2. Some new Bi- Measures of fuzzy entropy

Consider the measure

$$F_{a,b,k}(A) = -\sum_{i=1}^{n}\left(\mu_A(x_i)\ln\mu_A(x_i) + (1-\mu_A(x_i))\ln(1-\mu_A(x_i))\right) +$$
$$\frac{b}{a^k}\sum_{i=1}^{n}\left((1+a\mu_A(x_i))\ln(1+a\mu_A(x_i)) + (1+a(1-\mu_A(x_i)))\ln(1+a(1-\mu_A(x_i)))\right)$$
$$-\,2\frac{b}{a^k}(1+a)\ln(1+a)$$

where a, b, k are parameters.

Here $F_{a,b,k}(A)$ is defined for all $\mu_A(x_i)$, in the range $0 \le \mu_A(x_i) \le 1$, $i = 1, 2, \ldots\ldots, n$ and it should be continuous in this range.

$F_{a,b,k}(A) = 0$, *when* $\mu_A(x_i) = 0$ *or* $\mu_A(x_i) = 1$.

$F_{a,b,k}(\mu_A(x_i)) = F_{a,b,k}(1-\mu_A(x_i))$.

Now $F_{a,b,k}(A)$ can be written as $\sum_{i=1}^{n} f(x_i)$, where

$$f(x) = -x\ln x + (1-x)\ln(1-x) + \frac{b}{a^k}\left((1+ax)\ln(1+ax) + (1+a(1-x))\ln(1+a(1-x))\right)$$
$$-2\frac{b}{a^k}(1+a)\ln(1+a)$$

Then

$$f'(x) = -\ln x + \ln(1-x) + \frac{b}{a^k}\left(a\ln(1+ax) - a\ln(1+a(1-x))\right) \ \textit{and}$$

$$f''(x) = -\frac{1+(a-c)x}{x(1+ax)} - \frac{1+(a-c)(1-x)}{(1-x)(1-a(1-x))}\ ;\ c = \frac{b}{a^k}.$$

Hence $F_{a,b,k}(A)$ will be concave, if $1+(a-c)x \ge 0$ *and* $1+(a-c)(1-x) \ge 0$

But it is true when

$1 \ge c$ *or* $a \ge c$ *i.e. when* $a^{k-2} \ge b$ *or* $a^{k-1} \ge b$. ....(1)

$(k-2)\ln a \ge 0$ *or* $(k-1)\ln a \ge 0$. ....(2)

But $(k-2)\ln a \ge 0$, *if* $a \ge 1$ *or* $k \ge 2$ *or* $0 < a < 1$, $k < 2$;

$(k-1)\ln a \ge 0$, *if* $a \ge 1$ *or* $k \ge 1$ *or* $0 < a < 1$, $k < 1$.

3. Special Cases:

1) When $a^{k-2} \ge b$ or $a^{k-1} \ge b$, $F_{a,b,k}(A)$ represent a three parametric bi- measure of fuzzy entropy.
   When $b = 1, k = 1$,
   $F_{a,b,k}(A) = F_4(A)$.
2) When $b = 1, k = 2$,
   $F_{a,b,k}(A) = F_5(A)$.
3)
4) When $0 < b \le 1$, then (1) will be hold if
5) When $b > 1$, we can find suitable value of a, k using (2). Thus in any case, the family of bi-measure of fuzzy entropy is much larger than $F_4(A)$ *and* $F_5(A)$.

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