Goldie Pure Rickart Modules and Duality

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Abstract: ***Let R be a commutative ring with identity and M be an R-module. Let $Z_2(M)$ be the second singular submodule of M. In this research we introduce the concept of Goldie Pure Rickart modules and dual Goldie Pure Rickart modules as a generalization of Goldie Rickart modules and dual Goldie Rickart modules respectively. An R-module M is called Goldie Pure Rickart if $f^{-1}(Z_2(M))$is a pure(in sense of Anderson and Fuller) submodule of M for every $f \in End_R(M)$.An R-module M is called dual Goldie Pure Rickart if $\pi^{-1}(Im\bar{f})$ is a pure(in sense of Anderson and Fuller) submodule of M for every $f \in End_R(M)$. Various properties of this class of modules are given and some relationships between these modules and other related modules are studied.***

Keywords: Goldie Pure Rickart modules, Pure Rickart modules, dual Goldie Pure Rickart modules, dual Pure Rickart modules, Pure Submodules

1. Introduction

Throughout this research R denotes a commutative ring with identity. For a right R-module M, $S = End_R(M)$ will denote the endomorphism ring of M ; thus M can be viewed as a left S- right R-bimodule. For $f \in S$, the right annihilator of f in M is $r_M(Sf) = r_M(f) = Ker f = \{m \in M \mid f(m) = 0\}$.The singular submodule of M is $Z(M) = \{ m \in M \mid mI = 0$ for some essential ideal I of $R \}$. If $M = Z(M)$ then M is called singular and M is nonsingular provided $Z(M) = 0$. The second singular submodule(or the Goldie torsion submodule) $Z_2(M)$ is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. The module M is called Goldie torsion (orZ_2-torsion) if $M = Z_2(M)$. It is clear that every singular module is Goldie torsion. An R-module M is called Rickart if for every $f \in End_R(M)$, Ker f is a direct summand of M [12]. An R-module M is called dual Rickartif for every $f \in End_R(M)$, Im $f$is a direct summand of M [11].According to Ungor, Halicioglu and Harmanci [13], an R-module M is called Goldie Rickart if$f^{-1}(Z_2(M))$is a direct summand of M for every $f \in S$ where $f^{-1}(Z_2(M)) = \{ m \in M \mid f(m) \in Z_2(M)\}$. Asgari and Haghanyin [5] introduced the dual concept of Goldie Pure Rickart module under name dual t-Rickart module. An R-module M is called dual t-Rickartif $\pi^{-1}(Im\bar{f}) = \pi^{-1}(Im f + Z_2(M)) = Im f + Z_2(M)$ is a direct summand of M for every $f \in S$ where$\bar{f}: M \longrightarrow M / Z_2(M)$is the homomorphisim defined by $\bar{f}(m) = f(m) + Z_2(M)$ and$\pi$is the canonical epimorphism from M to $M / Z_2(M)$defined by $\pi(m) = m + Z_2(M)$.

Our aim of this work is to give and study a generalization of Goldie Rickart modules and dual Goldie Rickart modules respectively. An R-module M is called Goldie Pure Rickart if$f^{-1}(Z_2(M))$ is a pure (in sense of Anderson and Fuller) submodule of M for every $f \in S$. An R-module M is called dual Goldie Pure Rickart if $\pi^{-1}(Im\bar{f}) = Im f + Z_2(M)$ is a pure(in sense of Anderson and Fuller) submodule of M for every $f \in S$. A submodule N of an R-module M is called pure(in sense of Anderson and Fuller)if $N \cap MI = NI$ for every ideal I of R [4]. An R-module M is called regular if every submodule of M is a pure submodule[14].It is obvious that every Goldie Rickart module is Goldie Pure Rickart and every dual Goldie Rickart module is dual Goldie Pure Rickart, but not conversely Remarks and Exampels2.2.(3)and 3.2.(3)respectively.

This research consists of three sections. In Section two we supply some examples and properties of Goldie Pure Rickart modules(Remarks and Examples 2.3). We see that Goldie Pure Rickart modules coincide with Pure Rickart modules in non singular modules. It is shown that the direct sum of Goldie Pure Rickart modules is Goldie Pure Rickart. Section three deals with the concept of dual Goldie Pure Rickart modules. We show that dual Goldie Pure Rickart modules are exactly dual Pure Rickart modules when the modules are nonsingular. Many results about these concepts are investigated.

2. Goldie Pure Rickart Modules

In this section we give the concept of Goldie Pure Rickart modules. The basic properties are investigated. It is shown that every direct sum of Goldie Pure Rickart modules is again Goldie Pure Rickart. We begin by giving our definition.

Definition 2.1. An R-module M is called Goldie Pure Rickart if$f^{-1}(Z_2(M))$is a pure submodule of M for every $f \in S$. If $M = R$, then R is called Goldie Pure Rickart ring if R is Goldie Pure Rickart as R-module.

Lemma 2.2. Let M be an R-module, then for every $f \in S$, $f^{-1}(Z_2(M)) = \{m \in M \mid f(m) \in Z_2(M)\}$ and $Z_2(M) = \cap_{f \in S} f^{-1}(Z_2(M))$. Moreover the kernel Ker$f$ is a submodule of $f^{-1}(Z_2(M))$.

Proof. It is clear that $f^{-1}(Z_2(M)) = \{m \in M \mid f(m) \in Z_2(M)\}$. Let $m \in Z_2(M)$, then $f(m) \in f(Z_2(M))$for every $f \in S$. It is known that $f(Z(M)) \leq Z(M)$, then one can easily see that $f(Z_2(M)) \leq Z_2(M)$.Thus $f(m) \in Z_2(M)$, implies that $m \in f^{-1}(Z_2(M))$. Then $Z_2(M) \leq \cap_{f \in S} f^{-1}(Z_2(M))$. For the reverse inclusion, let $m \in \cap_{f \in S} f^{-1}(Z_2(M))$. Then $m \in f^{-1}(Z_2(M))$, implies that $f(m) \in Z_2(M)$ for every $f \in S$. Taking $f = 1$ the identity endomorphism of M, then we have $m \in Z_2(M)$. It follows that $\cap_{f \in S} f^{-1}(Z_2(M)) \leq Z_2(M)$ and hence

$Z_2(M) = \bigcap_{f \in S} f^{-1}(Z_2(M))$. Further, it is obvious that Ker f $=\{m \in M \mid f(m) = 0\}$ is a submodule of $f^{-1}(Z_2(M))$.

Recall that an R-module M is called Pure Rickart if for every $f \in S$, Ker f is a pure (in sense of Anderson and Fuller) submodule of M [1]. If $M = R$, then R is called Pure Rickart ring if R is Pure Rickart as R-module. In other words, R is Pure Rickart ring if $ann_R(a)$ of R is a pure ideal of R for each $a \in R$ [1].When $M = R$, the concept of Pure Rickart modules coincides with that of PF-rings [1]. R is called aPF-ring if every principal ideal is a flat ideal in R [10].

Remarks and Examples 2.3

(1) Let M be a nonsingular R-module. Then M is Pure Rickart$R$-module if and only if it is Goldie Pure Rickart.
Proof. It is clear.

(2) If M is Goldie Pure Rickart R-module, then M need not be Pure Rickart. For example, consider $\mathbb{Z}_4$ as a $\mathbb{Z}$-module. Since $Z_2(\mathbb{Z}_4) = \mathbb{Z}_4$ then $f^{-1}(Z_2(\mathbb{Z}_4))$ is a pure submodule of M for every $f \in End_{\mathbb{Z}}(\mathbb{Z}_4)$ implies that $\mathbb{Z}_4$ is Goldie Pure Rickart. On the other hand, for $f \in End_{\mathbb{Z}}(\mathbb{Z}_4)$ with $f(m) = m2$, Ker $f = \{\bar{0}, \bar{2}\}$is not a pure submodule of $\mathbb{Z}_4$ for each $m \in \mathbb{Z}_4$. Hence $\mathbb{Z}_4$ is not Pure Rickart.

(3) It is obvious that every Goldie Rickart module is Goldie Pure Rickart, but the reverse is not true in general. For example, consider the ring $R = (\prod_{i=1}^{\infty} \mathbb{Z}_2) / (\bigoplus_{i=1}^{\infty} \mathbb{Z}_2)$. By [15, Example 2.5], every principal ideal of the power series ring $R_1 = R[[x]]$ over R is flat implies that R_1 is a PF-ring. That is, R_1 is Pure Rickart and nonsingularR_1-module. So by (1), R_1 is Goldie Pure Rickart butR_1 is not Goldie Rickart because if R_1 is Goldie Rickart then by [13],it is Rickart which is a contradiction since R_1 is not Rickart by [15,Example 2.5].

(4) Clearly Goldie torsion (singular) module is Goldie Pure Rickart module. The converse is not true in general. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is Goldie Pure Rickart since $Z_2(\mathbb{Z}) = Z(\mathbb{Z}) = 0$, then $f^{-1}(Z_2(\mathbb{Z}))$= ker $f = 0$ is pure in $\mathbb{Z}$ but $\mathbb{Z}$ is neither Goldie torsion nor singular.

(5) Of course, every regular module is Goldie Pure Rickart, but the converse is not true in general. For example, the $\mathbb{Z}$ -module $\mathbb{Z}$ is Goldie Pure Rickart module but not regular. Also, one can easily see that the $\mathbb{Z}$-module $\mathbb{Z}_4$is Goldie torsion then by (1), it is Goldie Pure Rickart but not regular.

(6) If R is Goldie torsion ring, that is $Z_2(R) = R$. Then every R-module is Goldie Pure Rickart.
Proof. Let M be an R-module and $Z_2(R) = R$, it is not hard to see that $MZ_2(R) \leq Z_2(M)$.Then $Z_2(M) = M$, implies that $f^{-1}(Z_2(M))$is a pure submodule of M for every $f \in S$. That is M is Goldie Pure Rickart.

(7) If M / N is a Goldie Pure Rickart$R$-module for any non-zero submodule N of an R-module M, then Mmay not be a Goldie Pure Rickart. For example $\mathbb{Z}_{12}$ as a $\mathbb{Z}_{12}$-module, it can be easily shown that $Z(\mathbb{Z}_{12}) = \{\bar{0}, \bar{6}\} = Z_2(\mathbb{Z}_{12})$. Then $\mathbb{Z}_{12}$ is not Goldie Pure Rickart because for the identity endomorphism $1 \in End_{\mathbb{Z}}(\mathbb{Z}_{12})$, we have $1^{-1}(Z_2(\mathbb{Z}_{12})) = \{ m \in \mathbb{Z}_{12} \mid 1(m) \in Z(\mathbb{Z}_{12}) \} = \{\bar{0}, \bar{6}\}$ is not a pure submodule of $\mathbb{Z}_{12}$,while $\mathbb{Z}_{12} / \{\bar{0}, \bar{6}\}$ is regular module and hence are Goldie Pure Rickart by(5).

The following fact is needed throughout the paper, which can be found in [3]

Lemma 2.4.let$M$ be an R-module, then we have
(1) If A is a pure submodule of N, and N is a pure submodule of M. then A is a pure submodule of M.
(2) If A is a pure submodule of M and $N$is a submodule of M containing A, then A is a pure submodule of N.

Recall that an R-module M is called Pure simple if $M \neq$<0> and it has no pure submodules except <0> and M [8].

Proposition 2.5. Every Pure simple Goldie Pure Rickart R-module is Pure Rickartor Goldie torsion module.

Proof. Let M be pure simple Goldie Pure Rickart R-module and f be the identity endomorphism of M. Then $f^{-1}(Z_2(M)) = Z_2(M)$ is a pure submodule of M. But M is pure simple, implies that $Z_2(M) = M$ or$Z_2(M) = 0$.That is M is Goldie torsion or it is nonsingular and so by Remark and Example 2.3 (1), M is PureRickart.

Recall that a submodule A of an R-module M is called an essential submodule of M(or M is an essential extension of$A$) if $A \cap B \neq 0$, for every submodule B of M. If A has no proper essential extension in M, then A is said to be closed[7].An R-module M is called Purely extending module if every closed submodule in M is a pure submodule in M[6].

Proposition 2.6. If M is a Pure simple and Purely extending R-module, then M is nonsingular or Goldie Pure Rickart.

Proof. Let M be a Pure simple and Purely extending R-module. Since $Z_2(M)$ is a closed submodule in M, then $Z_2(M)$ is a pure submodule in M. Hence $Z_2(M)$ =0 or $Z_2(M)$=M. It follows that M is nonsingular or Goldie Pure Rickart because $f^{-1}(Z_2(M)) = M$ for every $f \in S$.

Recall that a submodule A of an R-module M is called y-closed if M/A is nonsingular module [9].

Proposition 2.7. Every Purely extending R-module is Goldie Pure Rickart.

Proof. Let M be a Purely extending R-module. Since $Z_2(M)$ is y-closed submodule in M, then$f^{-1}(Z_2(M))$ is again y-closed submodule in M for every $f \in S$. Thus $f^{-1}(Z_2(M))$ is closed submodule in M, and hence $f^{-1}(Z_2(M)$is pure submodule in M. Hence M is Goldie Pure Rickart.

Remark2.8. The converse of Proposition 2.7 does not hold in general, for example; Consider $\mathbb{Z}$-module M=$\mathbb{Z}_8 \oplus \mathbb{Z}_2$.Clearly that $Z_2(M)$= M, then $M$is Goldie torsion as $\mathbb{Z}$-module, and hence it is Goldie Pure Rickart. Let N= $(\bar{2}, \bar{1})$ $\mathbb{Z}$ be the submodule generated by$(\bar{2}, \bar{1})$, it is not hard to see that N is closed submodule in M. But N is not pure in M, since$(\bar{4}, \bar{0}) = (\bar{1}, \bar{0})\, 4 \in M4 \cap N$.On the other hand$(\bar{4}, \bar{0}) \notin N4 = \{(\bar{0}, \bar{0})\}$implies that$M4 \cap N \neq N4$. That is M is not Purely extending.

Proposition 2.9. Let M be an R-module. Then M is a Goldie Pure Rickart and $Z_2(M)$ a direct summand of M if and only if $M = Z_2(M) \oplus N$ where N is a nonsingular Pure Rickart module.

Proof.($\Longrightarrow$) Assume that M is Goldie Pure Rickart and $M = Z_2(M) \oplus N$ for some submodule N of M. Let $f \in End_R(N)$ and $1_{Z_2(M)}$ be the identity endomorphism of $Z_2(M)$. Then $1_{Z_2(M)} \oplus f \in S$. Put $g = 1_{Z_2(M)} \oplus f$, so $g^{-1}(Z_2(M)) = \{ (m_1, m_2) \in M | g(m_1, m_2) = (m_1, f(m_2)) \in Z_2(M) \}$ where $m_1 \in Z_2(M)$ and $m_2 \in N$, $f(m_2) \in N$. But $Z_2(M) \cap N = 0$, it follows that $g^{-1}(Z_2(M)) = \{(m_1, m_2) \in M | m_1 \in Z_2(M)$ and $m_2 \in$ Ker $f\}$ $= Z_2(M) \oplus$ Ker f. But $g^{-1}(Z_2(M))$ is a pure submodule of M, that is $Z_2(M) \oplus$ Ker f is a pure submodule of M and since Ker f is a direct summand of $Z_2(M) \oplus$ Ker f, implies that Ker f is a pure submodule of $g^{-1}(Z_2(M))$. Hence by Lemma 2.4(1), Ker f is a pure submodule of M. But N is containing Ker f, thus by Lemma 2.4(2), Ker f is a pure submodule of N. Hence N is Pure Rickart R-module. Also, N is a nonsingular R-module because $M/Z_2(M)$ is nonsingular.

($\Longleftarrow$) Assume $M = Z_2(M) \oplus N$ where N is a nonsingular Pure Rickart R-module. To show that M is Goldie Pure Rickart, let $f \in S$ and ρ_N be the projection map of M on to N. Then $\rho_N f|_N \in End_R(N)$, implies that $1_{Z_2(M)} \oplus \rho_N f|_N \in End_R(M)$. Since $f^{-1}(Z_2(M)) = \{(m_1, m_2) \in M | f(m_1, m_2) \in Z_2(M)\}$ where $m_1 \in Z_2(M)$, $m_2 \in N$ and $Z_2(M) \cap N = 0$, then $f^{-1}(Z_2(M)) = \{(m_1, m_2) \in M |$ where $m_1 \in Z_2(M)$ and $m_2 \in$ Ker $f\}$. It is easy to see that Ker f = Ker $\rho_N f|_N$. Hence $f^{-1}(Z_2(M)) = Z_2(M) \oplus$ Ker $\rho_N f|_N$. Since N is Pure Rickart R-module implies that Ker $\rho_N f|_N$ is a pure submodule of N, but a direct sum of pure submodules is a pure submodule [3]. It follows $f^{-1}(Z_2(M))$ is a pure submodule of M, therefore M is Goldie Pure Rickart.

Proposition 2.10. Let M be Goldie Pure Rickart R-module and Ker f is a pure submodule of $f^{-1}(Z_2(M))$ for any $f \in S$. Then M is Pure Rickart and $Z_2(M)$ is a pure submodule of M.

Proof. Let M be a Goldie Pure Rickart R-module and $f \in S$, then $f^{-1}(Z_2(M))$ is a pure submodule of M and by hypothesis, Ker f is a pure submodule of $f^{-1}(Z_2(M))$. So Ker f is a pure submodule of M. That is M is Pure Rickart. Further $Z_2(M)$ is a pure submodule of M, because $1^{-1}(Z_2(M)) = Z_2(M)$ is a pure submodule of M, where 1 is the identity endomorphism of M.

Proposition 2.11. Let M be a Pure Rickart R-module and $Z_2(M)$ be a direct summand of M. Then M is a Goldie Pure Rickart and Ker f is a pure submodule of $f^{-1}(Z_2(M))$ for any $f \in S$.

Proof: Let M be a Pure Rickart R-module and $M = Z_2(M) \oplus N$ for some submodule N of M. Then by [1, Proposition 2.6], N is Pure Rickart and by Proposition 2.9, M is Goldie Pure Rickart. In addition, for any $f \in S$, Ker f is a pure submodule of M implies that Ker f is a pure submodule of $f^{-1}(Z_2(M))$.

Lemma 2.12. Let $\{M_i\}_{i \in \Lambda}$ be a class of R-modules for an arbitrary index set Λ. For any $f = (f_i)_{i \in \Lambda} \in End_R(\oplus_{i \in \Lambda} M_i)$, then $f^{-1}(Z_2(\oplus_{i \in \Lambda} M_i)) = \oplus_{i \in \Lambda} f_i^{-1}(Z_2(M_i))$ where $f_i \in End_R(M_i)$.

Proof. Let $m \in f^{-1}(Z_2(\oplus_{i \in \Lambda} M_i))$, then $m \in \oplus_{i \in \Lambda} M_i$. Let $m = (m_i)_{i \in \Lambda}$ where $m_i \in M_i$ for every $i \in \Lambda$, implies $f((m_i)_{i \in \Lambda}) \in Z_2(\oplus_{i \in \Lambda} M_i)$. But $Z_2(\oplus_{i \in \Lambda} M_i) = \oplus_{i \in \Lambda} Z_2(M_i)$ it follows that $f((m_i)_{i \in \Lambda}) \in \oplus_{i \in \Lambda} Z_2(M_i)$ and so $f_i(m_i) \in Z_2(M_i)$. Thus $m_i \in f_i^{-1}(Z_2(M_i))$ for every $i \in \Lambda$ and hence $m = (m_i)_{i \in \Lambda} \in \oplus_{i \in \Lambda} f_i^{-1}(Z_2(M_i))$. Similarly for the reverse.

Theorem 2.13. Let $\{M_i\}_{i \in \Lambda}$ be a class of R-modules for an arbitrary index set Λ. Then M_i is a Goldie Pure Rickart R-module for all $i \in \Lambda$ if and only if $\oplus_{i \in \Lambda} M_i$ is Goldie Pure Rickart.

Proof.($\Longrightarrow$) Assume M_i is Goldie Pure Rickart R-module for all $i \in \Lambda$ and $M = \oplus_{i \in \Lambda} M_i$. Let I be an ideal of R and $f = (f_i)_{i \in \Lambda} \in End_R(\oplus_{i \in \Lambda} M_i)$ where $f_i \in End_R(M_i)$. To show that $MI \cap f^{-1}(Z_2(M)) = f^{-1}(Z_2(M))\, I$. Since $MI \cap f^{-1}(Z_2(M)) = (\oplus_{i \in \Lambda} M_i I) \cap (\oplus_{i \in \Lambda} f_i^{-1}(Z_2(M_i)) = \oplus_{i \in \Lambda}(M_i\, I \cap f_i^{-1}(Z_2(M_i))$. But M_i is Goldie Pure Rickart R-module for all $i \in \Lambda$. It follows that $M_i I \cap f_i^{-1}(Z_2(M_i) = f_i^{-1}(Z_2(M_i)I$, and hence $M I \cap f^{-1}(Z_2(M)) = \oplus_{i \in \Lambda} f_i^{-1}(Z_2(M_i)I = (\oplus_{i \in \Lambda} f_i^{-1}(Z_2(M_i))I = f^{-1}(Z_2(M))$. That is $f^{-1}(Z_2(M))$ is a pure submodule of M.
($\Longleftarrow$) Similarly.

Corollary 2.14. Every direct summand of a Goldie Pure Rickart module is again Goldie Pure Rickart.

Proof. It follows directly by Theorem 2.13.
The following theorem gives a characterization of Goldie Pure Rickart rings in terms of Goldie Pure Rickart modules.

Proposition 2.15. Let R be a ring. The following statements are equivalent.
(1) $\oplus_\Lambda R$ is Goldie Pure Rickart R-module for each index set Λ.
(2) Every projective R-module is Goldie Pure Rickart module.
(3) Every free R-module is Goldie Rickart.
(4) R is a Goldie Pure Rickart ring.

Proof.(1)$\Longrightarrow$ (2) Let M be a projective R-module, then there exists a free R- module F and an R-epimorphism $f: F \longrightarrow M$, and $F \cong \oplus_\Lambda R$ where Λ is an index set. We have the following short exact sequence $0 \longrightarrow$ Ker $f \xrightarrow{i} \oplus_\Lambda R \xrightarrow{f} M \longrightarrow 0$ where i is the inclusion mapping. Since M is projective, the sequence is split implies that $\oplus_\Lambda R \cong$ Ker $f \oplus M$. But $\oplus_\Lambda R$ is Goldie Pure Rickart R-module. Therefore by Corollary 2.14, M is Goldie Pure Rickart module.

(2)$\Longrightarrow$(4) Assume that ever projective R-module is Goldie Pure Rickart module. Since R is a projective R-module, then R is a Goldie Pure Rickart R-module.

(4)⟹(1) Assume that R is a Goldie Pure Rickart ring, then by Theorem 2.13,for any index set Λ , $\oplus_\Lambda R$ is a Goldie Pure Rickart R-module.

(1)⟹(3) and (3)⟹(4)Follow by similar proof of (1) ⟹ (2) and (2) ⟹ (4)

Dual Goldie Pure Rickart Modules

This section is devoted to study the concept of dual Goldie Pure Rickart modules. Basic properties of this type of modules are investigated. We start with the following definition.

Definition 3.1. An R-module $M$is called dual Goldie Pure Rickart if $\pi^{-1}(\mathrm{Im}\bar{f}) = \pi^{-1}(\mathrm{Im}f + Z_2(M)) = \mathrm{Im}f + Z_2(M)$ is a pure (in sense of Anderson and Fuller) submodule of M for every $f\in S$, where$\bar{f}: M \longrightarrow M/Z_2(M)$.If$M= R$, then R is called dual Goldie Pure Rickart ring if R is Goldie Pure Rickart as R-module.

Recall that an R-module M is called dual Pure Rickart if for every $f\in\mathrm{End}_R(M)$, Imf is a pure (in sense of Anderson and Fuller) submodule of M [2]. If $M = R$, then R is called dual Pure Rickart ring if R is dual Pure Rickart as R-module.

Remarks and Examples 3.2.

(1) Let M be a nonsingular R-module. Then M is dual Pure Rickart R-module if and only if it is dual Goldie Pure Rickart.
Proof.(⟹) Assume that M is dual Pure Rickart$R$-module. Then for every $f\in\mathrm{End}_R(M)$, $\pi^{-1}(\mathrm{Im}\bar{f}) = \pi^{-1}(\mathrm{Im}f + Z_2(M)) = \mathrm{Im}f + 0 = \mathrm{Im}f$. But M is dual Pure Rickart, then $\pi^{-1}(\mathrm{Im}\bar{f})$= Im$f$ is pure in M. So M is dual Goldie Pure Rickart.
(⟸) By similar proof.

(2) If M is dual Goldie Pure Rickart$R$-module, then M need not be dual Pure Rickart. For example, the ℤ-moduleℤ$_{12}$ is dual Goldie Pure Rickart because $Z_2(\mathbb{Z}_{12}) = \mathbb{Z}_{12}$ implies $\pi^{-1}(\mathrm{Im}f + Z_2(\mathbb{Z}_{12})) = \mathrm{Im}f + \mathbb{Z}_{12} = \mathbb{Z}_{12}$ is pure of ℤ$_{12}$ for each $f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{12})$ while ℤ$_{12}$ is notdual Pure Rickart since Im$f = \{\bar{0}, \bar{6}\}$ is not a pure submodule of ℤ$_{12}$ where $f(m)=m$ 6 for each $m \in\mathbb{Z}_{12}$.

(3) It is clear that every dual Goldie Rickart module is dual Goldie Pure Rickart. But the converse is not true in general. For example, consider the ring $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$and the R-module $M = R^{(R)}$.
By [2, Example 2.2(2)], M isdual Pure Rickart and since M is nonsingular implies that M is dual Goldie Pure Rickart by (1). But M is not dual Goldie Rickart module because if M is dual Goldie Rickart, it follows that M is dual Rickart.

(4) Obviously every Goldie torsion (singular) module is dualGoldie Pure Rickart module. The converse is not true in general. For example, in the ℤ$_6$-module ℤ$_6$, for any $f\in\mathrm{End}_R(M)$,$\pi^{-1}(\mathrm{Im}f + Z_2(\mathbb{Z}_6)) = \mathrm{Im}f + \bar{0} = \mathrm{Im}f$is pure in ℤ$_6$.

(5) It is clear that regular module is dual Goldie Pure Rickart. The converse is not truein general. For example, the ℤ-moduleℚ is dual Goldie Pure Rickart module but not regular.

(6) If R is Goldie torsion ring, that is $Z_2(R) = R$. Then every R-module is dual Goldie Pure Rickart.
Proof. Let M be an R-module and$Z_2(R) = R$, By the same argument of Remark and Example l 2.3.(6), $Z_2(M) = M$, implies that $\pi^{-1}(\mathrm{Im}f + Z_2(M)) = \mathrm{Im}f + M = M$is a pure submodule of M for every $f\in\mathrm{End}_R(M)$.That is M is dual Goldie Pure Rickart.

(7) If M / N is a dual Goldie Pure Rickart$R$-module for any non-zero submodule N of an R-module M, then M need not be dual Goldie Pure Rickart. For example ℤas a ℤ-module, for every $f\in\mathrm{End}_R(\mathbb{Z})$, where $0\neq f \neq 1$, then$\pi^{-1}(\mathrm{Im}f + Z_2(\mathbb{Z})) = \mathrm{Im}f + 0 = \mathrm{Im}f$ is not pure ofℤ. It follows thatℤ is not dual Goldie Pure Rickart.On the other hand, $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ as ℤ-module is Goldie torsion module, for each positive integer n and hence it is dual Goldie Pure Rickart.

Proposition 3.3. Every Pure simple dual Goldie Pure Rickart R-module is dual Pure Rickartor Goldietorsion module.

Proof. Let M be pure simple dual Goldie Pure Rickart R-module and f be the zero endomorphism of M. Then $\pi^{-1}(\mathrm{Im}f + Z_2(M)) = 0 + Z_2(M) = Z_2(M)$ is a pure submodule of M. But $M$is pure simple, implies $Z_2(M) = 0$ or $Z_2(M) = M$. That is M is Goldie torsion or it is nonsingular and so by Remark and Example 3.2 (1), $M$is dual PureRickart.

Proposition 3.4. If M is Pure simple and Purely extending R-module, then M is a nonsingular or dual Goldie Pure Rickart.

Proof. Let M be Pure simple and Purely extending R-module. Since $Z_2(M)$ is a closed submodule in M, then $Z_2(M)$ is a pure submodule in M. Hence $Z_2(M) = 0$ or $Z_2(M) = M$. It follows that M is nonsingular or dual Goldie Pure Rickart becauce.

Proposition 3.5. Let M be an R-module. Then M is dual Goldie Pure Rickart and $Z_2(M)$ a direct summand of M if and only if $M = Z_2(M)\oplus N$ where N is a nonsingular dual Pure Rickart module.

Proof. (⟹) Assume that M is dual Goldie Pure Rickart and $M = Z_2(M)\oplus N$ for some submodule N of M. Let $f\in\mathrm{End}_R(N)$ and$1_{Z_2(M)}$be the identity endomorphism of $Z_2(M)$.Then $g = 1_{Z_2(M)}\oplus f\in\mathrm{End}_R(M)$. So $\pi^{-1}(\mathrm{Im}g + Z_2(M)) = \mathrm{Im}g + Z_2(M) = g(M) + Z_2(M) = (1_{Z_2(M)}\oplus f)(Z_2(M)\oplus N) + Z_2(M) = (Z_2(M)\oplus f(N)) + (Z_2(M)\oplus 0) = Z_2(M)\oplus f(N)$ which is a pure submodule in M. Since Imf is a direct summand of $Z_2(M)\oplus f(N)$, then Imf is a pure submodule of$\pi^{-1}(\mathrm{Im}g + Z_2(M))$. Hence Im$f$ is pure in M. But N is containing Imf , thus Im$f$is pure in N. Hence N is a dual Pure Rickart R-module. Also Nisa non singular R-module because $M/Z_2(M)$ is nonsingular.

(⟸) Assume $M = Z_2(M)\oplus N$ where N isa nonsingular dual Pure Rickart R-module. To show that M is dual Goldie Pure Rickart, let $f\in\mathrm{End}_R(M)$, $\rho_N$bethe projection map of M onto N and i be the inclusion map of N into M. Then $(\rho_N fi)\in\mathrm{End}_R(N)$, implies that $1_{Z_2(M)}\oplus(\rho_N fi)\in\mathrm{End}_R(M)$. Since $\pi^{-1}(\mathrm{Im}f + Z_2(M)) = \mathrm{Im}f + Z_2(M)$. It is easy to see that Ker f =Ker

$\rho_N f|_N$. Hence π^{-1}(Imf +Z_2(M)) = Z_2(M) $\oplus$ Ker $\rho_N f|_N$. Since N is dual Pure Rickart, implies that Ker $\rho_N f|_N$ is a pure submodule of N, but a direct sum of pure submodules is pure submodule. Therefore π^{-1} (Imf +Z_2(M))is a pure submodule of M, it follows that M is dual Goldie Pure Rickart.

Proposition 3.6. Let M bedual Goldie Pure Rickart R-module and Imf is a pure submodule of π^{-1} (Imf + Z_2(M)) for any $f \in End_R(M)$. Then M is dual Pure Rickart and $Z_2(M)$ is a pure submodule of M.

Proof. Let M be a dual Goldie Pure Rickart R-module and $f \in End_R(M)$. Thenπ^{-1}(Imf + Z_2(M)) is a pure submodule of M and by hypothesis, Imf is a pure submodule of π^{-1}(Imf +Z_2(M)).It follows that I mf is pure submodule of M, thus M is dual Pure Rickart. Also $Z_2(M)$ is a pure submodule of M, since π^{-1}(Im0 + Z_2(M)) = Z_2(M) is a pure submodule of M, where0is the zero endomorphism of M.

Theorem 3.7. Let $\{ M_i \}_{i\in\Lambda}$be a class of R-modules for an arbitrary index setΛ. Then M_i is dual Goldie Pure Rickart modules for all $i \in \Lambda$if and only if$\oplus_{i\in\Lambda} M_i$is dual Goldie Pure Rickart module.

Proof. ($\Longrightarrow$) Assume M_i is dual Goldie Pure Rickart R-module for all $i \in \Lambda$ and $M = \oplus_{i\in\Lambda} M_i$. Let I be an ideal of R and $f = (f_i)_{i\in\Lambda} \in End_R(\oplus_{i\in\Lambda} M_i)$ where $f_i \in End_R(M_i)$.To show that $MI \cap \pi^{-1}$(Imf +Z_2(M)) = π^{-1}(Imf + Z_2(M))I. Since $MI \cap \pi^{-1}$(Imf + Z_2(M)) = ($\oplus_{i\in\Lambda} M_i I$)$\cap$($\oplus_{i\in\Lambda}$(Imf_i+ $Z_2(M_i)$) = $\oplus_{i\in\Lambda}$($M_i I \cap$(Imf_i + $Z_2(M_i)$). But M_i is dual Goldie Pure Rickart R-module for all $i \in \Lambda$. So $M_i I \cap$(Imf_i + $Z_2(M_i)$) =(Imf_i + $Z_2(M_i)$)I = ,and hence $MI \cap f^{-1}$(Z_2(M)) = $\oplus_{i\in\Lambda}$(Imf_i + $Z_2(M_i)$) I = ($\oplus_{i\in\Lambda}$(Imf_i + $Z_2(M_i)$) I= π^{-1}(Imf + Z_2(M))I. That isπ^{-1}(Imf + Z_2(M))is a pure submodule in M.
($\Longleftarrow$)It follows by a similar proof.

Corollary 3.8. Every direct summand of a dual Goldie Pure Rickart module is again dual Goldie Pure Rickart.

Proof. It follows directly by Theorem 3.6.
The following theorem gives a characterization of dual Goldie Pure Rickart rings in terms of dual Goldie Pure Rickart modules.

Proposition 3.9. Let R be a ring. The following statements are equivalent.

(1) $\oplus_\Lambda R$ is dual Goldie Pure Rickart R-module for each index set Λ.
(2) Every projective R-module is dual Goldie Pure Rickartmodule.
(3) Every free R-module is dual Goldie Rickart.
(4) R is a dual Goldie Pure Rickart ring.

Proof. (1) $\Longrightarrow$ (2) Let M be a projective R-module, then there exists a free R- module F and an R-epimorphism $f : F \longrightarrow M$, and $F \cong \oplus_\Lambda R$ where Λ is an index set. We have the following short exact sequence $0 \longrightarrow$ Ker $f \xrightarrow{i} \oplus_\Lambda R \xrightarrow{f} M \longrightarrow 0$ where i is the inclusion mapping. Since M is projective, the sequence is split implies that $\oplus_\Lambda R \cong$ Ker $f \oplus M$.But $\oplus_\Lambda R$ is a dual Goldie Pure Rickart R-module. Therefore by Corollary 3.7, M is a dual Goldie Pure Rickart module.

(2) $\Longrightarrow$ (4) Assume that ever projective R-module is dual Goldie Pure Rickart. Since R is a projective R-module, then R is a dual Goldie Pure Rickart R-module.

(4) $\Longrightarrow$ (1) Assume that R is a dual Goldie Pure Rickart ring, then by Theorem 3.6,for any index set Λ, $\oplus_\Lambda R$ is a dual Goldie Pure Rickart R-module.

(1) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) by similar proof of (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (4) respectively.

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