
\begin{align*}
& p(u_{2n_1}, u_{2n_1}), \ p(v_{2n_1}, v_{2n_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2n_1}, u_{2n_1}), \\
& p(v_{2n_1}, v_{2n_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2n_1}, u_{2n_1}), \ p(v_{2n_1}, v_{2n_1}), \\
& p(w_{2n_1}, w_{2n_1}), \ p(u_{2n_1}, u_{2n_1}), \ p(v_{2n_1}, v_{2n_1}), \ p(w_{2m_1}, w_{2m_1})
\end{align*}$$

since \( \psi \) is increasing.

Similarly, we have

$$\psi(p(v_{2m_1}, v_{2n_1})) = \psi(p(S(x_{2m_1}, y_{2m_1}, z_{2m_1}), T(x_{2n_1}, y_{2n_1}, z_{2n_1})))$$

$$\psi(M(x_{2m_1}, y_{2m_1}, z_{2m_1}, x_{2n_1}, y_{2n_1}, z_{2n_1})) - \phi(M(x_{2m_1}, y_{2m_1}, z_{2m_1}, x_{2n_1}, y_{2n_1}, z_{2n_1}))$$

$$\leq \psi_{\max}
\begin{align*}
& p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2m_1}, u_{2m_1}), \\
& p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \\
& p(w_{2n_1}, w_{2n_1}), \ p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1})
\end{align*}$$

$$- \phi_{\max}
\begin{align*}
& p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2m_1}, u_{2m_1}), \\
& p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \\
& p(w_{2n_1}, w_{2n_1}), \ p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1})
\end{align*}$$

and

$$\psi(p(w_{2m_1}, w_{2n_1})) = \psi(p(S(x_{2m_1}, y_{2m_1}, z_{2m_1}), T(x_{2n_1}, y_{2n_1}, z_{2n_1})))$$

$$\psi(M(x_{2m_1}, y_{2m_1}, z_{2m_1}, x_{2n_1}, y_{2n_1}, z_{2n_1})) - \phi(M(x_{2m_1}, y_{2m_1}, z_{2m_1}, x_{2n_1}, y_{2n_1}, z_{2n_1}))$$

$$\leq \psi_{\max}
\begin{align*}
& p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2m_1}, u_{2m_1}), \\
& p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \\
& p(w_{2n_1}, w_{2n_1}), \ p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1})
\end{align*}$$

$$- \phi_{\max}
\begin{align*}
& p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2m_1}, u_{2m_1}), \\
& p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1}), \ p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \\
& p(w_{2n_1}, w_{2n_1}), \ p(u_{2m_1}, u_{2m_1}), \ p(v_{2m_1}, v_{2m_1}), \ p(w_{2m_1}, w_{2m_1})
\end{align*}$$
Letting $n,m \rightarrow \infty$ from (17), we have

$$\psi\left(\frac{\varepsilon}{2}\right) \leq \lim_{k \rightarrow \infty} \max \left\{ \begin{array}{l}
p(u_{2m_{k-1}}, u_{2n_k}), \quad p(v_{2m_{k-1}}, v_{2n_k}), \quad p(w_{2m_{k-1}}, w_{2n_k}), \quad p(u_{2m_{k-1}}, u_{2m_k}), \\
p(v_{2m_{k-1}}, v_{2m_k}), \quad p(w_{2m_{k-1}}, w_{2m_k}), \quad p(u_{2m_{k-1}}, u_{2n_k}), \quad p(v_{2m_{k-1}}, v_{2n_k}), \\
p(u_{2n_{k-1}}, u_{2n_k}), \quad p(v_{2n_{k-1}}, v_{2n_k}), \quad p(w_{2n_{k-1}}, w_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2m_k}), \\
p(v_{2n_{k-1}}, v_{2m_k}), \quad p(w_{2n_{k-1}}, w_{2m_k}), \quad p(u_{2n_{k-1}}, u_{2n_k}), \quad p(v_{2n_{k-1}}, v_{2n_k}), \\
p(v_{2n_{k-1}}, v_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2n_k}), \quad p(w_{2n_{k-1}}, w_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2m_k}), \\
p(w_{2n_{k-1}}, w_{2n_k}), \quad \frac{1}{2}[p(u_{2n_{k-1}}, u_{2n_k}) + p(u_{2n_{k-1}}, u_{2n_k})] \\
\end{array} \right\}$$

Hence from (17), we have

$$\psi\left(\frac{\varepsilon}{2}\right) \leq \lim_{k \rightarrow \infty} \max \left\{ \begin{array}{l}
p(u_{2m_{k-1}}, u_{2n_k}), \quad p(v_{2m_{k-1}}, v_{2n_k}), \quad p(w_{2m_{k-1}}, w_{2n_k}), \quad p(u_{2m_{k-1}}, u_{2m_k}), \\
p(v_{2m_{k-1}}, v_{2m_k}), \quad p(w_{2m_{k-1}}, w_{2m_k}), \quad p(u_{2m_{k-1}}, u_{2n_k}), \quad p(v_{2m_{k-1}}, v_{2n_k}), \\
p(u_{2n_{k-1}}, u_{2n_k}), \quad p(v_{2n_{k-1}}, v_{2n_k}), \quad p(w_{2n_{k-1}}, w_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2m_k}), \\
p(v_{2n_{k-1}}, v_{2m_k}), \quad p(w_{2n_{k-1}}, w_{2m_k}), \quad p(u_{2n_{k-1}}, u_{2n_k}), \quad p(v_{2n_{k-1}}, v_{2n_k}), \\
p(v_{2n_{k-1}}, v_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2n_k}), \quad p(w_{2n_{k-1}}, w_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2m_k}), \\
p(w_{2n_{k-1}}, w_{2n_k}), \quad \frac{1}{2}[p(u_{2n_{k-1}}, u_{2n_k}) + p(u_{2n_{k-1}}, u_{2n_k})] \\
\end{array} \right\}$$

$$-\phi \max \left\{ \begin{array}{l}
p(u_{2m_{k-1}}, u_{2n_k}), \quad p(v_{2m_{k-1}}, v_{2n_k}), \quad p(w_{2m_{k-1}}, w_{2n_k}), \quad p(u_{2m_{k-1}}, u_{2m_k}), \\
p(v_{2m_{k-1}}, v_{2m_k}), \quad p(w_{2m_{k-1}}, w_{2m_k}), \quad p(u_{2m_{k-1}}, u_{2n_k}), \quad p(v_{2m_{k-1}}, v_{2n_k}), \\
p(u_{2n_{k-1}}, u_{2n_k}), \quad p(v_{2n_{k-1}}, v_{2n_k}), \quad p(w_{2n_{k-1}}, w_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2m_k}), \\
p(v_{2n_{k-1}}, v_{2m_k}), \quad p(w_{2n_{k-1}}, w_{2m_k}), \quad p(u_{2n_{k-1}}, u_{2n_k}), \quad p(v_{2n_{k-1}}, v_{2n_k}), \\
p(v_{2n_{k-1}}, v_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2n_k}), \quad p(w_{2n_{k-1}}, w_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2m_k}), \\
p(w_{2n_{k-1}}, w_{2n_k}), \quad \frac{1}{2}[p(u_{2n_{k-1}}, u_{2n_k}) + p(u_{2n_{k-1}}, u_{2n_k})] \\
\end{array} \right\}$$

$$= \psi\left(\frac{\varepsilon}{2}\right) - \phi(t),$$

$$t = \lim_{k \rightarrow \infty} \max \left\{ \begin{array}{l}
p(u_{2m_{k-1}}, u_{2n_k}), \quad p(v_{2m_{k-1}}, v_{2n_k}), \quad p(w_{2m_{k-1}}, w_{2n_k}), \quad p(u_{2m_{k-1}}, u_{2m_k}), \\
p(v_{2m_{k-1}}, v_{2m_k}), \quad p(w_{2m_{k-1}}, w_{2m_k}), \quad p(u_{2m_{k-1}}, u_{2n_k}), \quad p(v_{2m_{k-1}}, v_{2n_k}), \\
p(u_{2n_{k-1}}, u_{2n_k}), \quad p(v_{2n_{k-1}}, v_{2n_k}), \quad p(w_{2n_{k-1}}, w_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2m_k}), \\
p(v_{2n_{k-1}}, v_{2m_k}), \quad p(w_{2n_{k-1}}, w_{2m_k}), \quad p(u_{2n_{k-1}}, u_{2n_k}), \quad p(v_{2n_{k-1}}, v_{2n_k}), \\
p(v_{2n_{k-1}}, v_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2n_k}), \quad p(w_{2n_{k-1}}, w_{2n_k}), \quad p(u_{2n_{k-1}}, u_{2m_k}), \\
p(w_{2n_{k-1}}, w_{2n_k}), \quad \frac{1}{2}[p(u_{2n_{k-1}}, u_{2n_k}) + p(u_{2n_{k-1}}, u_{2n_k})] \\
\end{array} \right\} > 0$$

$$< \psi\left(\frac{\varepsilon}{2}\right),$$

since $\phi(t) > 0$ for $t > 0$.

Is a contradiction.
Hence $\{u_{2n}\}, \{v_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in the metric space $(X, p^\delta)$.

Letting $n,m \rightarrow \infty$ in

$$p^\delta(u_{2n+1}, u_{2m+1}) - p^\delta(u_{2n}, u_{2m}) \leq p^\delta(u_{2n+1}, u_{2n}) + p^\delta(u_{2m}, u_{2m+1})$$

we have

$$\lim_{n,m \rightarrow \infty} p^\delta(u_{2n+1}, u_{2m+1}) = 0.$$

Similarly we have

$$\lim_{n,m \rightarrow \infty} p^\delta(v_{2n+1}, v_{2m+1}) = 0$$

and

$$\lim_{n,m \rightarrow \infty} p^\delta(w_{2n+1}, w_{2m+1}) = 0.$$

Thus $\{u_{2n+1}\}, \{v_{2n+1}\}$ and $\{w_{2n+1}\}$ are Cauchy sequences in the metric space $(X, p^\delta)$.
Hence $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are Cauchy sequences in the metric space $(X, p^\delta)$.
Hence, we have $\lim_{n,m \rightarrow \infty} p^\delta(u_n, u_m) = \lim_{n,m \rightarrow \infty} p^\delta(v_n, v_m) = \lim_{n,m \rightarrow \infty} p^\delta(w_n, w_m) = 0$.

Now, from the definition of $p^\delta$ and from (3), we obtain
\[
\lim_{n,m \to \infty} p(u_n, u_m) = 0. \tag{18}
\]

\[
\lim_{n,m \to \infty} p(v_n, v_m) = 0. \tag{19}
\]

and

\[
\lim_{n,m \to \infty} p(w_n, w_m) = 0. \tag{20}
\]

Suppose \( g(X) \) is complete. Since \( u_{2n} = \text{St}(x_{2n}, y_{2n}, z_{2n}) = g(x_{2n+1}) \), it follows \( \{u_{2n}\} \subseteq g(X) \), \( \{v_{2n}\} \subseteq g(X) \) and \( \{w_{2n}\} \subseteq g(X) \) are Cauchy sequences in the complete metric space \( (g(X), p') \), it follows that \( \{u_{2n}\}, \{v_{2n}\} \) and \( \{w_{2n}\} \) are convergent in \( (g(X), p') \).

Thus

\[
\lim_{n \to \infty} p^\prime(u_{2n}, \alpha) = 0,
\]

and

\[
\lim_{n \to \infty} p^\prime(v_{2n}, \beta) = 0,
\]

and

\[
\lim_{n \to \infty} p^\prime(w_{2n}, \gamma) = 0, \text{ for some } \alpha, \beta \text{ and } \gamma \in g(X).
\]

Since \( \alpha, \beta, \gamma \in g(X) \), there exist \( s, t, r \in X \) such that \( \alpha = gs \), \( \beta = gt \) and \( \gamma = gr \).

Since \( \{u_{2n}\}, \{v_{2n}\} \) and \( \{w_{2n}\} \) are Cauchy sequences in \( X \) and \( \{u_{2n}\} \rightarrow \alpha \), \( \{v_{2n}\} \rightarrow \beta \) and \( \{w_{2n}\} \rightarrow \gamma \), it follows that \( \{u_{2n+1}\} \rightarrow \alpha \), \( \{v_{2n+1}\} \rightarrow \beta \) and \( \{w_{2n+1}\} \rightarrow \gamma \).

From Lemma 1.5 (b), we have

\[
p(\alpha, \alpha) = \lim_{n \to \infty} p(u_{2n+1}, \alpha) = \lim_{n,m \to \infty} p(u_n, u_m). \tag{21}
\]

\[
p(\beta, \beta) = \lim_{n \to \infty} p(v_{2n+1}, \beta) = \lim_{n,m \to \infty} p(v_n, v_m) \tag{22}
\]

and

\[
p(\gamma, \gamma) = \lim_{n \to \infty} p(w_{2n+1}, \gamma) = \lim_{n,m \to \infty} p(w_n, w_m). \tag{23}
\]

From (23),(22), (21), (20), (19) and (18) we obtain

\[
p(\alpha, \alpha) = \lim_{n \to \infty} p(u_{2n+1}, \alpha) = \lim_{n \to \infty} p(u_{2n}, \alpha) = 0. \tag{24}
\]

\[
p(\beta, \beta) = \lim_{n \to \infty} p(v_{2n+1}, \beta) = \lim_{n \to \infty} p(v_{2n}, \beta) = 0 \tag{25}
\]

and

\[
p(\gamma, \gamma) = \lim_{n \to \infty} p(w_{2n+1}, \gamma) = \lim_{n \to \infty} p(w_{2n}, \gamma) = 0. \tag{26}
\]

Now we claim that, for each \( n \geq 1 \), at least one of the following assertions holds.

\[
\frac{1}{2} p(u_{2n-1}, u_{2n}) \leq \max \{ p(u_{2n-1}, \alpha), \ p(v_{2n-1}, \beta), \ p(w_{2n-1}, \gamma) \}
\]

or

\[
\frac{1}{2} p(u_{2n+1}, u_{2n}) \leq \max \{ p(u_{2n}, \alpha), \ p(v_{2n}, \beta), \ p(w_{2n}, \gamma) \}
\]

Suppose to the contrary that

\[
\frac{1}{2} p(u_{2n-1}, u_{2n}) > \max \{ p(u_{2n-1}, \alpha), \ p(v_{2n-1}, \beta), \ p(w_{2n-1}, \gamma) \}
\]

and

\[
\frac{1}{2} p(u_{2n+1}, u_{2n}) > \max \{ p(u_{2n}, \alpha), \ p(v_{2n}, \beta), \ p(w_{2n}, \gamma) \}
\]
\[ p(u_{2n-1}, u_{2n}) \leq p(u_{2n-1}, \alpha) + p(\alpha, u_{2n}) - p(\alpha, \alpha) \]
\[ \leq \frac{1}{2} \left[ p(u_{2n-1}, u_{2n}) + p(u_{2n}, u_{2n+1}) \right] \]
\[ \leq p(u_{2n-1}, u_{2n}) \]

is a contradiction .
Hence claim holds.

Sub case(a) :
Claim : Show that \( \alpha = T(s, t, r) \), \( \beta = T(t, r, s) \) and \( \gamma = T(r, s, t) \).

Suppose \( \frac{1}{2} p(u_{2n-1}, u_{2n}) \leq \max \{ p(u_{2n-1}, \alpha), \ p(v_{2n-1}, \beta), \ p(w_{2n-1}, \gamma) \} \)

Suppose \( \alpha \neq T(s, t, r) \) or \( \beta \neq T(t, r, s) \) or \( \gamma \neq T(r, s, t) \).

From (2.1.1), we get
\[ \psi (p(S(x_{2n}, y_{2n}, z_{2n}), T(s, t, r))) \]
\[ \leq \psi \left( \max \left\{ \begin{array}{c}
\{ p(fx_{2n}, \alpha), \ p(fy_{2n}, \beta), \ p(fz_{2n}, \gamma), \\
p(fx_{2n}, S(x_{2n}, y_{2n}, z_{2n})), \ p(fy_{2n}, S(y_{2n}, z_{2n}, x_{2n})), \\
p(fz_{2n}, S(z_{2n}, x_{2n}, y_{2n})), \ p(\alpha, T(s, t, r)), \\
p(\beta, T(t, r, s)), \ p(\gamma, T(r, s, t)), \\
p(\alpha, S(x_{2n}, y_{2n}, z_{2n})), \ p(\beta, S(y_{2n}, z_{2n}, x_{2n})), \ p(\gamma, S(z_{2n}, x_{2n}, y_{2n}))
\end{array} \right\} \right) \]
\[ - \phi \left( \max \left\{ \begin{array}{c}
\frac{1}{2} \left[ p(fu_{2n}, T(s, t, r)) + p(\alpha, S(x_{2n}, y_{2n}, z_{2n})) \right], \\
\frac{1}{2} [p(fy_{2n}, T(t, r, s)) + p(\beta, S(y_{2n}, z_{2n}, x_{2n}))], \\
\frac{1}{2} [p(fz_{2n}, T(r, s, t)) + p(\gamma, S(z_{2n}, x_{2n}, y_{2n}))]
\end{array} \right\} \right) \]

Letting \( n \rightarrow \infty \).
\[ \psi (p(\alpha, T(s, t, r))) \leq \psi \left( \max \{ p(\alpha, T(s, t, r)), \ p(\beta, T(t, r, s)), \ p(\gamma, T(r, s, t)) \} \right) \]
\[ - \phi \left( \max \{ p(\alpha, T(s, t, r)), \ p(\beta, T(t, r, s)), \ p(\gamma, T(r, s, t)) \} \right) \]

Similarly ,
\[ \psi (p(\beta, T(t, r, s))) \leq \psi \left( \max \{ p(\alpha, T(s, t, r)), \ p(\beta, T(t, r, s)), \ p(\gamma, T(r, s, t)) \} \right) \]
\[ - \phi \left( \max \{ p(\alpha, T(s, t, r)), \ p(\beta, T(t, r, s)), \ p(\gamma, T(r, s, t)) \} \right) \]

and
\[ \psi (p(\gamma, T(r, s, t))) \leq \psi \left( \max \{ p(\alpha, T(s, t, r)), \ p(\beta, T(t, r, s)), \ p(\gamma, T(r, s, t)) \} \right) \]
\[ - \phi \left( \max \{ p(\alpha, T(s, t, r)), \ p(\beta, T(t, r, s)), \ p(\gamma, T(r, s, t)) \} \right) \]

Now,
\[\psi(\max\{p(\alpha, T(s, t, r)), p(\beta, T(t, r, s)), p(\gamma, T(r, s, t))\})\]
\[= \max\{\psi(p(\alpha, T(s, t, r))), \psi(p(\beta, T(t, r, s))), \psi(p(\gamma, T(r, s, t)))\}\]
\[\leq \psi(\max\{p(\alpha, T(s, t, r)), p(\beta, T(t, r, s)), p(\gamma, T(r, s, t))\} - \phi(\max\{p(\alpha, T(s, t, r)), p(\beta, T(t, r, s)), p(\gamma, T(r, s, t))\}).\]

It follows that
\[\max\{p(\alpha, T(s, t, r)), p(\beta, T(t, r, s)), p(\gamma, T(r, s, t))\} = 0.\]

Therefore \(T(s, t, r) = \alpha = gs, T(t, r, s) = \beta = gt\) and \(T(r, s, t) = \gamma = gr\).

Since \((g, T)\) is weakly compatible, we have
\[T(\alpha, \beta, \gamma) = g\alpha, T(\beta, \gamma, \alpha) = g\beta\) and \(T(\gamma, \alpha, \beta) = g\gamma.\)

We claim that
\[T(\alpha, \beta, \gamma) = \alpha, T(\beta, \gamma, \alpha) = \beta\) and \(T(\gamma, \alpha, \beta) = \gamma.\)

Let us suppose that
\[T(\alpha, \beta, \gamma) \neq \alpha\) or \(T(\beta, \gamma, \alpha) \neq \beta\) or \(T(\gamma, \alpha, \beta) \neq \gamma.\)

Since
\[\frac{1}{2} \min[p(fz2n, S(x2n, y2n, z2n)), p(g\alpha, T(\alpha, \beta, \gamma))], \max(p(fz2n, g\alpha), p(fy2n, g\beta), p(fz2n, g\gamma)].\]

From (2.1.1), we get
\[\psi(p(S(x2n, y2n, z2n), T(\alpha, \beta, \gamma)))\]
\[\leq \psi\left(\max\left\{\begin{array}{l}
p(fz2n, g\alpha), p(fy2n, g\beta), p(fz2n, g\gamma), 
p(fz2n, S(x2n, y2n, z2n)), p(fy2n, S(y2n, z2n, x2n)), 
p(fz2n, S(z2n, x2n, y2n)), p(\alpha, T(\alpha, \beta, \gamma)), 
p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta)), 
p(fz2n, T(\alpha, \beta, \gamma)), p(fy2n, T(\beta, \gamma, \alpha)), p(fz2n, T(\gamma, \alpha, \beta)), 
p(g\alpha, S(x2n, y2n, z2n)), p(g\beta, S(y2n, z2n, x2n)), p(g\gamma, S(z2n, x2n, y2n))
\end{array}\right\}\right)\]
\[\leq \psi\left(\max\left\{\begin{array}{l}
p(fz2n, g\alpha), p(fy2n, g\beta), p(fz2n, g\gamma), p(fz2n, S(x2n, y2n, z2n)), p(fy2n, S(y2n, z2n, x2n)), p(fz2n, S(z2n, x2n, y2n)), p(\alpha, T(\alpha, \beta, \gamma)), p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta)), 
p(g\gamma, T(\gamma, \alpha, \beta)), \frac{1}{2}[p(fz2n, T(\alpha, \beta, \gamma)) + p(g\alpha, S(x2n, y2n, z2n))]
\end{array}\right\}\right)\]
\[- \phi\left(\max\left\{\begin{array}{l}
p(fz2n, g\alpha), p(fy2n, g\beta), p(fz2n, g\gamma), p(fz2n, S(x2n, y2n, z2n)), p(fy2n, S(y2n, z2n, x2n)), p(fz2n, S(z2n, x2n, y2n)), p(\alpha, T(\alpha, \beta, \gamma)), p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta)), 
p(g\gamma, T(\gamma, \alpha, \beta)), \frac{1}{2}[p(fz2n, T(\alpha, \beta, \gamma)) + p(g\alpha, S(x2n, y2n, z2n))]
\end{array}\right\}\right)\]
\[-\phi(\max\{p(\alpha, T(\alpha, \beta, \gamma)), p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta))\}).\]

Letting \(n \rightarrow \infty\),
\[\psi(p(\alpha, T(\alpha, \beta, \gamma))) \leq \psi(\max\{p(\alpha, T(\alpha, \beta, \gamma)), p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta))\})\]
\[= \phi(\max\{p(\alpha, T(\alpha, \beta, \gamma)), p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta))\}).\]

Similarly,
\[\psi(p(\beta, T(\beta, \gamma, \alpha))) \leq \psi(\max\{p(\alpha, T(\alpha, \beta, \gamma)), p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta))\})\]
\[= \phi(\max\{p(\alpha, T(\alpha, \beta, \gamma)), p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta))\}).\]

and
\[\psi(p(\gamma, T(\gamma, \alpha, \beta))) \leq \psi(\max\{p(\alpha, T(\alpha, \beta, \gamma)), p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta))\})\]
\[= \phi(\max\{p(\alpha, T(\alpha, \beta, \gamma)), p(\beta, T(\beta, \gamma, \alpha)), p(\gamma, T(\gamma, \alpha, \beta))\}).\]

Now,
\[ \psi \left( \max \{ p(\alpha, T(\alpha, \beta, \gamma)), \ p(\beta, T(\beta, \gamma, \alpha)), \ p(\gamma, T(\gamma, \alpha, \beta)) \} \right) \]
\[ = \max \{ \psi(p(\alpha, T(\alpha, \beta, \gamma))), \ \psi(p(\beta, T(\beta, \gamma, \alpha))), \ \psi(p(\gamma, T(\gamma, \alpha, \beta))) \} \leq \psi \left( \max \{ p(\alpha, T(\alpha, \beta, \gamma)), \ p(\beta, T(\beta, \gamma, \alpha)), \ p(\gamma, T(\gamma, \alpha, \beta)) \} \right) \]
\[ - \phi \left( \max \{ p(\alpha, T(\alpha, \beta, \gamma)), \ p(\beta, T(\beta, \gamma, \alpha)), \ p(\gamma, T(\gamma, \alpha, \beta)) \} \right). \]

It follows that:
\[ \text{max} \{ p(\alpha, T(\alpha, \beta, \gamma)), \ p(\beta, T(\beta, \gamma, \alpha)), \ p(\gamma, T(\gamma, \alpha, \beta)) \} = 0. \]

Hence
\[ T(\alpha, \beta, \gamma) = g_\alpha = \alpha, \ \ T(\beta, \gamma, \alpha) = g_\beta = \beta \quad \text{and} \quad T(\gamma, \alpha, \beta) = g_\gamma = \gamma. \quad (27) \]

Since \( T(X \times X \times X) \subseteq f(X) \), then there exist \( a, b, c \in X \) such that
\[ T(\alpha, \beta, \gamma) = fa = \alpha, \ \ T(\beta, \gamma, \alpha) = fb = \beta \quad \text{and} \quad T(\gamma, \alpha, \beta) = fc = \gamma. \]

Claim: if \( S(a, b, c) \neq \alpha \) or \( S(b, c, a) \neq \beta \) or \( S(c, a, b) \neq \gamma \).

Suppose that \( S(a, b, c) \neq \alpha \) or \( S(b, c, a) \neq \beta \) or \( S(c, a, b) \neq \gamma \).

Clearly,
\[ \frac{1}{2} \min \{ p(fa, S(a, b, c)), p(g\alpha, T(\alpha, \beta, \gamma)) \} \leq \max \{ p(fa, g\alpha), \ p(fb, g\beta), \ p(fc, g\gamma) \}. \]

From (2.1.1), we get
\[ \psi(p(S(a, b, c), fa)) = \psi(p(S(a, b, c), T(\alpha, \beta, \gamma))) \]
\[ \leq \psi(M(a, b, c, \alpha, \beta, \gamma)) \]
\[ - \phi(M(a, b, c, \alpha, \beta, \gamma)). \]

\[ M(a, b, c, \alpha, \beta, \gamma) = \max \left\{ \begin{array}{l} p(\alpha, a), \ \ p(\beta, b), \ \ p(\gamma, c), \ \ p(\alpha, S(a, b, c)), \ \ p(\beta, S(b, c, a)), \ \ p(\gamma, S(c, a, b)) \end{array} \right\} \]
\[ - \phi(M(a, b, c, \alpha, \beta, \gamma)). \]

Hence
\[ \psi(p(S(a, b, c), fa)) \leq \psi(M(a, b, c, \alpha, \beta, \gamma)) \]
\[ - \phi(M(a, b, c, \alpha, \beta, \gamma)). \]

Similarly,
\[ \psi(p(S(b, c, a), fb)) \leq \psi(M(a, b, c, \alpha, \beta, \gamma)) \]
\[ - \phi(M(a, b, c, \alpha, \beta, \gamma)). \]

and
\[ \psi(p(S(c, a, b), fc)) \leq \psi(M(a, b, c, \alpha, \beta, \gamma)) \]
\[ - \phi(M(a, b, c, \alpha, \beta, \gamma)). \]

Now,
\[ \psi(M(a, b, c, \alpha, \beta, \gamma)) \]
\[ = \max \left\{ \psi(p(\alpha, S(a, b, c)), \ \psi(p(\beta, S(b, c, a)), \ \psi(p(\gamma, S(c, a, b)))) \right\} \]
\[ \leq \psi(M(a, b, c, \alpha, \beta, \gamma)) \]
\[ - \phi(M(a, b, c, \alpha, \beta, \gamma)). \]

It follows that
\[ \max \{ p(\alpha, S(a, b, c)), \ p(\beta, S(b, c, a)), \ p(\gamma, S(c, a, b)) \} = 0. \]

Hence \( S(a, b, c) = fa = \alpha, \ S(b, c, a) = fb = \beta \), and \( S(c, a, b) = fc = \gamma. \)

Since (S.f) is weakly compatible.

Thus \( S(\alpha, \beta, \gamma) = fa, \ S(\beta, \gamma, \alpha) = fb, \) and \( S(\gamma, \alpha, \beta) = fc \).
Suppose $S(\alpha, \beta, \gamma) \neq \alpha$ or $S(\beta, \gamma, \alpha) \neq \beta$ or $S(\gamma, \alpha, \beta) \neq \gamma$.

Clearly,

$$\frac{1}{2} \min \{ p(f\alpha, S(\alpha, \beta, \gamma)), p(g\alpha, T(s, t, r)) \} \leq \max \{ p(f\alpha, g\alpha), p(f\beta, g\beta), p(f\gamma, g\gamma) \}.$$

From (2.1.1), we get

$$\psi(p(f\alpha, \alpha)) = \psi(p(S(\alpha, \beta, \gamma), T(s, t, r))) \leq \psi(M(\alpha, \beta, \gamma, s, t, r)) - \phi(M(\alpha, \beta, \gamma, s, t, r)).$$

Hence

$$M(\alpha, \beta, \gamma, \beta, \gamma, T, S, f, g) \leq \max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \} = \max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \}.$$

Similarly,

$$\psi(p(f\beta, \beta)) \leq \psi(\max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \}) - \phi(\max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \})$$

and

$$\psi(p(f\gamma, \gamma)) \leq \psi(\max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \}) - \phi(\max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \}).$$

Now,

$$\psi(\max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \}) = \max \{ \psi(p(f\alpha, \alpha)), \psi(p(f\beta, \beta)), \psi(p(f\gamma, \gamma)) \} \leq \psi(\max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \}) - \phi(\max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \}).$$

It follows that

$$\max \{ p(f\alpha, \alpha), p(f\beta, \beta), p(f\gamma, \gamma) \} = 0.$$ 

Therefore

$$S(\alpha, \beta, \gamma) = f\alpha = \alpha, \quad S(\beta, \gamma, \alpha) = f\beta = \beta \quad \text{and} \quad S(\gamma, \alpha, \beta) = f\gamma = \gamma. \quad (28)$$

From (27) and (28), $(\alpha, \beta, \gamma)$ is common tripled fixed point of $S, T, f$ and $g$. Suppose $(\alpha', \beta', \gamma')$ is another common tripled fixed point of $S, T, f$ and $g$.

Clearly

$$\frac{1}{2} \min \{ p(f\alpha, S(\alpha, \beta, \gamma)), p(g\alpha', T(\alpha', \beta', \gamma')) \} \leq \max \left\{ p(f\alpha, g\alpha'), p(f\beta, g\beta'), p(f\gamma, g\gamma') \right\}.$$
From (2.1.1), we get
\[
\psi(p(\alpha, \alpha')) = \psi(p(S(\alpha, \beta, \gamma), T(\alpha', \beta', \gamma'))) \\
\leq \psi(M(\alpha, \beta, \gamma, \alpha', \beta', \gamma')) \\
- \phi(M(\alpha, \beta, \gamma, \alpha', \beta', \gamma')).
\]

\[
M(\alpha, \beta, \gamma, \alpha', \beta', \gamma') = \max \left\{ \begin{array}{l}
p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma'), \ p(\alpha, \alpha), \\
p(\beta, \beta), \ p(\gamma, \gamma), \ p(\alpha', \alpha'), \ p(\beta', \beta').
\end{array} \right\}
\]

\[
= \max \left\{ \begin{array}{l}
p(\alpha', \alpha), \ p(\beta', \beta), \ p(\gamma, \gamma'), \ p(\gamma, \gamma') \end{array} \right\} \text{from (24), (25), (26) and (p_2)}.
\]

Hence
\[
\psi(p(\alpha, \alpha')) \leq \psi(\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\}) \\
- \phi(\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\}).
\]

Similarly,
\[
\psi(p(\beta, \beta')) \leq \psi(\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\}) \\
- \phi(\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\}).
\]

and
\[
\psi(p(\gamma, \gamma')) \leq \psi(\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\}) \\
- \phi(\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\}).
\]

Now,
\[
\psi(\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\}) = \max \{\psi(p(\alpha, \alpha')), \ \psi(p(\beta, \beta')), \ \psi(p(\gamma, \gamma'))\} \\
\leq \psi(\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\}) \\
- \phi(\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\}).
\]

It follows that.
\[
\max\{p(\alpha, \alpha'), \ p(\beta, \beta'), \ p(\gamma, \gamma')\} = 0.
\]

Therefore
\[
\alpha = \alpha', \ \beta = \beta' \quad \text{and} \quad \gamma = \gamma'.
\]

Therefore \((\alpha, \beta, \gamma)\) is a unique common tripled fixed point of \(S, T, f\) and \(g\).

Now we claim that \(\alpha = \beta, \ \beta = \gamma \quad \text{and} \quad \gamma = \alpha\).

Suppose \(\alpha \neq \beta\) or \(\beta \neq \gamma\) or \(\gamma \neq \alpha\).

Clearly
\[
\frac{1}{2} \min\{p(f\alpha, S(\alpha, \beta, \gamma)), p(g\beta, T(\beta, \gamma, \alpha))\} \leq \max \left\{ \begin{array}{l}
p(f\alpha, g\beta), \\
p(f\beta, g\gamma), \\
p(f\gamma, g\alpha)
\end{array} \right\}.
\]

From (2.1.1), we get
\[
\psi(p(\alpha, \beta)) = \psi(p(S(\alpha, \beta, \gamma), T(\beta, \gamma, \alpha)) \\
\leq \psi(M(\alpha, \beta, \gamma, \alpha)) \\
- \phi(M(\alpha, \beta, \gamma, \alpha)).
\]
\[
M(\alpha, \beta, \gamma, \beta, \gamma, \alpha) = \max \left\{ p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha), p(\alpha, \alpha), p(\beta, \beta), p(\gamma, \gamma), \frac{1}{2}[p(\alpha, \alpha) + p(\beta, \beta)], \frac{1}{2}[p(\beta, \gamma) + p(\gamma, \beta)], \frac{1}{2}[p(\gamma, \alpha) + p(\alpha, \gamma)] \right\}
\]

\[
= \max \left\{ p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha) \right\}
\]

\[
\psi(p(\alpha, \beta)) \leq \psi(\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\}) - \phi(\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\}).
\]

Similarly,
\[
\psi(p(\beta, \gamma)) \leq \psi(\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\}) - \phi(\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\})
\]

and
\[
\psi(p(\gamma, \alpha)) \leq \psi(\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\}) - \phi(\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\}).
\]

Now,
\[
\psi(\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\}) = \max\{\psi(p(\alpha, \beta)), \psi(p(\beta, \gamma)), \psi(p(\gamma, \alpha))\}
\]

\[
\leq \psi(\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\}) - \phi(\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\}).
\]

It follows that,
\[
\max\{p(\alpha, \beta), p(\beta, \gamma), p(\gamma, \alpha)\} = 0.
\]

Hence \( \alpha = \beta = \gamma \) and \( \gamma = \alpha \).

Therefore \( (\alpha, \alpha, \alpha) \) is a unique common tripled fixed point of \( S, T, f \) and \( g \). Sub case (b): There exist a unique common tripled fixed point of \( S, T, f \) and \( g \) when \( \frac{1}{2} p(u_{2n-1}, u_{2n}) \leq \max\{p(u_{2n}, \alpha), p(v_{2n}, \beta), p(w_{2n}, \gamma)\} \) holds.

References


[13] Berinde, V, Borcut, M Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces.

