Local Bifurcation Analysis for a Special Type of SIR Epidemic Model

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Abstract: An analytic study for the local bifurcation proposed for a special type of SIR epidemic model “holling type II treatment failure rate “, the occurrence of trans critical bifurcation proved depending on Sotomayor theorem, while Hopf bifurcation were not realized.

Keywords: Local bifurcation, Hopf bifurcation, Sotomayor theorem, SIR model, holling type II

1. Introduction

In scientific fields as diverse as fluid mechanics, electronics, chemistry and theoretical ecology, there is the application of what is referred to as bifurcation analysis. Bifurcation is the study of changes in the qualitative structures of the solution to a given system of differential equation, a bifurcation occurs when a small smooth change to the parameter values of a system causes a sudden qualitative change in its behavior, bifurcation occur in both continuous systems and discrete systems. The name “bifurcation “ was first introduced by “Henri Poincare” in 1885 in the first paper in mathematic showing such behavior. There are two principle types of bifurcation “Local bifurcation” and “Global bifurcation”, local bifurcation answer the two following questions:

(1) What can happen to the phase portrait near an equilibrium (fixed point) when a parameter passes a bifurcation value?
(2) How to determine which of the alternatives occurs in a concrete system in which one is interested?

Local bifurcation includes, which illustrated in figure no.(1.1):
1) Saddle-node (fold) bifurcation.
2) Trans critical bifurcation.
3) Pitchfork bifurcation.
4) Period-doubling (flip) bifurcation.
5) Hopf bifurcation.
6) Neimark-Sacker (secondary Hopf) bifurcation.


2. Mathematical Model

This research studied and modified model (1), taken from [7] from the classical simple SIR epidemiological model for a set of people with summation equal to N(t) at time t is breaker to three subsets, the susceptible individuals S(t), infected individuals I(t) and the removable individuals R(t).

Such model can be represented as a system of nonlinear differential equations in follows:

\[
\begin{align*}
\frac{dS}{dt} &= \Lambda - \beta SI - \mu S - \theta R - \psi(I-m)I - (\mu + \alpha)I \\
\frac{dI}{dt} &= \beta SI - K1 + I - \theta R - \psi(I-m)I - (\mu + \alpha)I \\
\frac{dR}{dt} &= \psi(I-m)I - (\theta + \mu)R 
\end{align*}
\]

Where \(\Lambda > 0\) is the natural birth rate of the population, \(\beta > 0\) is the incidence rate of the susceptible individuals because of parasitic disease transmitted by contact from the individual to the susceptible, \(\psi > 0\) is the recovery rate, \(m\) is the failure treatment rate such that \((0 \leq m \leq 1)\), \(\theta > 0\) is the loosing immunity rate of the recovered individuals, \(\mu > 0\) natural death rate, \(\alpha > 0\) is the disease related death, \(K1 > 0\) the half saturation constant (for more detail see [7]).

The system of differential equations (1) has two equilibrium points say \(E_0, E_1\) , where \(E_0 = (S_0, 0, 0)\) for \(S_0 = \frac{\Lambda}{\mu}\) called disease free equilibrium point, and \(E_1 = (S_1, I_1, R_1)\), is called endemic equilibrium point where \(S_1 = \frac{\Lambda(K_1+I_1)}{\beta(K_1+I_1)}\), \(I_1 = \frac{\psi(1-m)I_1}{(\theta + \mu)}\) and \(R_1\) is a positive root for the following equation \(D_1I^3 + D_2I^2 + D_3I + D_4I + D_5 = 0\) ....(2)(for more detail see [7]).

3. Local Bifurcation

According to system (1) we will study local bifurcation, those which happened in a small Neighborhood of...
equilibrium or fixed point in dynamical systems depending on one parameter, transcritical of equilibria happened when an exchange of stability occurred, Hopf bifurcation happens when not only change of stability happened but also a periodic solutions are born, see [8].

**Theorem 3.1** System (1) has transcritical bifurcation near the disease free equilibrium point $E_0$, but neither saddle-node bifurcation nor pitchfork bifurcation can occur on the parameter $a_0 = \frac{\beta S_0}{\nu_1} - \psi(1-m) - \mu$.

**Proof**
The Jacobin matrix of system (1) at $(E_0, a_0)$ is $J(E_0, a_0) = 
\begin{bmatrix}
-\mu & -\beta k S_0 & 0 \\
0 & \psi(1-m) & -(\theta + \mu) \\
0 & 0 & 0 \\
\end{bmatrix}$.
Clearly the eigenvalue $\lambda_i$ of system (1) in the I direction is $\lambda_i = 0$, note $\lambda_i = \frac{\beta S_0}{k_1} - (1-m) - \mu - \alpha$, further the eigenvector say $v = [v_1, v_2, v_3]^T$ corresponding to the eigenvalue $\lambda_i$ satisfy the following $Jv = \lambda v$, and because $\lambda = 0$ then $Jv = 0$ where $J = J(E_0, a_0)$.

$v = [v_1, v_2, v_3]^T$ with third eigen value is real and $\psi(1-m)$.

Similarly the eigenvector $\omega = (\omega_1, \omega_2, \omega_3)^T$ corresponding to $\lambda_i$ of $J^T$ can be written as

$J^T = 
\begin{bmatrix}
-\mu & 0 & 0 \\
0 & -\beta k S_0 & 0 \\
0 & \psi(1-m) & -(\theta + \mu) \\
\end{bmatrix}$

We get $\omega = [0, 0, 0]^T$, where $\omega_3$ is any non zero real number.

Now rewrite system (1) in vector form as:

$\frac{dx}{dt} = f(X)$, where $X = (S, I, R)^T$ and $f = (f_1, f_2, f_3)^T$ with $f_i, i = 1, 2, 3$ given in system (1), and then determine $\frac{df}{da} = f_a = \begin{bmatrix} 0 \\
0 \\
-1 \end{bmatrix}$ then $f_a(E_0, a_0) = 0$

Therefore : $\omega^T f_a = 0$, consequently according to Sotomayor theorem [5], the system has no saddle-node bifurcation near $E^o$ at $a_0$. Now in order to investigate the accruing of other types of bifurcation the derivative of respect to vector $X$, say $Df_a(E_0, a_0)$ is $Df_a(E_0, a_0) = 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}$

So $\omega^T[Df_a(E_0, a_0).v] = -\omega_3 v_3 \neq 0$
Again according to Sotomayor theorem, if in addition to the above the following holds:

$$\begin{bmatrix} D^2f_a(E_0, a_0).v \end{bmatrix} = \begin{bmatrix} \beta \xi v_2^2 & 0 \\
0 & 2\beta \xi v_3^2 \\
-\beta \xi v_1^2 & 0 \\
\end{bmatrix}$$

Therefore : $
\omega^T[D^2f_a(E_0, a_0).v] = \omega_3 \left( \frac{2\beta \xi v_3^2}{k_1} - \frac{2\beta \nu^2 \xi^2 S_0}{k_1^2} \right) \neq 0$

Then system (1) has a transcritical bifurcation at $E_0$ when the parameter $\alpha$ passes through the bifurcation value $a_0$, prove complete.

## 4. Hopf Bifurcation Analysis of System (1)
In this section the occurrence of Hopf bifurcation near the endemic equilibrium point $(E_1)$ of system (1) is studied below.

**Theorem 4.1:** System (1) has no Hopf bifurcation near endemic equilibrium point $E_1$.

**Proof**
According to the local stability analysis of system (1) at $(E_1)$, we have the coefficients of the characteristic equation

$$\lambda^3 + (\beta S_1) \lambda + (\beta S_1) = 0$$

with third eigen value is real and negative. In addition the following two conditions are holds at the specific parameter say $t = t^*$:

$\rho_1 (t^*) = 0$ ...

$\rho_2 (t^*) = 0$ ...

Depending on [7] we have:

$$J(E_1) = \begin{bmatrix} -\beta S_1 & -\beta S_1 & 0 \\
0 & -\beta S_1 & 0 \\
0 & 0 & -\beta S_1 \end{bmatrix}$$

Now from $\Delta = \Omega_1 \Omega_2 - \Omega_3 = 0$, see [7], i.e $\Delta = 0$ then according to [2] there is possibility to occurrence of Hopf bifurcation if and only if the Jacobin matrix of system (1) near $(E_1)$ has two complex conjugate eigenvalues, say $\lambda_k = \rho_1 \pm i \rho_2$ for $k = 1, 2$ with third eigen value is real and negative. In addition the two conditions are holds at the specific parameter say $t = t^*$ :

$\rho_1 (t^*) = 0$ ...

$\rho_2 (t^*) = 0$ ...

Hence we have:

$$J(E_1) = \begin{bmatrix} -\beta S_1 & -\beta S_1 & 0 \\
0 & -\beta S_1 & 0 \\
0 & 0 & -\beta S_1 \end{bmatrix}$$

Now from $\Delta = 0$ we obtain that $Lb_{11} + Tb_{11} + D = 0$ ...

Clearly for $D < 0$ we have the two real roots of the equation (2)

$$b_{11} = -\frac{T}{2L} + \frac{1}{2L} \sqrt{T^2 - 4LD}$$

Since $b_{11} \neq -\frac{T}{2L} - \mu < 0$, then we get

$$b_{11} = -\frac{T}{2L} - \frac{1}{2L} \sqrt{T^2 - 4LD}$$

and hence

$$b_{11} = -\frac{T}{2L} + \frac{1}{2L} \sqrt{T^2 - 4LD} = 0$$
Which gives \( f(\mu^*) = 0 \), and hence \( \mu = \mu^* \) represent a root of equation (2), consequently for \( \mu = \mu^* \) we have \( \Omega_1 \Omega_2 = \Omega_3 \), from which the characteristic equation can be written as:

\[
\rho_3(\lambda) = (\lambda + \Omega_1)(\lambda^2 + \Omega_2) = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots (3)
\]

Hence in such case (i.e. \( \mu = \mu^* \)) the eigen values are \( \lambda_1 = -\Omega_1 \), and \( \lambda_{2,3} = \pm i\sqrt{\Omega_2} \), so the first condition (a) for the Hopf' bifurcation is satisfied at \( \mu = \mu^* \), that is \( \rho_1(\mu^*) = 0 \), while \( \rho_2(\mu^*) = \sqrt{\Omega_2} \).

Let us now check the second condition (b).

In generals the complex eigen values for any value of \( \mu \) can be written as:

\[
\lambda_{2,3} = \rho_1(\mu) + i\rho_2(\mu) \quad \text{then by substituting } \lambda_2 \text{ in equation (3) and calculate the derivative with respect to the parameter } \mu, \text{ that is } \frac{d}{d\mu}\rho_3(\lambda) = \rho_3'(\lambda) = 0 \text{ and comparing the two sides of this equation with equating their real and imaginary parts, it is obtain that :}
\]

\[
\Psi(\mu)\rho_1'(\mu) - \Phi(\mu)\rho_2'(\mu) = -\Theta(\mu) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4)
\]

Where

\[
\Psi(\mu) = 3(\rho_1(\mu))^2 + 2\Omega_1(\mu)\rho_1(\mu) + \rho_2(\mu) - 3(\rho_2(\mu))^2
\]

\[
\Phi(\mu) = 6\rho_1(\mu)\rho_2(\mu) + 2\Omega_1(\mu)\rho_2(\mu)
\]

\[
\Theta(\mu) = (\rho_1(\mu))^2\Omega_1(\mu) + \Omega_2(\mu)\rho_1(\mu) + \Omega_3(\mu)
\]

\[
\Gamma(\mu) = 2\rho_1(\mu)\rho_2(\mu)\Omega_1(\mu) + \Omega_2(\mu)\rho_2(\mu)
\]

Solving the linear system (4) for the unknowns \( \rho_1'(\mu) \) and \( \rho_2'(\mu) \) it is obtain that:

\[
\rho_1'(\mu) = \frac{\Psi\Theta - \Phi\Gamma}{\Psi^2 + \Phi^2}, \quad \rho_2'(\mu) = \frac{\Phi\Theta - \Psi\Gamma}{\Psi^2 + \Phi^2}
\]

Hence the second condition (b) of Hopf bifurcation will be reduces to verifying that

\[
\Psi(\mu^*)\Theta(\mu^*) + \Gamma(\mu^*)\Phi(\mu^*) \neq 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5)
\]

Straight forward computation shows that, see [7]:

\[
\Omega_1 = -1, \quad \Omega_2 = -(b_{22} + b_{32} + b_{33}), \quad \Omega_3 = -\Omega_2 - \Omega_1(b_{22} + b_{12} + b_{33})
\]

Thus for \( \mu = \mu^* \) we have:

\[
\Psi = -2\Omega_2, \quad \Phi = 2\Omega_1\sqrt{\Omega_2}, \quad \Theta = -\Omega_1(b_{22} + b_{12} + b_{33}) \quad \text{and } \Gamma = -(b_{22} + b_{12} + b_{33})\sqrt{\Omega_2}
\]

Therefore substituting in equation (5) we get that:

\[
\Psi\Theta + \Gamma\Phi = 0 \quad \text{i.e.} \quad \rho_1'(\mu) = \frac{d}{d\mu}\rho_1(\mu) = 0
\]

Hence the system has no “Hopf bifurcation around (E_f) “.prove complete

5. Conclusion

In this research we successfully prove that

1) System (1) has transcritical bifurcation near the disease free equilibrium point (E_0) at the parameter \( \alpha_0 = \frac{\beta k_0}{k_1} \), \( \psi(1-m) - \mu \), but no saddle-node bifurcation or pitchfork bifurcation.

2) System (1) has no Hopf bifurcation occur near the endemic equilibrium point (E_i) at the parameter \( \mu \).

3) It will be a good opportunity to study another bifurcations types over the same system via the parameters includes.

References


Figure 1.1: Local bifurcation Diagram