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Local Bifurcation Analysis for a Special Type of SIR Epidemic Model

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Abstract: ***An analytic study for the local bifurcation proposed for a special type of SIR epidemic model "holling type II treatment failure rate ", the occurrence of trans critical bifurcation proved depending on Sotomayor theorem, while Hopf bifurcation were not realized.***

Keywords: Local bifurcation, Hopf bifurcation, Sotomayor theorem, SIR model, holling type II

1. Introduction

In scientific fields as diverse as fluid mechanics, electronics, chemistry and theoreticalecology, there is the application of what is referred to as bifurcation analysis. Bifurcation is the study of changes in the qualitative structures of the solution to a given system of differential equation, a bifurcation occurs when a small smooth change to the parameter values of a system causes a sudden qualitative change in its behavior, bifurcation occur in both continuous systems and discrete systems. The name "bifurcation "was first introduced by "Henri Poincare" in 1885 in the first paper in mathematic showing such behavior. There are two principle types of bifurcation "Local bifurcation" and "Global bifurcation", local bifurcation answer the two following questions:

(1) What can happen to the phase portrait near an equilibrium (fixed point)when a parameter passes a bifurcation value?
(2) How to determine which of the alternatives occurs in a concrete system in which one is interested?

Local bifurcation includes, which illustrated in figure no.(1.1) :

1) Saddle- node (fold) bifurcation.
2) Trans critical bifurcation.
3) Pitchfork bifurcation.
4) Period-doubling (flip) bifurcation.
5) Hopf bifurcation.
6) Neimark-Sacker (secondary Hopf) bifurcation.

Many researcher had been study bifurcation in dynamical systems in accurate in ordinary differential equation in both linear and nonlinear, autonomous and non-autonomous. In [1] David Orrell and Leonard Smith studied visualizing bifurcation in high dimensional systems, in [2]Gustavo Revel, Diego Alonso, and Jorgel Moiola studied bifurcation theory applied to the analysis of power systems, in [3] Paul Phillipson and Peter investigated bifurcation dynamics of three-dimensional systems in their tutorial and reviews, in [4] Anthony Leung studied Bifurcation of Reaction Diffusion Systems in epidemic system, in [5] Ahmed Ali studied local bifurcation for many different systems of SVIR and SVIRS in epidemic diseases models, in [6] Azhar Majeed prove the local bifurcation of an ecological system consisting of a predator and stage structured prey. In our research we showed two main theorems on local bifurcation of special type of epidemic model called "SIR Holling Type II "with frailer of treatment, the model was studied in [7], and never deal before with its bifurcation analysis which we present and prove.

2. Mathematical Model

This research studied and modified model (1), taken from [7] from the classical simple **SIR** epidemiological model for a set of people with summation equal to N(t) at time **t** is breaker to three subsets, the susceptible individuals S(t), infected individuals I(t) and the removable individuals R(t).

Such model can be represented as a system of nonlinear differential equations in follows:

$$\frac{dS}{dt} = \Lambda - \frac{\beta SI}{K1 + I} - \mu S$$
$$\frac{dI}{dt} = \frac{\beta SI}{K1+I} + \theta R - \psi(1-m)I - (\mu+\alpha)I \qquad .....(1)$$
$$\frac{dR}{dt} = \psi(1-m)I - (\theta+\mu)R$$

Where $\Lambda>0$ is the natural birth rate of the population, $\beta>0$ is the incidence rate of the susceptible individuals because of parasitic disease transmitted by contact from the individual to the susceptible, $\psi>0$ is the recovery rate, m is the failure treatment rate such that $(0 \le m \le 1)$, $\theta>0$ is the loosing immunity rate of the recovered individuals, $\mu>0$ natural death rate, $\alpha>0$ is the disease related death, K1 >0 the half saturation constant(for more detail see [7]).

The system of differential equations (1) has two equilibrium points say E_0, E_1 , where $E_0 = (S_0, 0, 0)$ for $S_0 = \frac{\Lambda}{\mu}$, E_0 iscalled *disease free equilibrium point*, and $E_1 = (S_1, I_1, R_1)$, is called *endemic equilibrium point* where $S_1 = \frac{\Lambda(K_1+I_1)}{\beta I_1+\mu(K_1+I_1)}$, $R_1 = \frac{\psi(1-m)I_1}{(\theta+\mu)}$ and I_1 is a positive root for the following equation $D_1I^4 + D_2I^3 + D_3I^2 + D_4I + D_5 = 0$ .....(2)(for more detail see [7]).

3. Local Bifurcation

According to system (1) we will study local bifurcation, those which happened in a small Neighborhood of equilibrium or fixed point in dynamical systems depending on one parameter, transcritical of equilibria happened when an exchange of stability occurred, Hopf bifurcation happens when not only change of stability happened but also a periodic solutions are born, see [8].

Theorem 3.1 System (1) has transcritical bifurcation near the disease free equilibrium point E_0, but neither saddle-node bifurcation nor pitchfork bifurcation can occur on the parameter $\alpha_0 = \frac{\beta S_0}{k_1} - \psi(1-m) - \mu$.

Proof

The Jacobin matrix of system (1) at (E_0, α_0) is $J(E_0, \alpha_0) = \begin{bmatrix} -\mu & -\beta k_1 S_0 & 0 \\ 0 & \psi(1-m) & -(\theta+\mu) \\ 0 & 0 & 0 \end{bmatrix}$, Clearly the eigenvalue λ_I of system (1) in the I direction is λ_I=0, note $\lambda_I = \frac{\beta S_0}{k_1} - (1-m) - \mu - \alpha$, further the eigenvector say $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = [v_1 \quad v_2 \quad v_3]^T$ corresponding to the eigenvalue to λ_I satisfy the following $Jv = \lambda v$, and because $\lambda = 0$ then $Jv = 0$ where $(J = J(E_0, \alpha_0))$,

$$\begin{bmatrix} -\mu & -\beta k_1 S_0 & 0 \\ 0 & \psi(1-m) & -(\theta+\mu) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0,$$

we got $v = \begin{bmatrix} \tau v_3 \\ \xi v_3 \\ v_3 \end{bmatrix}$, where $\tau = \left(\frac{-\beta k_1 S_0}{\mu}\right)\left(\frac{\theta+\mu}{\psi(1-m)}\right), \xi = \frac{\theta+\mu}{\psi(1-m)}$

similarly the eigenvector $\omega = (\omega_1, \omega_2, \omega_3)^T$ corresponding λ_I of J^T can be written

$$\begin{bmatrix} -\mu & 0 & 0 \\ -\beta k_1 S_0 & \psi(1-m) & 0 \\ 0 & -(\theta+\mu) & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 0$$

We get $\omega = \begin{bmatrix} 0 \\ 0 \\ \omega_3 \end{bmatrix}$, here ω_3 *is any non zero real number.*

Now rewrite system (1) in vector form as:

$\frac{dX}{dt} = f(X)$, where $X = (S, I, R)^T$ and $f = (f_1, f_2, f_3)^T$ with $f_i, i = 1,2,3$ given in system (1), and then determine $\frac{df}{d\alpha} = f_\alpha = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ *then* $f_\alpha(E^0, \alpha_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Therefore : $\omega^T f_\alpha = 0$, consequently according to Sotomayor Theorem [5], the system has no saddle-node bifurcation near E^0 at α_0, Now in order to investigate the accruing of other types of bifurcation the derivative of f_α with respect of vector X, say $Df_\alpha(E^0, \alpha_0)$ is

$$Df_\alpha(E^0, \alpha_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

So $\omega^T [Df_\alpha(E^0, \alpha_0).v] = -\omega_3 v_3 \neq 0$

Again according to Sotomayor theorem, if in addition to the above the following holds:

$$[D^2 f_\alpha(E^0, \alpha_0).(v,v)] = \begin{bmatrix} \frac{\tau\beta\xi v_3^2}{k_1} + \frac{2\beta\xi v_3^2 \tau S_0}{k_1^2} \\ 0 \\ \frac{2\beta\tau\xi v_3^2}{k_1} - \frac{2\beta v_3^2 \xi^2 S_0}{k_1^2} \end{bmatrix}$$

Therefore :

$$\omega^T.[D^2 f_\alpha(E^0, \alpha_0).(v,v)] = \omega_3.\left(\frac{2\beta\tau\xi v_3^2}{k_1} - \frac{2\beta v_3^2 \xi^2 S_0}{k_1^2}\right) \neq 0$$

Then system (1) has a transcritical bifurcation at E_0 when the parameter $\boldsymbol{\alpha}$ passes through the bifurcation value $\boldsymbol{\alpha}_0$, prove complete

4. Hopf Bifurcation Analysis of System (1)

In this section the occurrence of Hopf bifurcation near the endemic equilibrium point (***E₁***) of system (1) is studied below

Theorem 4.1: System (1) has no Hopf bifurcation near the endemic equilibrium point E_1.

Proof

According to the local stability analysis of system (1) at (E_1), we have the coefficients of the characteristic equation $\Omega_i, i = 1,2,3$ at positive, from theorem (4.2) in [7], we got $\Delta > 0$ with condition (4a) is holds then system(1) has no Hopf bifurcation.

Suppose that $\Delta = \Omega_1\Omega_2 - \Omega_3 = 0$, see [7], i.e. $\Delta = 0$ then according to [2] there is possibility to occurrence of Hopf bifurcation if and only if the Jacobin matrix of system (1) near (E_1) has two complex conjugate eigenvalues, say $\lambda_k = \rho_1 \mp i\rho_2$ *for* $k = 1,2$ with third eigen value is real and negative. In addition the following two conditions are holds at the specific parameter say t =t*:

$$\rho_1(t^*) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

$$\left.\frac{d\rho_1}{dt}\right|_{t=t^*} \neq 0 \ldots\ldots\ldots\ldots\ldots\ldots (b)$$

Depending on [7] we have:

$$J(E_1) = \begin{bmatrix} \frac{-\beta I_1}{K_1+I_1} - \mu & \frac{-K_1\beta S_1}{(K_1+I_1)^2} & 0 \\ \frac{\beta I_1}{K_1+I_1} & \frac{K_1\beta S_1}{(K_1+I_1)^2} - \mu - \alpha & 0 \\ 0 & 0 & -(\theta+\mu) \end{bmatrix} = [b_{mn}]_{3\times3}$$ for *m, n= 1, 2, 3*

Now from Δ= 0 we obtain that

$Lb_{11} + Tb_{11} + D = 0$ ……….(2), where

$$L = -(b_{22} + b_{33}) > 0$$
$$T = -(b_{22}^2 + b_{33}^2) < 0$$
$$D = b_{12}b_{21}b_{22} - b_{33}b_{22}^2 - b_{33}^2 < 0$$

Clearly for D <0 we have the two real roots of the equation (2)

$$b_{11} = \frac{-T}{2L} \pm \frac{1}{2L}\sqrt{T^2 - 4LD}$$

Since $b_{11} = \frac{-\beta I_1}{K_1+I_1} - \mu < 0$, then we get $b_{11} = \frac{-T}{2L} - \frac{1}{2L}\sqrt{T^2 - 4LD}$ and hence

$$\frac{\beta I_1}{K_1 + I_1} + \mu - \left(\frac{T}{2L} + \frac{1}{2L}\sqrt{T^2 - 4LD}\right) = 0$$

Which gives $f(\mu^*) = 0$, and hence $\mu = \mu^*$ represent a root of equation (2), consequently for $\mu = \mu^*$we have $\Omega_1\Omega_2 = \Omega_3$, from which the characteristic equation can be written as :

$$\rho_3(\lambda) = (\lambda + \Omega_1)(\lambda^2 + \Omega_2) = 0 \quad \dots\dots\dots\dots\dots\dots\dots.(3)$$

Hence in such case (i.e. $\mu = \mu^*$) the eigen values are $\lambda_1 = -\Omega_1 < 0$ and$\lambda_{2,3} = \pm i\sqrt{\Omega_2}$, so the first condition (a) for the Hopf bifurcation is satisfied at $\mu = \mu^*$, that is

$\rho_1(\mu^*) = 0, while\ \rho_2(\mu^*) = \sqrt{\Omega_2}.$

let us now check the second condition (b).

In generals the complex eigen values for any value of μ can be written as:

$\lambda_{2,3} = \rho_1(\mu) + i\rho_2(\mu)$ then by substituting λ_2 in equation (3) and calculate the derivative with respect to the parameter μ, that is $\frac{d}{d\mu}\rho_3(\lambda) = \rho_3'(\lambda) = 0$ and comparing the two sides of this equation with equating their real and imaginary parts, it is obtain that :

$$\Psi(\mu)\rho_1'(\mu) - \Phi(\mu)\rho_2'(\mu) = -\Theta(\mu) \quad \dots\dots\dots\dots\dots (4)$$
$$\Phi(\mu)\rho_1'(\mu) + \Psi(\mu)\rho_2'(\mu) = -\Gamma(\mu)$$

Where

$$\Psi(\mu) = 3(\rho_1(\mu))^2 + 2\Omega_1(\mu)\rho_1(\mu) + \rho_2(\mu) - 3(\rho_2(\mu))^2$$
$$\Phi(\mu) = 6\rho_1(\mu)\rho_2(\mu) + 2\Omega_1(\mu)\rho_2(\mu)$$
$$\Theta(\mu) = (\rho_1(\mu))^2\Omega_1'(\mu) + \Omega_2'(\mu)\rho_1(\mu) + \Omega_3'(\mu) - \Omega_1'(\mu)(\rho_2(\mu))^2$$
$$\Gamma(\mu) = 2\rho_1(\mu)\rho_2(\mu)\Omega_1'(\mu) + \Omega_2'(\mu)\rho_2(\mu)$$

Solving the linear system (4) for the unknowns $\rho_1'(\mu)$ and $\rho_2'(\mu)$ it is obtain that

$$\rho_1'(\mu) = \frac{\Psi\Theta + \Gamma\Phi}{\Psi^2 + \Phi^2} \quad , \quad \rho_2'(\mu) = \frac{\Phi\Theta - \Psi\Gamma}{\Psi^2 + \Phi^2}$$

Hence the second condition (b) of Hopf bifurcation will be reduces to verifying that

$$\Psi(\mu^*)\Theta(\mu^*) + \Gamma(\mu^*)\Phi(\mu^*) \neq 0 \dots\dots\dots\dots\dots\dots (5)$$

Strait forward computation shows that, see [7] :

$\Omega_1' = -1$, $\Omega_2' = -(b_{22} + b_{12} + b_{33})$,

$\Omega_3' = -\Omega_2 - \Omega_1(b_{22} + b_{12} + b_{33})$

Thus for $\mu = \mu^*$ we have :

$\Psi = -2\Omega_2$, $\Phi = 2\Omega_1\sqrt{\Omega_2}$, $\Theta = -\Omega_1(b_{22} + b_{12} + b_{33})$ and $\Gamma = -(b_{22} + b_{12} + b_{33})\sqrt{\Omega_2}$

Therefore substituting in equation (5) we get that:

$\Psi\Theta + \Gamma\Phi = 0$ i.e. $\rho_1'(\mu) = \frac{d}{d\mu}\rho_1(\mu) = 0$

Hence the system has no "Hopf bifurcation around (***E₁***) ".prove complete

5. Conclusion

In this research we successfully prove that

1) System (1) has transcritical bifurcation near the disease free equilibrium point (***E₀***) at the parameter $\alpha_0 = \frac{\beta S_0}{k_1} - \psi(1-m) - \mu$, but no saddle-node bifurcation or pitchfork bifurcation.
2) System (1) has no Hopf bifurcation occur near the endemic equilibrium point ***(E₁)*** at the parameter μ.
3) It will be a good opportunity to study another bifurcations types over the same system via the parameters includes.

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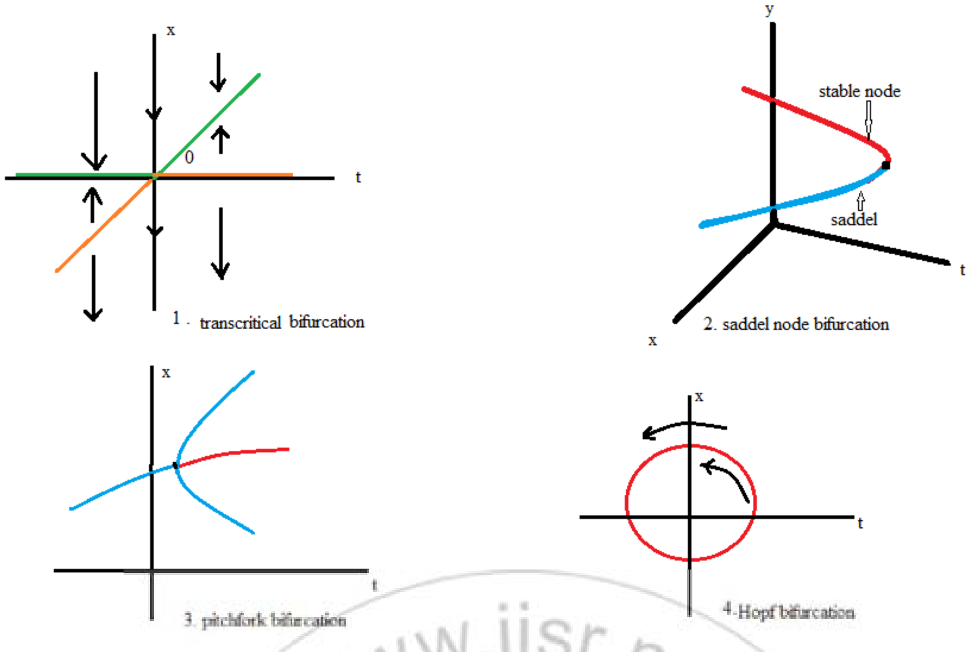

Figure 1.1: Local bifurcation Diagram
1. transcritical bifurcation 2. saddle node bifurcation 3. pitchfork bifurcation and 4. Hopf bifurcation.

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