Goldie Pure Rickart Modules and Duality

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Abstract: Let R be a commutative ring with identity and M be an R-module. Let $Z_2(M)$ be the second singular submodule of M. In this research we introduce the concept of Goldie Pure Rickart modules and dual Goldie Pure Rickart modules as a generalization of Goldie Rickart modules and dual Goldie Rickart modules respectively. An R-module M is called Goldie Pure Rickart if $f^{-1}(Z_2(M))$ is a pure submodule of M for every $f \in \text{End}(M)$. An R-module M is called dual Goldie Pure Rickart if $f^{-1}(Z_2(M))$ is a pure submodule of M for every $f \in \text{End}(M)$. Various properties of this class of modules and other related modules are studied.

Keywords: Goldie Pure Rickart modules, Pure Rickart modules, dual Goldie Pure Rickart modules, dual Pure Rickart modules, Pure Submodules

1. Introduction

Throughout this research R denotes a commutative ring with identity. For a right R-module $M$, $S=\text{End}(M)$ will denote the endomorphism ring of M; thus M can be viewed as a left S-module. For $f \in S$, the right annihilator of $f$ in $M$, denoted by $\text{Ann}_R(f) = \{m \in M | f(m) = 0\}$. The singular submodule of $Z(M) = \{m \in M | mf = 0\}$ for some essential ideal I of R. If $M = Z(M)$ then M is called singular and M is nonsingular provided $Z(M) = 0$. The second singular submodule (or the Goldie torsion submodule) $Z_2(M)$ is defined by $Z_2(M) = Z(M)(Z(M))/Z(M)$. The module $Z_2(M)$ is called Goldie torsion (or torsion-free) M if $Z_2(M) = 0$. It is clear that every singular module is Goldie torsion. An R-module M is called Rickart if for every $f \in \text{End}(M)$, Ker $f$ is a direct summand of M. An R-module M is called dual Rickart if for every $f \in \text{End}(M)$, $\text{Im}(f)$ is a direct summand of M [11]. According to Ungor, Halicioglu and Harmanci [13], an R-module M is called Goldie Rickart if $f^{-1}(Z_2(M))$ is a direct summand of M for every $f \in S$ where $f^{-1}(Z_2(M)) = \{m \in M | f(m) \in Z_2(M)\}$. Asgari and Haghanayin [5] introduced the dual concept of Goldie Pure Rickart module under name dual torsion module. An R-module M is called dual torsion module if $\pi^{-1}(f^{-1}(Z_2(M))) = \text{Im}(f) + Z_2(M)$ is a direct summand of M for every $f \in S$ where $f, M \to M$. This is the homomorphism defined by $f(m) = f(m) + Z_2(M)$ and it is the canonical epimorphism from M to $M/Z_2(M)$. Our aim of this work is to give and study a generalization of Goldie Rickart modules and dual Goldie Rickart modules respectively. An R-module M is called Goldie Pure Rickart if $f^{-1}(Z_2(M))$ is a pure submodule of M for every $f \in S$. An R-module M is called dual Goldie Pure Rickart if $f^{-1}(Z_2(M))$ is a pure submodule of M for every $f \in S$. A submodule N of an R-module M is called pure if every submodule of M is a pure submodule [14]. It is obvious that every Goldie Rickart module and every dual Goldie Rickart module is dual Goldie Pure Rickart, but not conversely. Remarks and Examples 2.2.3 and 3.2.3 respectively.

This research consists of three sections. In Section two we supply some examples and properties of Goldie Pure Rickart modules (Definitions and Examples 2.3). We see that Goldie Pure Rickart modules coincide with Pure Rickart modules in non singular modules. It is shown that the direct sum of Goldie Pure Rickart modules is Goldie Pure Rickart. Section three deals with the concept of dual Pure Rickart modules. We show that dual Goldie Pure Rickart modules are exactly dual Pure Rickart modules when the modules are nonsingular. Many results about these concepts are investigated.

2. Goldie Pure Rickart Modules

In this section we give the concept of Goldie Pure Rickart modules. The basic properties are investigated. It is shown that every direct sum of Goldie Pure Rickart modules is again Goldie Pure Rickart. We begin by giving our definition.

Definition 2.1. An R-module M is called Goldie Pure Rickart if $f^{-1}(Z_2(M))$ is a pure submodule of M for every $f \in S$. If $M = R$, then R is called Goldie Pure Rickart ring if R is Goldie Pure Rickart as R-module.

Lemma 2.2. Let M be an R-module, then for every $f \in S$, $f^{-1}(Z_2(M)) = \{m \in M | f(m) \in Z_2(M)\}$ and $Z_2(M) = \{0\}$. Moreover, the kernel $\ker f$ is a submodule of $f^{-1}(Z_2(M))$.

Proof. It is clear that $f^{-1}(Z_2(M)) = \{m \in M | f(m) \in Z_2(M)\}$. Let $m \in Z_2(M)$, then $f(m) \in (Z_2(M))$ for every $f \in S$. It is known that $f(Z_2(M)) \subseteq Z_2(M)$, then one can easily see that $f(Z_2(M)) \subseteq Z_2(M)$. Thus $f(m) \in Z_2(M)$, which implies that $m \in f^{-1}(Z_2(M))$. Then $Z_2(M) \subseteq \ker f \subseteq sf^{-1}(Z_2(M))$. For the reverse inclusion, let $m \in sf^{-1}(Z_2(M))$. Then $m \in f^{-1}(Z_2(M))$, which implies that $f(m) \in Z_2(M)$ for every $f \in S$. Taking $f = 1$ the identity endomorphism of M, then we have $m \in Z_2(M)$. It follows that $\ker f \subseteq sf^{-1}(Z_2(M)) \subseteq Z_2(M)$ and hence
Recall that an $R$-module $M$ is called Pure Rickart if for every $f \in S$, $\text{Ker } f = \{ m \in M \mid f(m) = 0 \}$ is a submodule of $f^{-1}(Z(M))$.

The following fact is needed throughout the paper, which can be found in [3]

**Lemma 2.4.** Let $M$ be an $R$-module, then we have

1. If $A$ is a pure submodule of $N$, and $N$ is a pure submodule of $M$, then $A$ is a pure submodule of $M$.
2. If $A$ is a pure submodule of $M$ and $N$ is a submodule of $M$ containing $A$, then $A$ is a pure submodule of $N$.

Recall that an $R$-module $M$ is called Pure simple if $M \neq 0$ and it has no pure submodules except $\{0\}$ and $M$ [8].

**Proposition 2.5.** Every Pure simple Goldie Pure Rickart $R$-module is Pure Rickart or Goldie torsion module.

**Proof.** Let $M$ be pure simple Goldie Pure Rickart $R$-module and $f$ be the identity endomorphism of $M$. Then $f^{-1}(Z(M)) = Z(M)$ is a pure submodule of $M$. But $M$ is pure simple, implies that $Z(M) = M$ or $Z(M) = 0$. That is $M$ is Goldie torsion or it is nonsingular and so by Remark and Example 2.3 (1), $M$ is Pure Rickart.

Recall that a submodule $A$ of an $R$-module $M$ is called an essential submodule of $M$ (or $M$ is an essential extension of $A$) if $A \cap B \neq 0$, for every submodule $B$ of $M$. If $A$ has no proper essential extension in $M$, then $A$ is said to be closed [7]. An $R$-module $M$ is called Purely extending module if every closed submodule of $M$ is a pure submodule of $M$.

**Proposition 2.6.** If $M$ is a Pure simple and Purely extending $R$-module, then $M$ is nonsingular or Goldie Pure Rickart.

**Proof.** Let $M$ be a Pure simple and Purely extending $R$-module. Since $Z(M)$ is a closed submodule in $M$, then $Z(M)$ is a pure submodule in $M$. Hence $Z(M) = 0$ or $Z(M) = M$. It follows that $M$ is nonsingular or Goldie Pure Rickart because $f^{-1}(Z(M)) = M$ for every $f \in S$.

Recall that a submodule $A$ of an $R$-module $M$ is called $y$-closed if $M/A$ is nonsingular module [9].

**Proposition 2.7.** Every Purely extending $R$-module is Goldie Pure Rickart.

**Proof.** Let $M$ be a Purely extending $R$-module. Since $Z(M)$ is $y$-closed submodule in $M$, then $f^{-1}(Z(M))$ is again $y$-closed submodule in $M$ for every $f \in S$. Thus $f^{-1}(Z(M))$ is closed submodule in $M$, and hence $f^{-1}(Z(M))$ is pure submodule in $M$. Hence $M$ is Goldie Pure Rickart.

**Remark 2.8.** The converse of Proposition 2.7 does not hold in general, for example; Consider $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}$. Clearly that $Z(M) = M$, then $M$ is Goldie torsion as $\mathbb{Z}$-module, and hence it is Goldie Pure Rickart. Let $N = \langle 2, 1 \rangle \mathbb{Z}$ be the submodule generated by $\langle 2, 1 \rangle$, it is not hard to see that $N$ is closed submodule in $M$. But $N$ is not pure in $M$, since $\langle 2, 0 \rangle \notin M / N$. On the other hand, $\langle 4, 0 \rangle \notin M / N \iff N = \{0\}$ implies that $M / N \neq N$. That is $M$ is not Purely extending.
Proposition 2.9. Let \( M \) be an \( R \)-module. Then \( M \) is a Goldie Pure Rickart if and only if \( M \cong Z(M) \oplus N \) where \( N \) is a nonsingular Pure Rickart module.

Proof. (\( \iff \)) Assume that \( M \) is Goldie Pure Rickart and \( M = Z(M) \oplus N \) for some submodule \( N \) of \( M \). Let \( f \in \text{End}(M) \) and let \( 1_{Z(M)} \oplus f \in \text{End}(M) \). Then \( 1_{Z(M)} \oplus f \in \text{End}(M) \), so \( g^{-1}(Z(M)) = \{(m_1, m_2) \in M \mid f(m_1) = m_2 \} \) where \( m_1 \in Z(M) \) and \( m_2 \in N \). But \( Z(M) \cap N = 0 \), so it follows that \( g^{-1}(Z(M)) = \{(m_1, m_2) \in M \mid f(m_1) = m_2 \} \) and \( f \mid g^{-1}(Z(M)) \). Hence \( M \) is a Goldie Pure Rickart module.

Lemma 2.12. Let \( \{M_i\}_{\alpha \in \Lambda} \) be a class of \( R \)-modules for an arbitrary index set \( \Lambda \). Let \( M = \bigoplus_{\alpha \in \Lambda} M_i \). Then \( M \) is a projective module if and only if \( \bigoplus_{\alpha \in \Lambda} M_i \) is projective for every \( \alpha \in \Lambda \).

Proof. Let \( M = \bigoplus_{\alpha \in \Lambda} M_i \). Then \( M \) is projective if and only if \( \bigoplus_{\alpha \in \Lambda} M_i \) is projective for every \( \alpha \in \Lambda \).

Theorem 2.13. Let \( \{M_i\}_{\alpha \in \Lambda} \) be a class of \( R \)-modules for an arbitrary index set \( \Lambda \). Let \( M = \bigoplus_{\alpha \in \Lambda} M_i \). Then \( M \) is a projective module if and only if \( \bigoplus_{\alpha \in \Lambda} M_i \) is projective for every \( \alpha \in \Lambda \).

Proof. Let \( M = \bigoplus_{\alpha \in \Lambda} M_i \). Then \( M \) is projective if and only if \( \bigoplus_{\alpha \in \Lambda} M_i \) is projective for every \( \alpha \in \Lambda \).
(4)⇒(1) Assume that $R$ is a Goldie Pure Rickart ring, then by Theorem 2.13, for any index set $A$, $\phi_j R$ is a Goldie Pure Rickart $R$-module.

(1)⇒(3) and (3)⇒(4) Follow by similar proof of (1)⇒(2) and (2)⇒(4)

**Dual Goldie Pure Rickart Modules**

This section is devoted to study the concept of dual Goldie Pure Rickart modules. Basic properties of this type of modules are investigated. We start with the following definition.

**Definition 3.1.** An $R$-module $M$ is called dual Goldie Pure Rickart if $\pi^{-1}(\text{Im}(f)) = \text{Im}(f + Z_2(M))$ is a pure (in sense of Anderson and Fuller) submodule of $M$ for every $f \in \text{End}_R(M)$, where $f : R M \rightarrow M \otimes Z_2(M), \text{Im}(f) = R$, then $R$ is called dual Goldie Pure Rickart ring if $R$ is Goldie Pure Rickart as $R$-module.

Recall that an $R$-module $M$ is called dual Pure Rickart if for every $f \in \text{End}_R(M)$, $\text{Im}(f)$ is a pure (in sense of Anderson and Fuller) submodule of $M$ [2]. If $M = R$, then $R$ is called dual Pure Rickart ring if $R$ is dual Pure Rickart as $R$-module.

**Remarks and Examples 3.2.**

1. Let $M$ be a nonsingular $R$-module. Then $M$ is dual Pure Rickart $R$-module if and only if it is dual Goldie Pure Rickart.

   **Proof** ($\Rightarrow$) Assume that $M$ is dual Pure Rickart $R$-module. Then for every $f \in \text{End}_R(M)$, $\pi^{-1}(\text{Im}(f)) = \text{Im}(f + Z_2(M)) = \text{Im}(f)$. But $M$ is dual Pure Rickart, then $\pi^{-1}(\text{Im}(f)) = \text{Im}(f)$ is pure in $M$. So $M$ is dual Goldie Pure Rickart.

   ($\Leftarrow$) By similar proof.

2. If $M$ is a dual Goldie Pure Rickart $R$-module, then $M$ need not be dual Pure Rickart. For example, the $\mathbb{Z}$-module $\mathbb{Z}_{12}$ is dual Goldie Pure Rickart because $Z_2(\mathbb{Z}_{12}) = \mathbb{Z}_{12}$ implies $\pi^{-1}(\text{Im}(f + Z_2(\mathbb{Z}_{12}))) = \text{Im}(f + Z_2(\mathbb{Z}_{12}))$ is pure $12$ for each $f \in \text{End}_R(\mathbb{Z}_{12})$ while $Z_2(\mathbb{Z}_{12})$ is not dual Pure Rickart since $\pi^{-1}(\mathbb{Z}_{12}) = \mathbb{Z}_{12}$ where $f(m) = (m, 0)$ is not a pure submodule of $\mathbb{Z}_{12}$.

3. It is clear that every dual Goldie Pure Rickart module is dual Goldie Pure Rickart. But the converse is not true in general. For example, consider the ring $R = \prod_{i=1}^{\infty} \mathbb{Z}$ and the $R$-module $M = R(f R)$.

   By [2, Example 2.2(2)], $M$ is dual Pure Rickart, and since $M$ is nonsingular implies that $M$ is dual Goldie Pure Rickart by (1). But $M$ is not dual Goldie Pure Rickart module because if $M$ is dual Goldie Rickart, it follows that $M$ is dual Rickart.

4. Obviously every Goldie torsion (singular) module is dual Goldie Pure Rickart module. The converse is not true in general. For example, in the $\mathbb{Z}_n$-module $\mathbb{Z}_n$, for any $f \in \text{End}_R(M)$, $\pi^{-1}(\text{Im}(f + Z_2(\mathbb{Z}_n))) = \text{Im}(f + \mathbb{Z}_n) = \text{Im}(f)$ is pure in $\mathbb{Z}_n$.

5. It is clear that regular module is dual Goldie Pure Rickart. The converse is not true in general. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ is dual Goldie Pure Rickart module but not regular.

6. If $R$ is Goldie torsion ring, that is $Z_2(R) = R$. Then every $R$-module is dual Goldie Pure Rickart.

   **Proof.** Let $M$ be an $R$-module and $Z_2(\mathbb{R}) = R$. By the same argument of Remark and Example 1 2.3, $(6)$, $Z_2(\mathbb{M}) = M$, implies that $\pi^{-1}(\text{Im}(f + Z_2(\mathbb{M}))) = \text{Im}(f + M)$ is a pure submodule of $M$ for every $f \in \text{End}_R(\mathbb{M})$. That is $M$ is dual Goldie Pure Rickart.

7. If $M / N$ is a dual Goldie Pure Rickart or dual Goldie Pure Rickart $R$-module for any non-zero submodule $N$ of an $R$-module $M$, then $N$ need not be dual Goldie Pure Rickart. For example, consider the ring $\mathbb{Z}$-module, for every $f : \mathbb{Z} \rightarrow \mathbb{Z}$, where $0 \neq f \neq 1$, then $\pi^{-1}(\text{Im}(f + Z_2(\mathbb{Z}))) = \text{Im}(f) = \text{Im}(f)$ is not pure of $\mathbb{Z}$. It follows that $\mathbb{Z}$ is not dual Goldie Pure Rickart. On the other hand, $\mathbb{Z} / \mathbb{Z} = \mathbb{Z}$ as a module is torsion module, for each positive integer $n$ and hence it is dual Goldie Pure Rickart.

**Proposition 3.3.** Every pure simple dual Goldie Pure Rickart $R$-module is dual Goldie Pure Rickart module.

**Proof.** Let $M$ be pure simple dual Goldie Pure Rickart $R$-module and $f$ be the zero endomorphism of $M$. Then $\pi^{-1}(\text{Im}(f + Z_2(\mathbb{M}))) = 0$. But $M$ is dual Goldie Pure Rickart, then $\pi^{-1}(\text{Im}(f)) = \text{Im}(f)$ is pure in $M$. Hence $M$ is dual Goldie Pure Rickart because.

**Proposition 3.4.** If $M$ is pure simple and purely extending $R$-module, then $M$ is a nonsingular or dual Goldie Pure Rickart.

**Proof.** Let $M$ be pure simple and purely extending $R$-module. Since $Z_2(\mathbb{M}) = 0$ is a closed submodule in $M$, then $Z_2(\mathbb{M})$ is a pure submodule in $M$. Hence $Z_2(\mathbb{M}) = 0$ or $Z_2(\mathbb{M}) = M$. That follows that $M$ is nonsingular or dual Goldie Pure Rickart because.

**Proposition 3.5.** Let $M$ be an $R$-module. Then $M$ is dual Goldie Pure Rickart and $Z_2(\mathbb{M})$ a direct summand of $M$ if and only if $M = Z_2(\mathbb{M}) \oplus N$ where $N$ is a nonsingular dual Goldie Pure Rickart module.

**Proof.** ($\Rightarrow$) Assume that $M$ is dual Goldie Pure Rickart and $M \cong Z_2(\mathbb{M}) \oplus N$ for some submodule $N$ of $M$. Let $f \in \text{End}_R(M)$ and $Z_2(\mathbb{M})$ be the identity endomorphism of $Z_2(\mathbb{M})$. Then $g = 1_{Z_2(\mathbb{M})} \oplus f \in \text{End}_R(M)$. So $\pi^{-1}(\text{Im}(f + Z_2(\mathbb{M}))) = \text{Im}(f + Z_2(\mathbb{M})) = (Z_2(\mathbb{M}) \oplus f(N)) = (Z_2(\mathbb{M}) \oplus f(N)) = (Z_2(\mathbb{M}) \oplus f(N))$ which is a pure submodule in $M$. Since $f$ is a direct summand of $Z_2(\mathbb{M}) \oplus f(N)$, then $f$ is a pure submodule of $\pi^{-1}(\text{Im}(f + Z_2(\mathbb{M})))$. Hence $f$ is pure in $M$. But $N$ is containing $\pi^{-1}(\text{Im}(f)$, thus $f$ is pure in $N$. Hence $N$ is a dual Goldie Pure Rickart $R$-module. Also $\text{N}$ is not singular $R$-module because $M / Z_2(\mathbb{M})$ is nonsingular.

($\Leftarrow$) Assume $M = Z_2(\mathbb{M}) \oplus N$ where $N$ is nonsingular dual Goldie Pure Rickart $R$-module. To show that $M$ is dual Goldie Pure Rickart, let $f \in \text{End}_R(M)$, $\rho_b$, the projection map of $M$ onto $N$ and $i$ be the inclusion map of $N$ into $M$. Then $(\rho_b \circ f) \in \text{End}_R(N)$ implies that $1_{Z_2(\mathbb{M})} \oplus (\rho_b \circ f) \in \text{End}_R(M)$. Since $\pi^{-1}(\text{Im}(f + Z_2(\mathbb{M}))) = \text{Im}(f + Z_2(\mathbb{M}))$. It is easy to see that $\ker f = \ker(\rho_b \circ f)$.
Proposition 3.6. Let $M$ be dual Goldie Pure Rickart $R$-module and $\text{Im}f$ is a pure submodule of $\pi^{-1}(\text{Im}f + Z_2(M))$ for any $f \in \text{End}_R(M)$. Then $M$ is dual Pure Rickart and $Z_2(M)$ is a pure submodule of $M$.

Proof. Let $M$ be a dual Goldie Pure Rickart $R$-module and $f \in \text{End}_R(M)$. Then $\pi^{-1}(\text{Im}f + Z_2(M))$ is a pure submodule of $M$ and by hypothesis, $\text{Im}f$ is a pure submodule of $\pi^{-1}(\text{Im}f + Z_2(M))$. It follows that $\text{Im}f$ is a pure submodule of $M$, thus $M$ is dual Pure Rickart. Also $Z_2(M)$ is a pure submodule of $M$, where $0$ is the zero endomorphism of $M$.

Theorem 3.7. Let $\{M_i\}_{\in \mathcal{I}}$ be a class of $R$-modules for an arbitrary index set $\mathcal{I}$. Then $M$ is Goldie Pure Rickart modules for all $i \in \mathcal{I}$ and only if $\bigoplus_{i \in \mathcal{I}} M_i$ is dual Goldie Pure Rickart module.

Proof. ($\Rightarrow$) Assume $M_i$ is dual Goldie Pure Rickart $R$-module for all $i \in \mathcal{I}$ and $M = \bigoplus_{i \in \mathcal{I}} M_i$. Let $I$ be an ideal of $R$ and $f = (f_i)_{i \in \mathcal{I}} \in \text{End}_R(\bigoplus_{i \in \mathcal{I}} M_i)$ where $f_i \in \text{End}_R(M_i)$. To show that $Mf = \pi^{-1}(\text{Im}f + Z_2(M))$, it follows that $\text{Im}f$ is a pure submodule of $M$, thus $M$ is dual Pure Rickart. Also $Z_2(M)$ is a pure submodule of $M$, where $0$ is the zero endomorphism of $M$.

Corollary 3.8. Every direct summand of a dual Goldie Pure Rickart module is again dual Goldie Pure Rickart.

Proof. It follows directly by Theorem 3.6.

The following theorem gives a characterization of dual Goldie Pure Rickart rings in terms of dual Goldie Pure Rickart modules.

Proposition 3.9. Let $R$ be a ring. The following statements are equivalent.

1. $\bigoplus_{i \in \mathcal{I}} R$ is dual Goldie Pure Rickart $R$-module for each index set $\mathcal{I}$.
2. Every projective $R$-module is dual Goldie Pure Rickart module.
3. Every free $R$-module is dual Goldie Rickart.
4. $R$ is a dual Goldie Pure Rickart ring.

Proof. (1) $\Rightarrow$ (2) Let $M$ be a projective $R$-module, then there exists a free $R$-module $F$ and an $R$-epimorphism $f: F \to M$ and $F \cong \bigoplus_{i \in \mathcal{I}} R$ where $\mathcal{I}$ is an index set. We have the following short exact sequence $0 \to \text{Ker} f$.