International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

Assets Valuation Using a Contingent Claim

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Abstract: ***In this paper we consider the price dynamics of a portfolio consisting of risk-free and risky assets. The paper discusses the pricing process of a contingent claim, the pricing equation and the risk-neutral valuation under the Martingale representation property. A partial differential equation with an unknown price function is formulated. The solution of this PDE gives a unique pricing formula.***

Keywords: Continent claim valuation, Option pricing, Martingale representation, Risk-Neutral Valuation and Stochastic Integrals.

1. Introduction

The study of pricing on financial markets in continuous time is anchored on the use of stochastic processes and stochastic differential eq uations as the building blocks. A stochastic process is X is a diffussion if its local dynamics can be approximated by a stochastic difference equation of the following type,

$$X(t+\Delta t) - X(t) = \mu(t, X(t))\Delta t + \sigma(t, X(t))Z(t) \quad (1)$$

Here Z is normally distributed disturbance term which is independent of everything which has happened up to time t, while μ and σ are given ɗven deterministic functions. The intuitive content of (1) is that, over the time interval $[t, t+\Delta t]$ the X- process is driven by two separate terms namely

- A local deterministic velocity $\mu(t, X(t))$
- A Gaussian disturbance term, amplified by the factor $\sigma(t, X(t))$

The function μ is called the (local) (drift) term of the process, whereas σ is callled the (diffusion) term. In order to model the Gaussian disturbance terms we need the concept of a Weiner process.

Definition 1[1] A stochastic process **W** is called a **Weiner process** if the following conditions hold.

1) $W(0) = 0$
2) The process W has independent increments, i.e if $r < s \leq t < u$ then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.
3) For s< t the stochastic variable $W(t) - W(s)$ has the Gaussian distribution $N[0, \sqrt{t, -s}]$
4) W has continuous trajectories

We may now use the Weiner process to in order to write (1) as

$$X(t+\Delta t) - X(t) = \mu(t, X(t))\Delta t + \sigma(t, X(t))\Delta W(t) \quad (2)$$

Where $\Delta W(t)$ is defined by

$$\Delta W(t) = W(t+\Delta t) - W(t)$$

Making (2) more precise, we divide the equation by $\Delta(t)$ and let Δt tend to zero.

Formally we would obtain

$$\begin{cases} \dot{X}(t) = \mu(t,X(t)) + \sigma(t,X(t))v(t) \\ X(0) = a \end{cases}$$

where we have added an initial condition and where

$$v(t) = \frac{dW}{dt} \quad (3)$$

is the formal time derivative of the Weiner process W.

Another possibility of making equation (2) more precise is to let Δt tend to zero without first dividing the equation by Δt. We will then obtain the expression

$$\begin{cases} dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) \\ X(0) = a \end{cases} \quad (4)$$

and it is now natural to interpret equation(4)as a shorthand version of the following integral equation

$$X(t) = a + \int_0^t \mu(s, X(s))\, ds + \int_0^t \sigma(s, X(s))dW(s) \quad (5)$$

In equation (5) we may interpret the $d(s)$-integral as an ordinary Riemann integral. The natural interpretation of dW-integral is to view it as a Riemann-Stieljes integral for each

W-trajectory, but unfortunately this is not possible since one can show that the W-trajectories are of locally unbounded variation.

2. Diffusion Case

We now introduce the diffusioprocess that follows a Brownian motion process on a probability space $(\Omega, \mathcal{F}, P)$.We further proceed to show an equivalent probability measure Q similar to P also has a martingale property.Faymann-Kac theorem is applied to a price of an attainable contingent claim value process the result of which is demonstrated by the Black-Scholes [2] example. The steps in developing a pricing model follow.

Let us assume that there is a K'-dimensional Brownian motion W on the filtered probability space. The security prices

$$S_t = (S_t^0, S_0^1, S_0^2, \cdots, S_0^K)^T \text{ ,where } (\cdot^T \text{ means transpose}) \quad (7)$$

follows a diffusion process

$$\frac{dS_t}{S_t} = \alpha(S_t, t)dt + \sigma(S_t, t)dW_t \quad (8)$$

Where

$$\frac{dS_t}{S_t} = \left(\frac{dS_0}{S_t}, \frac{dS_t^1}{S_1}, \cdots, \frac{dS_t^K}{S_t}\right), \tag{9}$$

And $\alpha(x, t)$ is a K-dimensional column vector, and σ(x, t) is a $K \times K'$-matrix. Both α and σ are assumed to satisfy some regularity and growth conditions which make (8) have a solution and the Ito's integral part is square integrable. Similarly, we assume that S^0 is strictly positive and absolutely continuous, so we can define

$$r_t = \frac{d}{dt} lnS_t^0 \tag{10}$$

or

$$S_t^0 = exp\left(\int_0^t r_u du\right) \quad (11$$

Let us now define the discounted price

$$Z_t = (Z_t^1, \cdots, Z_t^K)^T = (\beta_t S_t^1, \cdots, \beta_t S_t^K)^T \tag{12}$$

The dynamics for the discounted security price process can be easily shown as the following:

$$\frac{dZ_t}{dt} = (\alpha_t - r_t 1_K)dt + \sigma_t dW_t, \tag{13}$$

where

$$\frac{dZ_t}{Z_t} = \left(\frac{dZ_t^1}{Z_t^1}, \cdots, \frac{dZ_t^K}{Z_t^K}\right) \tag{14}$$

and 1_K is a K-dimensional vector with each components equal to 1.

Consider the linear equation

$$-(\alpha_t - r_t 1_K) = 0_K \tag{15}$$

If there is a solution bt for (15) for each t (a.s.), we often refer b_t as the market price of risk process.

Furthermore, if b_t satisfies the following (Novikov) condition:

$$E\left[exp\left(\frac{1}{2}\int_0^T \|b_t\|^2 dt\right)\right] < \infty$$

Then

$$M_t = exp\left(-\sum_{i=1}^K \int_0^t b^i(s) dW_s^i - \frac{1}{2}\int_0^t \|b(s)\|^2 ds\right)$$

is a martingale, and the Q defined by

$$Q(A) = E(M_T 1_A), \qquad A \in \mathcal{F}$$

is a probability measure. Moreover, the process defined by

$$\widetilde{W}_t = W_t + \int_0^t b_u du$$

is a BM under the probability measure Q.

We show that Q is in fact an equivalent martingale measure. It is obvious that Q is P.

To show that the discounted security price process Z is a martingale under Q, we write

$$\begin{aligned}\frac{dZ_t}{Z_t} &= (\alpha_t - r_t 1_K)dt + \sigma_t dW_t \\ &= (\alpha_t - r_t 1_K)dt + \sigma_t d(W_t - b_u du) \\ &= (\alpha_t - r_t 1_K)dt - \sigma_t b_t dt + \sigma_t d\widetilde{W}_t \\ &= \sigma_t d\widetilde{W}_t\end{aligned}$$

Therefore Z is a martingale under Q (since $\widetilde{W}$ is a BM under Q). We have thus shown that Q is indeed an equivalent martingale measure.

Recall that $S_t = \frac{Z_t}{\beta_t}$ abusing the notations, we use S here to represent $(S_t^1, \cdots, S_t^K)$.

So we have

$$\frac{dS_t}{S_t} = r_t 1_K dt + \sigma_t dW_t.$$

We can conclude that under an equivalent martingale measure, the security prices have drifts rt, the instantaneous risk-free interest rate, and the volatility matrix σ does not change.

Applying the result $V_t = \beta^{-1} E_Q(\beta_T X | \mathcal{F}_{\sqcup})$, we can write the time$-t$ price of an attain- able contingent claim X as

$$V_t = E_Q\left[\exp\left(-\int_t^T r_t dt\right) X | \mathcal{F}\right]$$

In particular, if we also assume that it can be written as a function of (S_t, t) (e.g.,$r_t = r$ is a constant), then for an contingent claim $X = g(S_T)$ Feynman-Kac formula suggests that the time$-t$ price $V(S, t)$ satisfies the following deferential equation:

$$\frac{\partial V}{\partial t} + r(S,t) \sum_{i,j=1}^K S^i S^j a^{ij}(S,t) \frac{\partial^2 V}{\partial S^i \partial S^i} = r(S,t)V$$

where $a(S,t) = \sigma(S,t).\sigma(S,t)^T$ with the boundary condition

$$V(S,T) = g(S).$$

3. Black-Scholes formula

Assume $K = 1$ and the 2 securities are given by the following processes:

$$dS_t^0 = r(t)dt, \qquad S_0^0 = 1$$

$$\frac{dS_t^1}{S_t^1} = \mu(t)dt + \sigma(t)dW_t,$$

where$r(.), \mu(.), \sigma(.)$ are bounded deterministic functions with $r(t) > 0, \sigma(t) > 0, \forall t \geq 0$, and W is a 1-dimensional BM. We have

$$\beta_t = \exp\left(-\int_0^t r(s)ds\right)$$

$$Z_t = \beta_t S_t^1,$$

and using the integration by parts formula,

$$\frac{dZ_t}{Z_t} = (\mu(t) - r(t))dt + \sigma(t)dW_t.$$

Let us define

$$X_t = exp\left(-\int_0^t \frac{\mu(s) - r(s)}{\sigma(s)} dW_s - \frac{1}{2}\int_0^t \left(\frac{\mu(s) - r(s)}{\sigma(s)}\right)^2 ds\right)$$

and define a probability Q as $dQ = X_T dP$. Girsanov's Theorem implies that

$$\widetilde{W} \equiv W_t + \int_0^t \frac{\mu(s) - r(s)}{\sigma(s)} ds$$

is a 1-dimensional BM under the probability$Q$, and we can write

$$dZ_t = Z_t \sigma(t) d\widetilde{W}(t)$$

Paper ID: ART2017799 DOI: 10.21275/ART2017799

which implies that the discounted price process Z is a martingale under the probability measure Q. In other words, Q is an equivalent martingale measure.

An European call option on S^1 with a strike price X and expiration T is a contingent claim$\left(S_T^1 - X\right)^+$. Therefore the value of the option at time 0 is

$$= E_Q\left[exp\left(-\int_0^T r(s)ds\right)\right]$$

Notice that under the probability Q,

$$\frac{dS_t^1}{S_t^1} = r(t)dt + \sigma(t)\,d\bar{W}_t$$

and using Ito's formula,

$$dlnS = \left(r(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)\,d\bar{W}(t)$$

By computing the expectation in (8) we can get the Black-Scholes formula

$$C(S,t) = S_1^0\Phi(d) - e^{-rt}X\Phi(d - \sigma\sqrt{T}),$$

with

$$d = \frac{ln\left(\frac{S_t^0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

4. The Pricing of a Contingent Claim

We consider a fixed T claim of the form

$\chi = \Phi(S_T)$.

where δ is some given deterministic function. The price dynamics of the risky assets are

$$dS = S_t(\beta(S_t)dt + \sigma(S_t)dWt)$$

The guaranteed income from the risky asset investment is assumed in this case to be of the form

$$dI = S_t\delta[S_t]$$

where δ is constant.

The problem to be solved is that of determining the arbitrage free price for a T -claim of the form $\Phi(S_T)$. We therefore assume that the pricing function for the claim is a function of income I as well as S i.e

$$\pi(t:\chi) = R(t, S_t, I_t)$$

This is then determined in the following steps

- Assume that the pricing function is of the form $R(t; S_t)$
- Consider β, σ, δ and r as exogenously given
- Use the general results from the value of a self financed portfolio based on a a derivative instrument and the underlying stock
- Form a self-financing portfolio whose value process P has a stochastic differential without any driving Weiner process of the form
$$dP(t) = P(t)z(t)dt$$
- Since we assume absence of arbitrage we must have $z = r$
- The condition $z = r$ will have the form of a partial differential equation with R as the unknown function.
- The equation has a unique solution, thus giving a unique pricing formula for the derivative which is consistent with the absence of arbitrage.

Let w_S and w_R be the weights of the portfolio invested in the stock and the derivative respectively, we obtain the value of the dynamics as

$$dP = P\left\{ws\frac{dG_s}{S} + w_r\frac{dR}{R}\right\}$$

where the gain differential $dG_S$is given by

$$dG_s = dS + dI$$

i.e.

$dG_s = S(\beta + \delta)dt + \sigma SdW$

From the Ito formula we have the usual expression for the derivative dynamics

$$dR = \beta_R Rdt + \sigma_R\bar{R}dW$$

where

$$\beta_R = \frac{1}{R}\left\{\frac{\partial R}{\partial t} + \beta S\frac{\partial R}{\partial s} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 R}{\partial s}\right\}$$

Collecting terms in the value equation above

$$dP = P.\{w_s(\beta + \delta) + w_R\beta_R\}dt + P.\{w_s\sigma + W_R\}dW$$

and determining the portfolio weights in order to obtain a value process without a driving
Weiner process i.e. we define w_S and $w_R$as the solution to the system

$$w_s\sigma + w_R\sigma_R = 0$$
$$w_s\sigma + w_R = 1$$

Which has the solution

$$w_S = \frac{\sigma_R}{\sigma_R - \sigma}$$

and leaves us with the dynamics

$$dP = P.\{w_S(\beta + \delta) + w_R\beta_R\}dt$$

Absence of arbitrage implies that we must have the equation

$$w_S(\beta + \delta) + w_R\beta_R = r$$

with probability 1, for all t and, substituting for w_R, w_s, β_R and σ_R into this equation,
We get the equation

$$\frac{\partial R}{\partial t} + (r - \delta)S\frac{\partial S}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 R}{\partial s^2} - rR = 0$$

The value of the boundary condition is obvious giving us the following resulting pricing formula from which the pricing equation is derived.

The pricing function $R(T,s) = 0$ of the claim $\Phi(S_T)$ solves the boundary value problem

$$\begin{cases}\dfrac{\partial R}{\partial t} + (r - d)s\dfrac{\partial R}{\partial s} + \dfrac{1}{2}\sigma^2 s^2\dfrac{\partial^2 R}{\partial s^2} - rR = 0 \quad (14)\\ R(T,s) = \Phi(s)\end{cases}$$

Applying the Feynman-Kac representation theorem immediately gives us the risk-neutral valuation formula

5. Pricing Equation

The pricing function has the representation

$$R(s,t) = e^{-r(T-t)}E_{ts}^Q[\Phi(S_T)] \quad (15)$$

where the Qdynamics of S are given by

$$dSt = (r - \delta[S_t])S_t dt + \sigma(S_t)S_t dW_T \quad (16)$$

6. Risk-Neutral Valuation Formula

Under the martingale measure Q, the normalized gain process

$$G^z(t) = e^{-rt}S(t) + \int_0^t e^{-r\tau}\, dI(\tau) \qquad (17)$$

is a Qmartingale.

It is therefore expected that

$$S(0) = E^Q\left[\ \int_0^t e^{-r\tau} dI(\tau) + e^{-rt}S(t)\ \right] \qquad (18)$$

As in the discrete case, it is natural to analyse the pricing formulas for the special case when we have the standard Black-Scholes dynamics

$$dS = \beta S dt + \sigma S dW \qquad (19)$$

where α and σ are constants.

We also assume that the income function **δ is** a deterministic constant. This implies that the martinagale dynamics are given by

$$dS = (r - \delta)S dt + \ \ \sigma S dW \qquad (20)$$

7. Conclusion

We have demonstrated that to obtain a valuation formula, the attainable contingent claim depends directly on the definition of a self-financing strategy which also depends on the definition of the gains process. The solution to the boundary value problem then formulated from the portfolio dynamics in equation (14) gives the pricing equation when the Faynman-Kac technique is applied to it. The explicit formula is thus obtained in equation (15) with the dynamics of the underlying assets given by in equation (16). The risk-neutral valuation is also stated when we have the special case of the Black-Scholes dynamics.

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Paper ID: ART2017799 DOI: 10.21275/ART2017799