International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

# Dual Pure Rickart Modules and Their Generalization

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**Abstract:** ***Let R be a commutative ring with identity and M be an R-module.In this paper we introducethe dual concepts of Pure Rickart modules and Pureπ-Rickart modulesas a generalization of dual Rickart modules anddual π-Rickart modules respectively. Further, dual Pure Rickart modules and dual Pure π-Rickart modules can be consideredas a generalization of regular rings and π-regular rings respectively. Furthermore, dual Pureπ-Rickart modulesis a generalization of PureRickart modules. An R-module M is calleddualPure Rickartif for every f∈End$_R$(M ),Im f is a pure(in sense of Anderson and Fuller) submodule of M. An R-module M is calleddual Pure π- Rickart if for every f∈End$_R$(M ), there exist a positive integer n such that$Im\ f^n$is a pure (in sense of Anderson and Fuller) submodule of M.We show that several results of dual Rickart modules anddual π-Rickart modules can be extended to dual Pure Rickart modules and dual Pureπ-Rickart modules for this general settings. Many results about these concepts are givenand some relationships between these modules and other related modules are investigated.***

**Keywords:** dual Pure Rickart modules,dualPureπ-Rickart modules, regular modules, Pure submodules, relatively dual Rickartmodules

## 1. Introduction

Throughout this paper $R$ denotes a commutative ring with identity. For a right $R$-module $M$, $S$ =End$_R$ ($M$) will denote the endomorphism ring of $M$; thus $M$ can be considered as a left $S$-right $R$-bimodule. Following Rizvi and Roman [6], an $R$-module $M$ is called *dual Rickart* if for every $f$∈ End$_R$ ( $M$ ), Im $f$ is a direct summand of $M$ . An $R$-module $M$ is called *dualπ-Rickart* if for every $f$∈ End$_R$( $M$ ), there exists a positive integer $n$ such that$Im\ f^n$ is a direct summand of $M$ [8].

The main purpose of this work is to introduce and study the concept dual Pure Rickart modules as a generalization ofdual Rickart modules as well as that of von Neumann regular rings . An $R$-module $M$ is calleddual Pure Rickartif for every $f$∈ End$_R$($M$ ),the Im$f$ is a pure (in sense of Anderson and Fuller) submodule of $M$. Following Anderson and Fuller, a submodule $N$ of an $R$-module $M$ is called *pure* if $N \cap MI = NI$ for every ideal $I$ of $R$[3]. An $R$-module $M$ is called *regular* if every submodule of $M$ is pure[9]. Furthermore, we introduce the concept dual π-Pure Rickart modules as a generalization ofdual Pure Rickart modules. An $R$-module $M$ is called dual Pure *π-Rickart* if for every $f$∈ End$_R$ ( $M$ ), there exists a positive integer $n$ such that$Im\ f^n$ is a pure (in sense of Anderson and Fuller) submodule of $M$.

The work consists of five sections. In Section two, we provide some examples and properties of Pure Rickart modules (Remarks and Examples 2.2).It is shown thatdual Pure Rickart rings are precisely von Neumann regular rings (Definition 2.1). We show that the dual Pure Rickart property does not always transfer from a module to each of its submodules or conversely(Examples 2.3).But (Proposition 2.5) shows that that the dual Pure Rickart property is inherited from a module to each of its direct summands. At this place,we have already observed that the direct sum ofdual Pure Rickart modules need not be dual Pure Rickart(Remark 2.6). This observation lead us to introduce the concept ofrelatively dual Pure Rickart Modules to study under what conditions the direct sum of dual Pure Rickart Modules is again dual Pure Rickart. Let $M$ and $N$ be $R$-modules. $M$ is called relatively dual Pure Rickart to $N$ if for every $f$ ∈ Hom$_R$ ($M$, $N$), Im $f$ is a pure (in sense of Anderson and Fuller) submodule of $M$. Thus, as special case $M$ is a dual Pure Rickart module if and only if $M$ is relativelydual Pure Rickart to $M$. The concept of relatively dual Pure Rickart is introduced and investigated in section three. We give many results which are useful in this study on direct sums. In section four, we introduce and study the concept of dualPure π-Rickart modules. Provided that $R$ is an integral domain, we see that the dual Pure π-Rickart rings coincide withπ-regular rings(Proposition 4.3). A ring $R$is called π-regular ring if for each $a$∈$R$, there exists b ∈$R$ and a positive integer $n$ such that.$a^n = a^n b a^n$[5].Further, it is shown that some results of dual Pure Rickart modules can be extended to dual Pure π-Rickart modules for this general settings.

## 2. Dual Pure Rickart Modules

In this section we study the concept of dual Pure Rickart modules. Basic properties of this type of modules are investigated.We start with the followingdefinition.

**Definition 2.1.** *An R-module M is called dual Pure Rickart if for every f ∈End$_R$(M ),*Im *f is a pure (in sense of Anderson and Fuller)submodule of M.If M = R, then R iscalled dual Pure Rickart ring if R is dual Pure Rickart as R-module.*

Since for every$a$∈ $R$ and $f$∈ End$_R$( $R$)≅ $R$.Then $f$ is defined by$f$($r$) = $ra$ for each $r$∈$R$. This implies that Im $f$ = < $a$> ,and hence $R$ is dual Pure Rickart if and only if every principal ideal of R is Pure ideal of $R$. Therefore when $M$ = $R$, the concept of dual Pure Rickart modules coincides with that von Neumann regularringsas well as that of dual Rickart modules.

**Remarks and Examples 2.2.**

(1) Clearly every regular module is dual Pure Rickart.But the converse is not true in general. For example, the module $\mathbb{Q}$ as $\mathbb{Z}$-module is dual Pure Rickart since every endomorphism of $\mathbb{Q}$ is either zero or an isomorphism. But $\mathbb{Q}$ is not regular.

(2) It is clear that every dual Rickart module is dual Pure Rickart but not conversely. For example, consider the ring $R=\prod_{i=1}^{\infty}\mathbb{Z}_2$ and the $R$-module $M = R^{(R)}$. It is well-known that $R$ is regular ring, and hence all $R$-modules are regular. It follows that $M$ isregular $R$-module. Therefore byRemark and Example (1), $M$ is dual Pure Rickart. On the other hand, by [6,Example 3.9], $M$ is not dual Rickart.

(3) If $M$ is Pure Rickart module then $M$ need not be a dual Pure Rickart, wherean $R$-module $M$ is called*Pure Rickart* if for every $f \in \mathrm{End}_R(M)$,Ker $f$ is a pure(in sense of Anderson and Fuller)submoduleof $M$[1]. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is a Pure Rickart since ker $f = 0$, for every$f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z})$. But $\mathbb{Z}$ is not dual Pure Rickart since for any endomorphism$f: \mathbb{Z}\longrightarrow\mathbb{Z}$ defined by $f(m) = nm$ for each $m\in\mathbb{Z}$, Im $f = n\mathbb{Z}$ is not a pure submodule in $\mathbb{Z}$, for anypositive integer $n$greater than one.

(4) If $M$ is a dual Pure Rickart $R$-module, then $M$ may not be a Pure Rickart. For example, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ where $p$ is a prime number. It is easily to see that Im$f = \mathbb{Z}_{p^\infty}$ for each $f\in\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$, then $\mathbb{Z}_{p^\infty}$ is a dual Pure Rickart module. But $\mathbb{Z}_{p^\infty}$ is not aPure Rickart $\mathbb{Z}$-module, since there exists an endomorphism$f: \mathbb{Z}_{p^\infty}\longrightarrow\mathbb{Z}_{p^\infty}$ defined by$f(n/P^m + \mathbb{Z}) = n/p^{m-1} + \mathbb{Z}$, for each $n\in\mathbb{Z}$ and $m$ is a positive integer. Then Ker $f = \langle 1/p + \mathbb{Z}\rangle$ is not apure submodulein $\mathbb{Z}_{p^\infty}$.To see this, let$G_t = \langle 1/p^t + \mathbb{Z}\rangle$ be any submodule of$\mathbb{Z}_{p^\infty}$, where t is a positive integer. Then$1/p^t + \mathbb{Z} = (1/p^{2t} + \mathbb{Z})p^t \in G_t \cap \mathbb{Z}_{p^\infty} p^t$. On the other hand $1/p^t + \mathbb{Z} \notin G_t\, p^t = 0$.

(5) If $M = R$ is a dual Pure Rickart $R$-module, then $M$ is Pure Rickart.
*Proof.* Since $M = R$ is von Neumann regular, then $R$ is a PF-ring.But the concept of Pure Rickart modules coincides with that of PF-rings. So $R$ is Pure Rickart.

(6) If $M$ is Pure simple $R$-module.Then $M$ need not be dual Pure Rickart module, wherean $R$-module $M$ is called *Pure simple* if $M\neq\{0\}$ and it has no pure submodules except $\{0\}$ and $M$ [4]. For example, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a Pure simple butbyRemark and Example(4), it is not dual Pure Rickart . Also the $\mathbb{Z}$-module $\mathbb{Z}_4$ is a Pure simple, but not dual Pure Rickart since there exists an endomorphism $f: \mathbb{Z}_4\longrightarrow\mathbb{Z}_4$ defined by $f(m) = m2$ for each $m\in\mathbb{Z}_4$, then Im $f = \{\bar{0},\bar{2}\}$ is not Pure submodule in $\mathbb{Z}_4$.

(7) If $M$ is coquasi-Dedekind $R$-module. Then $M$ is dual Pure Rickart, wherean $R$-module $M$ is called *coquasi-Dedekind* if for every $f \in \mathrm{End}_R(M)$, Im $f = M$ [10].The converse is not true.For example, the $\mathbb{Z}$-module$\mathbb{Z}_6$ is regular.Then $\mathbb{Z}_6$ is dual Pure Rickart but it is not coquasi-Dedekind.

(8) If M is a dual Pure Rickart and Pure simple $R$-module then $M$ is a coquasi-Dedekind.*Proof.*Let $M$ be dual Pure Rickart and $0 \neq f\in \mathrm{End}_R(M)$, so Im $f$is a pure submodule in $M$. But $M$ is Pure simple and$f\neq 0$.Therefore Im $f= M$, implies that $M$ is coquasi-Dedekind.

The next two examples show that the dual Pure Rickart property does not always transfer from a module to each of its submodules or conversely.

**Examples 2.3.**

(1) If $M$ is a dual Pure Rickart module and $N$ is any submodule of $M$.Then $N$ need not bedual Pure Rickart. For example, the $\mathbb{Z}$-module$\mathbb{Q}$is dual Pure Rickart.But the submodule $\mathbb{Z}$ of $\mathbb{Q}$ is not a dual Pure Rickart as $\mathbb{Z}$-module.

(2) If each proper submodule of an $R$-module $M$ is dual Pure Rickart, then $M$ need not be dual Pure Rickart. For example, the $\mathbb{Z}$-module$\mathbb{Z}_4$in which every proper submodule is regular module, and hence it is dual Pure Rickart. But $\mathbb{Z}_4$ is not dual Pure Rickart.

Now we recallknown the following lemma from [9].

**Lemma 2.4.***Let $M$ be an $R$-module and $A$, $N$ be submodules. Then we have*

(1) *If $A$ is a pure submodule in $N$, and $N$ is a pure submodule in $M$. Then $A$ is a pure submodule in $M$.*

(2) *If $A$ is a pure submodule in $M$ and $A \subseteq N$. Then $A$ is a pure submodule in $N$.*

The following proposition shows that the dual Pure Rickart property is inherited from a module to each of its direct summands.

**Proposition 2.5.** *Every direct summand of a dual Pure Rickart module is dual Pure Rickart.*

*Proof.* Let $M$ be a dual Pure Rickart $R$-module and $A$ be a direct summand of $M$, then $M = A\oplus B$ for some submodule $B$ of $M$. Let$f \in \mathrm{End}_R(A)$,then we have $M = A\oplus B \xrightarrow{\rho} A \xrightarrow{f} A \xrightarrow{i} M$, where $\rho$ is the natural projection map of $M$ onto $A$ and $i$ is the inclusion map.Say $= i f \rho$. Then $g\in \mathrm{End}_R(M)$, therefore Im $g$ is a pure submodule in $M$. But $g(M) = (i f\rho)(M) = (if)(A) = i(f(A)) = f(A)$. That is Im $g$ = Im $f$ implies that Im $f$ is pure submodule in $M$. But $A$ is containing Im $f$, therefore by lemma 2.5(2), Im $f$ is pure in $A$. Hence $A$ is a dual Pure Rickart $R$-module.

We end this section by the following observation

**Remark 2.6.** The direct sum of dual Pure Rickart modules may be not be dual Pure Rickart. For example, let P be a prime number and the $\mathbb{Z}$-module$M=\mathbb{Z}_{p^\infty}\oplus\mathbb{Z}_p$ .Consider the endomorphism $f: M\longrightarrow M$defined by$f(m/p^t + \mathbb{Z}, \bar{n}) = (n/p^t + \mathbb{Z}, \bar{0})$, where $m \in \mathbb{Z}$, $t = 0,1,2,..$ and $\bar{n} \in \mathbb{Z}_p$. Then Im$f=\mathbb{Z}_p\oplus\bar{0}$ which is not pure in $\mathbb{Z}_{p^\infty}\oplus\bar{0}$, and hence it is not pure in $M$. Therefore $M = \mathbb{Z}_{p^\infty}\oplus\mathbb{Z}_p$ is not dual Pure

Rickart,while $\mathbb{Z}_{p^{\infty}}$ and $\mathbb{Z}_{p}$ are both dual Pure Rickart modules.

## 3. Relatively Dual Pure Rickart Modules

Remark 2.6 shows that a direct sum of dual Pure Rickart modules need not be dual Pure Rickart. In this section we define relativelydual Pure Rickart property in order to investigate when are direct sums of dual Pure Rickart modules also dual Pure Rickart.

Recall an $R$-module $M$ is called *relatively dual Rickart to an R-module N* if for every homomorphism $f: M \longrightarrow N$, Im $f$ is a direct summand of $M$[6]. In view of the above definition, an $R$-module $M$ is dual Rickart if and only if $M$ is relatively dual Rickart to $M$.

**Definition 3.1.** *Let M and N be R-modules. M is called relativelydual Pure Rickart to N if for every $f \in Hom_R(M, N)$,Im f is a pure( in sense of Anderson and Fuller ) submodule of M.* Thus, as special case $M$ is dual Pure Rickart if and only if $M$ isdual relatively Pure Rickart to $M$.

**Remarks and Examples 3.2**

(1) It is clear every relatively dual Rickart module is relatively dual Pure Rickart, but the converse is not true in general. For example, the module $M= R^{(R)}$ as $R$-module, where $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$. Then by Remark and Example 2.2(2),$M$is a dual Pure Rickart, then it is relatively dual Pure Rickart. But $M$ is not dual Rickart and hence it is not relatively dual Rickart.

(2) One can easily see that when $N$ is regular $R$-module, then every $R$-module $M$ is relatively dual Pure Rickart to $N$.

(3) If $M$ is regular $R$-module then $M$ need not be relatively dual Pure Rickart to an $R$-module $N$. For example, the $\mathbb{Z}$-module$\mathbb{Z}_2$ is regular but it is not relatively dual Pure Rickart to $\mathbb{Z}_4$ as$\mathbb{Z}$-module,since there exists the homomorphism $f: \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ defined by $f(m) = m2$ for each$m \in \mathbb{Z}_2$. ThenIm$f = \{\overline{0}, \overline{2}\}$ which is not pure in $\mathbb{Z}_4$.

(4) If $M$ is a dual Pure Rickart $R$-module, then $M$ need not be relatively dual Pure Rickart to an $R$-module $N$. For example, the $\mathbb{Z}$-module $\mathbb{Z}_2$ is dual Pure Rickart. But $\mathbb{Z}_2$ is not relatively Pure Rickart to $\mathbb{Z}_4$as $\mathbb{Z}$-module.

An $R$-module $M$ is said to have the *Pure sum Property*( briefly PSP ) if the sum of any two pure submodules is again pure [7].

**Lemma 3.3.***Let M be an R-module. Then M has the PSP if and only if every pure submodule of M has the PSP* [7].

**Lemma 3.4.***Let M be an R-module with the PSP, then for every decomposition $M = A \oplus B$ and for every $f \in Hom_R(A, B)$,Im f is a pure submodule in M* [7].

**Theorem 3.5.** *Let M be an R-module with the PSP and $A \oplus B$ is a pure submodule of M. Then A is relatively dual Pure Rickart module to B.*

*Proof.* Assume that $M$ has the PSP.Then by Lemma 3.3, every pure submodule of $M$has the PSP. So $A \oplus B$ has the PSP. By Lemma 3.4, for every $f \in$ Hom $_R(A, B)$, Im $f$ is a pure submodule in $A \oplus B$. But Im $f \subseteq A$ and $A$ is a direct summand in $A \oplus B$, so $A$ is pure in $A \oplus B$. Therefore by lemma 2.4(2), Im$f$is a pure submodule in $A$. Hence $A$ is relatively dual Pure Rickart to $B$.
As an immediate consequences we have

**Corollary 3.6.** *Let M and N be R-modules. If $M \oplus N$ has the PSP, then M is relatively dual Pure Rickart to N.*

**Corollary 3.7.***Let M be R-module. If $M \oplus M$ has the PSP, then M is dual Pure Rickart module.*

**Remark 3.8.** The converse of Corollary 3.6is not true in general. For example, the $\mathbb{Z}$-module$\mathbb{Z}_2$is regular.Then any $R$-module $N$ is relatively dual Pure Rickart to $\mathbb{Z}_2$. Let $N = \mathbb{Z}_4$ as $\mathbb{Z}$-module and let $M$ denote $\mathbb{Z}_4 \oplus \mathbb{Z}_2$as $\mathbb{Z}$-module. We show that $M$ does not have the PSP. Let $A = \overline{0} \oplus \mathbb{Z}_2$, and $B = (\overline{2}, \overline{1})$ be the submodule generated by $(\overline{2}, \overline{1})$. It is clear that each $A$ and$B$ is a pure submodule of $M$. But $A+B = \{(\overline{0},\overline{0}), (\overline{0},\overline{1}), (\overline{2},\overline{1}), (\overline{2},\overline{0})\}$ is not pure in $M$, since $(\overline{1},\overline{0})2 = (\overline{2},\overline{0}) \in (A+B) \cap M2$.On the other hand, $(A+B)2 = \{\overline{0},\overline{0}\}$.

Our next results on relatively dual Pure Rickart modules will be useful in this study on direct sums.

**Theorem 3.9.***Let M and N be R-modules. The following statements are equivalent*
*(1) M is relatively dual Pure Rickart to N.*
*(2) For every direct summand A of M and any submodule B of N, A is relatively dual Pure Rickart to B.*

*Proof.*(1) $\Rightarrow$(2) Assume $M$ is relatively dual Pure Rickart to $N$. Let $A$ be a direct summand of $M$ and $B$ is any submodule in $N$. Let $f \in$ Hom$_R$ $(A, B)$. Consider the following $M = A \oplus B \xrightarrow{\rho} A \xrightarrow{f} B \xrightarrow{i} N$ for a submodule$H$ of $M$ where $\rho$ is the natural projection map of $M$ onto $A$ and $i$ is the inclusion map.Say $g = i f\rho \in$ Hom$_R(M, N)$. This implies that Im$g$ is a pure submodule in $N$. Then we have $g(M) = (i f\rho)(M) = (if)(A) = i(f(A)) = f(A)$, and hence Im $f$ is pure in $N$. But $B$ is containing Im $f$, thus by lemma 2.4(2), Im $f$ is pure in $B$. Therefore $A$ is relatively dual Pure Rickart to $B$.

(2) $\Rightarrow$(1) It is clear by taking $A = M$ and $B = N$.

**Proposition 3.10.***Let $\{M_i\}_{i \in \Lambda}$ be a family of R-modules where $\Lambda = \{1,2,\ldots,n\}$ and N be R-module. The following statements are equivalent*
*(1) If N has the PSP, then $\oplus_{i=1}^{n} M_i$ is relatively dual Pure Rickart to N.*
*(2) $M_i$ is relatively dual Pure Rickart to N for all i = 1,2,…,n*

*Proof.*(1) $\Rightarrow$(2)It follows immediately from Theorem3.9.

International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

(2) $\Rightarrow$(1) Assume that $M_i$ is relatively dual Pure Rickart to $N$for all $i$ = 1,2,…,$n$ and $N$ has the PSP. To show that $\oplus_{i=1}^{n} M_i$ is relatively dual Pure Rickart to $N$, let $f\in \mathrm{Hom}_R(\oplus_{i=1}^{n} M_i, N)$, $f=\{f_i\}_{i=1}^{n}=(f_i)_{i\in\Lambda}$ and $f|_{M_i}=f_i : M_i \longrightarrow N$ is an $R$-homomorphismfor each $i$ = 1,2,…,$n$. Thus $Imf=\sum_{i=1}^{n} f_i$. But Im$f_i$isa pure submodulein $N$ and$N$ has the PSP, therefore $Imf=\sum_{i=1}^{n} f_i$ is purein $N$, andhence $\oplus_{i=1}^{n} M_i$ is relatively dual Pure Rickart to $N$.
As an immediate corollary one can see the following

**Corollary3.11.***Let* $\{M_i\}_{i\in\Lambda}$ *be a family of R-modules where* $\Lambda=\{1,2,\ldots,n\}$. *Then the following are equivalent*
(1) *If* $\oplus_{i=1}^{n} M_i$ *has the PSP, then* $\oplus_{i=1}^{n} M_{i\ i}$ *is relatively dual Pure Rickart to* $M_j$ *for all j* = 1,2,…,$n$.
(2) $M_i$ *is relatively dual Pure Rickart to* $M_j$ *for alli* = 1,2,…,$n$.

We end this section by the following two results
**Proposition 3.12.** *Let R be a ring. The following statements are equivalent*
(1) $\oplus_{\Lambda} R$ *is dual Pure Rickart R-module for any index set* $\Lambda$.
(2) *All projective R-modules are dual Pure Rickart modules.*
(3) *All free R-modules are dual Pure Rickart modules.*
*Proof.* (1) $\Rightarrow$ (2) Let $M$ be a projective $R$-module then there exists a free$R$-module $F$ and an epimorphism $f$ : $F \longrightarrow M$. Since $F\cong \oplus_{\Lambda} R$ for some index set $\Lambda$. We have the following short exact sequence $0 \longrightarrow \ker f \xrightarrow{i} \oplus_{\Lambda} R \xrightarrow{f} M \longrightarrow 0$ .But$M$ is projective then the sequence splits. Thus $\oplus_{\Lambda} R \cong \ker f \oplus M$. Since $\oplus_{\Lambda} R$ is dual Pure Rickart module.Therefore by Proposition 2.5, $M$ is dual Pure Rickart module.
(2) $\Rightarrow$ (1) It is clear and (1) $\Leftrightarrow$ (3) Similar proof of (2) $\Leftrightarrow$ (1).

**Proposition 3.13.** *Let R be a ring. The following statements are equivalent*
(1) R is regular.
(2) All R-modules are regular.
(3) All R-modules are relatively dual Pure Rickart to any R-module.
(4) All R-modules are dual Pure Rickart.
(5) All R-modules have the PSP.
(6) All projective R-modules have the PSP.
(7) All free R-modules have the PSP.

*Proof.*(1) $\Leftrightarrow$ (2)$\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) It follows by [7, Theorem 3.1].
(1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) It is clear.
(3) $\Rightarrow$ (1) Let $I$ be an ideal of $R$. Thus $I$ is relatively dual Pure Rickart as $R$-module to the $R$-module $R$. Hence for every $f\in \mathrm{Hom}_R$ $(I, R)$, Im $f$ is a pure ideal in $R$. Since the inclusion map $i\in \mathrm{Hom}_R$ $(I, R)$. Therefore Im $i = I$ is a pure ideal in $R$. Hence $R$ is regular.

(4) $\Rightarrow$ (1) Assume that all $R$-modules are dual Pure Rickart, then $R$ is dual Pure Rickart as $R$-module and hence $R$ is regular from the fact that the concept of dual Pure Rickart and regularity are equivalent in the ring by Definition 3.2.

## 4. Dual Pure $\pi$-Rickart Modules

In this section we introduce the concept ofdual Pure $\pi$-Rickart modules. Some basic properties of this class of modules are investigated. When $R$ is an integral domain, we see that the dual Pure $\pi$-Rickart rings coincide with$\pi$-regular rings. First, we give the following definition.

**Definition 4.1.***An R-module M is called dual Pure π-Rickart if for every* $f\in \mathrm{End}_R(M)$*there exists a positive integer n such that* $\mathrm{Im} f^n$ *is a pure ( in sense of Anderson and Fuller )submodule of M. If M = R, then R iscalled dual Pure π-Rickart ring if R is dual Pure π-Rickart as R-module.*

**Remarks and Examples 4.2.**

(1) Clearly every regular module is dual Pure $\pi$-Rickart.But the converse is not true in general. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ is dual Pure $\pi$-Rickart since every endomorphism of $\mathbb{Q}$ is either zero or an isomorphism. But $\mathbb{Q}$ is not regular. Furthermore, one can easily see that foreach positive integer $n$, the $\mathbb{Z}$-module $\mathbb{Z}_n$ isdual Pure $\pi$-Rickart.But $\mathbb{Z}_n$is not regular module for some positive integer $n$.
(2) It is clear that every dual $\pi$-Rickart module is dual Pure $\pi$-Rickart.However the converse true in the semisimple modules, wherean $R$-module $M$ is called *semisimple* if every submodule of M is a direct summand [3].
(3) If $M$ is Pure Rickart or Pure $\pi$-Rickart$R$-module then $M$ need not be dual Pure $\pi$-Rickart. For example, the $\mathbb{Z}$-module$\mathbb{Z}$ is Pure Rickart and Pure $\pi$-Rickart since for every$f\in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z})$, $\ker f = 0$. But $\mathbb{Z}$ is not dual Pure $\pi$-Rickart since for each positive integer $k$ greater than oneand any endomorphism$f : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(m) = km$where $m\in\mathbb{Z}$, Im $f = k\mathbb{Z}$. Then $\mathrm{Im} f^n = k^n\mathbb{Z}$is notpure in $\mathbb{Z}$, for anypositive integer $n$.
(4) Let $M$ dual Pure $\pi$-Rickart $R$-module, and $N$ be a submodule of $M$. Then N may not be dual Pure $\pi$-Rickart $R$-module. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ is dual Pure $\pi$-Rickart. But the submodule $\mathbb{Z}$ is notdual Pure $\pi$-Rickart $\mathbb{Z}$ -module.

Recall that an ideal $I$ of a ring $R$ is called a *pure ideal* of $R$ if for every $x\in I$, there exists $y\in I$ such that $xy=x$[4].

**Proposition 4.3.***Let M = R be an R-module. If R is dual π-Rickart then R is dual Pure π-Rickart.The convers is true if R is an integral domain.*

*Proof.* First statement is clear. Assume that $R$ is an integral domain. Let $R$ be a dual Pure $\pi$-Rickart, thenfor every $f\in \mathrm{End}_R(R)$, there exists a positiveinteger $n$ such that $\mathrm{Im} f^n$ isapure ideal of $R$. Since for every $0\neq a\in R$ and $f\in \mathrm{End}_R(R)\cong R$, then $f$is defined by $f(r) = ra$for each $r\in R$. This implies that Im $f = \langle a \rangle$ , and hence $\mathrm{Im} f^n = \langle a^n \rangle$ which is a pure ideal of $R$. Then for each $0\neq x \in <a^n>$, there exists $0\neq y\in <a^n>$ such that $xy = x$. Let $x = ra^n$ and $y = ba^n$, implies $ra^n ba^n = ba^n$. So $r(a^n ba^n - a^n) = 0$.But $r\neq 0$, implies that $a^n ba^n - a^n = 0$. Then $a^n ba^n = a^n$. Since $e = ba^n$ is an idempotent , it follows that $<a^n> = <e>$. Therefore $\mathrm{Im} f^n$ is a direct summand of $R$, and hence $R$ dual $\pi$-Rickart, in other words, $R$ is a $\pi$-regular rings.

**Proposition 4.4.** *Let M=R be an R-module. IfR is dual Pure π-Rickart,then R is Pure π-Rickart.*

*Proof.* Let $M = R$ bedual Pure $\pi$-Rickart $R$-module, then for every $0\neq a\in R$, $<a^n>$ is a pure ideal of $R$ for some positiveinteger $n$. Since every pure ideal is a flat ideal [9, Corollary 3.4], then $<a^n>$ is flat. But $R/ann_R(a^n)$ isomorphic to $<a^n>$, it follows that $ann_R(a^n)$ is a pure ideal of $R$. Hence $R$ is GPF-ring, that is a ring with the property that every $a\in R$,there exists a positive integer $n$ such that $ann_R(a^n)$ *isa pure ideal of R* [2], in other words, $R$ is Pure $\pi$-Rickart. The following proposition shows that the dual Pure $\pi$-Rickart property is inherited from a module to each of its direct summands.

**Proposition 4.5.** *Every direct summand of dual Pure π-Rickart module is dual Pure Rickart.*
*Proof.* By similar proof of Proposition 2.5.

**Proposition 4.6.** *Let R be a ring. The following statements are equivalent*

(1) $\oplus_\Lambda R$*is dual Pureπ-RickartR-module for any index setΛ.*
(2) *All projective R-modules are dual Pure π-Rickart modules.*
(3) *All free R-modules are dual Pure π-Rickart modules.*

*Proof.* By similar argument of Proposition 3.12.

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