Algorithm for a Modified Technique on Construction of Odd Magic Squares using Basic Latin Squares

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Abstract: ***The techniques for construction of odd-order normal magic square using basic Latin square, developed by Tomba (2012, 2013) have been studied further with a view to developing its suitable algorithm and program (MATLAB). Alternative techniques for construction of odd magic square using basic Latin square can also be generated applying single or double step/pass on basic Latin square construction that makes easier and suitable for computer application. The modified/ improved technique draws the same result as observed by Tomba and satisfies the property of T, for all pair numbers in the Magic Square.***

Keywords: Basic Latin square, normal magic square, Tomba's constant (T), single and double pass/step for Basic Latin Square construction etc.

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1. Introduction

Magic squares are practically important of the properties of equality in the sum of its rows, columns, diagonals. Latin squares and Greco-Latin squares are used in statistical research particularly in agricultural sciences and design of experiments whereas magic squares are used in puzzle games of cubes, pattern recognition and magic carpet constructions, magic square cipher in Cryptology etc.

A magic square of order n is an arrangement of integers in an (n x n) matrix such that the sums of all the elements in every row, column and along the two main diagonals are equal. A normal magic square contains the integers from 1 to n^2 and exists for all orders $n \geq 1$ except n= 2. Magic squares are classified into three types; odd, doubly-even (divisible by 4) and singly-even (not divisible by 4) The magic sum of a ($n \times n$) magic square is $S = \frac{1}{2}\left\{n(n^2+1)\right\}$.

The idea of magic squares has been introduced in junior level as part of recreational mathematics without any suitable formula or method to demonstrate the construction of it. There exist different methods or techniques for construction of magic squares based on algebra, graphs, computer oriented techniques etc. but not suitable for introducing in secondary levels. Tomba (May, 2012) developed a simple technique for construction odd-order magic squares using basic Latin squares [10]. Again, Tomba (July, 2012) developed a technique for constructing even order magic squares using basic Latin Square and observed that the method can provide magic squares for any doubly-even n but, generates weak magic squares in many cases for any singly-even n [11]. A similar approach was developed to construct magic squares using basic Latin squares for any singly-even n [12]. The paper introduces the techniques or methods for constructing odd, doubly-even and singly-even magic squares using basic Latin Squares. In singly-even cases, weak magic squares can be generated as far as possible, depending upon the choice of the central block and assignment of the pair-numbers satisfying T in selective positions.

1.1 Latin squares & basic Latin Squares

In a Latin square, Latin letters are seen once in each row and in each column and therefore the sums of rows and columns are equal but diagonal sums are unequal.

A basic (4×4) Latin square can be represented with Latin letters A, B, C and D as:

$$\begin{bmatrix} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{bmatrix} \text{ or: } \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad [1]$$

where, $\sum_i a_{ij} = \sum_j a_{ij}$ but

$\sum_i d_{ii} \neq \sum_j d_{j(n+1-j)}$ with diagonal notation d_{ii} [2]

1.2 Normal Magic Squares

A normal magic square contains the integers from 1 to n^2 and has the following properties

(a) Elements or numbers ($n \geq 1$) are consecutive and not repeated

(b) Sums of the rows, columns and diagonals are equal to magic sum, S

$$S = \sum_i b_{ij} = \sum_j b_{ij} = \sum_i d_{ii} = \sum_j d_{j(n+1-j)} \quad i, j = 1, 2, \ldots n \quad [3]$$

(c) Equality property of the rows, columns and diagonals remain unaltered for rotations and reflections.

1.3 Other Magic squares

There exists different (n × n) magic square not satisfying these properties. Examples of such magic squares, not satisfying the above properties are: magic squares (special or random, prime numbers etc.)

Examples:

(i) MS (special)

$$\begin{bmatrix} 1 & 14 & 14 & 4 \\ 11 & 7 & 6 & 9 \\ 8 & 10 & 10 & 5 \\ 13 & 2 & 3 & 15 \end{bmatrix}$$

(ii) MS (prime)

$$\begin{bmatrix} 17 & 39 & 71 \\ 113 & 59 & 5 \\ 47 & 29 & 101 \end{bmatrix}$$ [4]

It satisfies; $\sum_i b_{ij} = \sum_j b_{ij} = \sum_i d_{ii} = \sum_j d_{j(n+1-j)}$ but these magic squares are not normal because, in (i) The elements are repeated and non-consecutive and in (ii) the numbers (prime) are not repeated but non-consecutive.

1.4 Weak magic squares (normal)

A (n x n) array (with diagonal notation d_{ij}) satisfying the properties

(a) Elements (n ≥ 1) are consecutive and not repeated

(b) Sums of diagonals are equal to S

$$S = \sum_i d_{ii} = \sum_j d_{j(n+1-j)} \quad i, j = 1, 2, \ldots n$$

(c) Sums of the rows, columns are equal to S, except for some i and j

$$\Rightarrow S = \sum_i b_{ij} = \sum_j b_{ij} \quad i, j = 1, 2, \ldots\ldots, n \quad \text{except for}$$

some i and j

(d) Equality or un-equality properties in rows, columns and diagonals remain unaltered for rotations and reflections.

1.5 Alternate Structures (M S)

Let $\{a_{ij}\}$ be a magic square satisfying the properties (a) to (c). The alternate structures of a magic square can be expressed (if the rotation is clockwise or anticlockwise as $\{a_{ij}(k)\}$ for $(k\frac{\pi}{2})$; $(k = \pm 1, \pm 2, \ldots \pm m)$ where $\{a_{ij}\}$ = $\{a_{ij}(k)\}$ for all $i = 0, 4, 8, \ldots\ldots$

1.6 Matrix Representation (basic LS)

Let a (n x n) matrix be

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdot & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ \cdots\cdot & \cdots\cdot & \cdots & \cdots\cdot & \cdots\cdot \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}$$

The presentation of this matrix in Latin square format can be done in two forms;

Form 1: $$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdot & a_{1n-1} & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} & a_{21} \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots\cdots\cdot \\ a_{n-1,n-1} & a_{n-1,n} & \cdots & a_{n-1,n-3} & a_{n-1,n-2} \\ a_{nn} & a_{n1} & a_{n2} \cdots & a_{n,n-2} & a_{n,n-1} \end{bmatrix}$$

Form 2: $$\begin{bmatrix} a_{11} & a_{22} & \cdots\cdot & a_{n-1,,n-1} & a_{nn} \\ a_{12} & a_{23} & \cdots & a_{n-1,n} & a_{n1} \\ \cdots & \cdots & \cdots\cdot & \cdots & \cdots \\ a_{1,n-1} & a_{2,n,} & \cdots & a_{n-1,n-3} & a_{n,,n-2} \\ a_{1n} & a_{21} & \cdots & a_{n-1,n,-2} & a_{n,n-1} \end{bmatrix}$$

A simplified method for representing a given matrix in basic Latin Square format is expressed in two forms;

Given matrix — Presentation in basic LS

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} (3\times3)$$

Form-I $$\begin{bmatrix} A_1 & B_1 & C_1 \\ B_2 & C_2 & A_2 \\ C_3 & A_3 & B_3 \end{bmatrix}$$

Form-II $$\begin{bmatrix} A_1 & B_2 & C_3 \\ B_1 & C_2 & A_3 \\ C_1 & A_2 & B_3 \end{bmatrix}$$

$$\begin{bmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 \\ A_3 & B_3 & C_3 & D_3 & E_3 \\ A_4 & B_4 & C_4 & D_4 & E_4 \\ A_5 & B_5 & C_5 & D_5 & E_5 \end{bmatrix} (5\times5)$$

Form-I $$\begin{bmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ B_2 & C_2 & D_2 & E_2 & A_2 \\ C_3 & D_3 & E_3 & A_3 & B_3 \\ D_4 & E_4 & A_4 & B_4 & C_4 \\ E_5 & A_5 & B_5 & C_5 & D_5 \end{bmatrix}$$

Form-II $$\begin{bmatrix} A_1 & B_2 & C_3 & D_4 & E_5 \\ B_1 & C_2 & D_3 & E_4 & A_5 \\ C_1 & D_2 & E_3 & A_4 & B_5 \\ D_1 & E_2 & A_3 & B_4 & C_5 \\ E_1 & A_2 & B_3 & C_4 & D_5 \end{bmatrix}$$

2. Methodology

For constructing (n × n) magic square (n is odd) using basic Latin square

In this paper, techniques for construction of magic square using basic Latin square are considered. Theorem on construction of odd order magic squares is available in 10] whereas the theorems on construction of even-order magic squares (doubly even and singly even cases) are available in [11] and [12]. In singly-even cases, it may generate weak magic squares. The technique for construction of odd-magic squares using basic Latin square is described as follows:

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2.1 For any odd n, the construction process is expressed as follows:

Step-1: Represent the consecutive numbers 1 to n^2 in n rows and n columns. Find P = $\frac{(1+n^2)}{2}$ and the magic sum, S = $\frac{n(n^2+1)}{2}$

Step-2: Arrange it in basic Latin square format (i) to give the column sums equal.

Step-3: Select the row associated with P, assign this row as main diagonal elements (keeping the pivot element in the middle cell) in ascending or descending order and arrange other (column) elements in an orderly manner to give the desired magic square.

Alternately

Step-1: Represent the consecutive numbers 1 to n^2 in n rows and n columns. Find P = $\frac{(1+n^2)}{2}$ and the magic sum, S = $\frac{n(n^2+1)}{2}$

Step-2: Arrange it in basic Latin square format (ii) to give the row sums equal.

Step-3: Select the column associated with P, assign this column as main diagonal elements (keeping the pivot element in the middle cell) in ascending or descending order and arrange other (row) elements in an orderly manner to give the desired magic square.

3. Examples (For any odd n)

Example 1: (3 × 3) Magic square

S-1: Let the consecutive numbers (1 to 9) arranged in 3 rows and 3 columns,

Find P = $\frac{(n^2+1)}{2}$ = 5 and S = $\frac{n(n^2+1)}{2}$ =15

1	2	3
4	5	6
7	8	9

S-2: Arrange the matrix in basic Latin Square format (form-1). The arrangement gives the column totals equal.

1	2	3
5	6	4
9	7	8

S-3: Select the row associated with the pivot element (say 5, 6, 4) and assign it as main diagonal elements (keeping pivot element in the middle) in ascending/ descending order. Rearrange the other (column) elements in an orderly manner to get a new matrix representing the desired (3×3) magic square.

8	1	6
3	5	7
4	9	2

Alternately

S-1:Let the consecutive numbers (1 to 9) arranged in 3 rows and 3 columns.

1	4	7
2	5	8
3	6	9

Find P = $\frac{(n^2+1)}{2}$ = 5 and S = $\frac{n(n^2+1)}{2}$ =15

S-2: Arrange the matrix in basic Latin Square format (form-2). The arrangement gives the row totals equal.

1	5	9
2	6	7
3	4	8

S-3: Select the column associated with the pivot element (say 1, 5, 9) and assign it as main diagonal elements (keeping 5 in the middle) in ascending/descending order. Rearrange the other (row) elements in an orderly manner to get a new matrix representing the desired (3×3) magic square

8	1	6
3	5	7
4	9	2

Example-2 (5 × 5) Magic Square

S-1 Let the consecutive numbers (1 to 25) in 5 rows and 5 columns be represented.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Here, P = $\frac{(n^2+1)}{2}$ = 13 and S = $\frac{n(n^2+1)}{2}$ = 65

S-2: Arrange the matrix in basic Latin Square format:
The arrangement gives the column totals equal.

1	2	3	4	5
7	8	9	10	6
13	14	15	11	12
19	20	16	17	18
25	21	22	23	24

S-3: Select the row associated with the pivot element, P =13, assigning this row as main diagonal elements (fixing P in the middle) and arranging the other (column) elements in an orderly manner to get a new matrix representing the desired (5 × 5) magic square.

1	8	15	17	24
7	14	16	23	5
13	20	22	4	6
19	21	3	10	12
25	2	9	11	18

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Example-3: (7×7) Magic Square

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

It satisfies $\sum_i a_{ij} = \sum_j a_{ij} = \sum_i d_{ii} = \sum_j d_{j(n+1-j)}$

where P = 25 and S = 175.

The same practice can be used for construction of magic squares of any odd-n (n > 3)

4. Modified Technique

The technique developed by Tomba (2012) can be modified slightly to give the desired magic square by changing Step-3 only as:

Step-3 (modified): Fixed the column (or row) associated with the Pivot element. Perform double pass /steps of Basic Latin square construction (form-II) with the remaining columns (or rows) and shift the fixed column (or row) in the middle gives the desired magic square.

Examples with modified method

Example-1: (3×3) Magic square
Step-1: Same as shown before
Step-2: Same as shown before

1	2	3
5	6	4
9	7	8

Step-3 (modified): Fix the column associated with the pivot element (first column) and perform double pass for basic Latin square, shifting the columns $> \frac{n+1}{2}$ on the left and assigning the first column as the middle column gives the desired magic square.

1	6	8
5	7	3
9	2	4

8	1	6
3	5	7
4	9	2

Example-2: (5 × 5) Magic Square
Step-1. Same as shown above
Step-2: same as shown earlier.

1	2	3	4	5
7	8	9	10	6
13	14	15	11	12
19	20	16	17	18
25	21	22	23	24

Step-3 (modified): Fix the column associated with the pivot element (say 1, 5, 9) and perform double Latin square construction (form-II) for the first and second columns then shift the fixed column as middle column. It represents the desired (5×5) magic square satisfying T-26 for all pair numbers:

1	8	15	17	18
7	14	16	23	24
13	20	22	4	5
19	21	3	10	6
25	2	9	11	12

17	18	1	8	15
23	24	7	14	16
4	5	13	20	22
10	6	19	21	3
11	12	25	2	9

5. Advantages

The advantages of introducing single and double pass/ step can be discussed in the following heads:
i) Easy to develop the algorithm
ii) More time saving
iii) Convenient for generalization

6. Algorithm

Program Coding (MATLAB)

```
% Initialization of input matrix with 1 to n^2
 n=any odd number ≥3;
m=1;
for i=1:n
   for j=1:n
      a(i,j)=m;
      m=m+1;
   end
end

% Formation of basic Latin Square matrix

% shifting elements to left by i-1 position for each row
b=a;
for i=2:n
   for j=i:n
      l=i;
      for k=1:n
         b(i,k)=a(i,l);
         l=l+1;
         if l>n
            l=1;
         end
      end
   end
end

% shifting up of elements by j-1 position for each column
a=b;
for j=2:n
   for i=j:n
      l=j;
      for k=1:n
         b(k,j)=a(l,j);
         l=l+1;
         if l>n
            l=1;
         end
      end
   end

end

% Shifting to right by  (n+1)/2 position along row
a=b;
for i=1:n
    l=(n+1)/2;
      for j=1:n
         b(i,l)=a(i,j);
         l=l+1;
         if l>n
            l=1;
```

```
        end
      end
   end
% Displaying  b as the required magic square
b
```

7. Magic Squares

Generated with the program for n = 3, 5, 7, 9 and 13 (inputs and outputs):

1	2	3
4	5	6
7	8	9

8	1	6
3	5	7
4	9	2

(3×3)

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

(5×5)

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

(7×7 input)

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

(7×7 Output)

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81

(9×9 input)

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

(9×9 Output)

1	2	3	4	5	6	7	8	9	10	11	12	13
14	15	16	17	18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49	50	51	52
53	54	55	56	57	58	59	60	61	62	63	64	65
66	67	68	69	70	71	72	73	74	75	76	77	78
79	80	81	82	83	84	85	86	87	88	89	90	91
92	93	94	95	96	97	98	99	100	101	102	103	104
105	106	107	108	109	110	111	112	113	114	115	116	117
118	119	120	121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140	141	142	143
'44	145	146	147	148	148	150	151	152	153	154	155	156
157	158	159	160	161	162	163	164	165	166	167	168	169

(13×13 Input)

93	108	123	138	153	168	1	16	31	46	61	76	91
107	122	137	152	167	13	15	30	45	60	75	90	92
121	136	151	166	12	14	29	44	59	74	89	104	106
135	150	165	11	26	28	43	58	73	88	103	105	120
149	164	10	25	27	42	57	72	87	102	117	119	134
163	9	24	39	41	56	71	86	101	116	118	133	148
8	23	38	40	55	70	85	100	115	130	132	147	162
22	37	52	54	69	84	99	114	129	131	146	161	7
36	51	53	68	83	98	113	128	143	145	160	6	21
50	65	67	82	97	112	127	142	144	159	5	20	35
64	66	81	96	111	126	141	156	158	4	19	34	49
78	80	95	110	125	140	155	157	3	18	33	48	63
79	94	109	124	139	154	169	2	17	32	47	62	77

(13×13 Output)

(ii) More time can be saved with the double step Latin Square construction in lieu of selecting the row associated with the pivot element, assigning as main diagonal elements. Keeping P in the middle and arranging the other (column/row) elements in an orderly manner. In fact, arranging other elements in an orderly manner is complicated and many times may lead to difficult situations.

(iii) Convenient for further generalization:
Consider {5x5} magic square. Different magic squares can be developed by selecting the pair numbers satisfying T=26

(a) Consider the pair number (15, 11) and (17, 9) $\Longrightarrow$ taking column -2 and column-4 as column-1 and column-5 gives

15	8	1	24	17
16	14	7	5	23
22	20	13	6	4
3	21	19	12	10
9	2	25	18	11

8	15	1	17	24
14	16	7	23	5
20	22	13	4	6
21	3	19	10	12
2	9	25	11	18

(b) Consider the pair number (21, 5) and (14, 12) $\Longrightarrow$ taking row -2 and row-4 as row-1 and row-5 gives

16	14	7	5	23
15	8	1	24	17
22	20	13	6	4
9	2	25	18	11
3	21	19	12	10

The process can be continued selecting pair numbers satisfying T in different positions.

8. Conclusion

The modified technique for construction of magic squares using basic Latin Squares of any order ($n \geq 3$, n is odd) seems to be easier, suitable for developing algorithm and easy for extension. There exist different methods or techniques for construction of magic squares based on algebra, graphs etc. and many of these methods are not suitable for introducing in secondary school level.

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