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Fibrewise Bitopological Spaces

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Abstract: ***We introduce and discus recent type of fibrewise topological spaces, namely fibrewise bitopological spaces, Also, we introduce the concepts of fibrewise closed bitopological spaces, fibrewise open bitopological spaces, fibrewise locally sliceable bitopological spaces and fibrewise locally sectionable bitopological spaces. Furthermore, we state and prove several propositions concerning with these concepts.***

Keywords: Fibrewise bitopological spaces, fibrewise closed bitopological spaces, fibrewise open bitopological spaces, fibrewise locally sliceable bitopological spaces and fibrewise locally sectionable bitopological spaces.

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1. Introduction

In order to began the category in the classification of fibrewise (briefly F.W.) sets over a given set, named the base set, which say B. A F.W. set over B consest of a set M with a function $p: M \to B$, that is named the projection. The fibre over b for every point b of B is the subset $M_b = p^{-1}(b)$ of M. Perhaps, fibre will be empty since we do not require p is surjectve, also, for every subset B^* of B we considered $M_{B^*} = p^{-1}(B^*)$ as a F.W. set over B^* with the projection determined by p. The alternative notation $M \mid B^*$ is some time convenient. We considered the Cartesian product $B \times T$, for every set T, like a F.W. set B by the first projection.

Definition 1.1. [4] If M and N are F.W. sets over B, with projections p_M and p_N, respectively, a function $\varphi: M \to N$ is said to be F.W. function if $p_N \circ \varphi = p_M$, or $\varphi(M_b) \subset N_b$ for every point b of B.

Observe that a F.W. function $\varphi: M \to N$ over B limited by restriction, a F.W. function $\varphi_{B^*}: M_{B^*} \to N_{B^*}$ over B^* for every subset B^* of B.

Definition 1.2. [4] Let (B, Λ) be a topological space. The F.W. topology on a F.W. set M over B mean any topology on M for which the projection p is continuous.

Definition 1.3. [4] The F.W. function $\varphi: M \to N$, where M and N are F.W. topological spaces over B is named:

(a) Continuous if for every point $m \in M_B$; $b \in B$, the inverse image of every open set of $\varphi(m)$ is an open set of m.

(b) Open if for every point $m \in M_B$; $b \in B$, the direct image of every open set of m is an open set of $\varphi(m)$.

Definition 1.4. [4] The F.W. topological space (M, τ) over (B, Λ) is named F.W. closed, (resp. F.W. open) if the projection p is closed (resp. open).

The bitopological space study was first created by Kelly [5] in 1963 and after that a large number of researches have been completed to generalize the topological ideas to bitopological setting. In this research (M, τ_1, τ_2) and (N, σ_1, σ_2) (or briefly, M and N) always mean bitopological spaces on which no separation axioms are supposed unless clearly stated. By τ_i-open (resp., τ_i-closed), we shall mean the open (resp., closed) set with respect to τ_i in M, where $i = 1,2$. A is open (resp., closed) in M if it is both τ_1-open (resp., τ_1-closed), τ_2-open (resp., τ_2-closed). As well as, we built on some of the result in [1, 6, 7, 8]. For other notations or notions which are not mentioned here we go behind closely I.M.James [4], R.Engelking [3] and N. Bourbaki [2].

Definition 1.5. [5] A triple (M, τ_1, τ_2) where M is a non-empty set and τ_1 and τ_2 are topologies on M is named bitopological space.

Definition 1.6. [5] A function $\varphi: (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2)$ is said to be τ_i-continuous (resp. τ_i-open, τ_i-closed), if the function $\varphi: (M, \tau_i) \to (N, \sigma_i)$ is continuous (resp. open, closed). φ is named continuous (resp. open, closed) if it is τ_i-continuous (resp. τ_i-open, τ_i-closed) for every $i = 1,2$.

2. Fibrewise bitopological spaces

We will introduce the ideas of F.W. bitopological spaces, several topological properties on the obtained F.W. concepts are studied.

Definition 2.1. Let $(B, \Lambda_1, \Lambda_2)$ be a bitopological space. The F.W. bitopology on a F.W. set M over B mean any bitopology on M for which the projection p is continuous.

For example, we consider $(B, \Lambda_1, \Lambda_2)$ like a F.W. bitopological spaces over itself by the identity as projection. Also, the bitopological product $B \times T$, for every bitopological spaces T, can be regarded like a F.W. bitopological spaces over B, by the first projection and in the same way for every subspace of $B \times T$.

Remark 2.2.

(a) In F.W. bitopology we work over bitopological base space $(B, \Lambda_1, \Lambda_2)$. If B is a point–space the theory changes to that of ordinary bitopology.

(b) A F.W. bitopological spaces over B is just a bitopological space (M, τ_1, τ_2) with a continuous projection $p : (M, \tau_1, \tau_2) \rightarrow (B, \Lambda_1, \Lambda_2)$.

(c) The coarsest such bitopology got by p, in which the τ_i −open set of (M, τ_1, τ_2) are the exactly the inverse image of the Λ_i −open set of $(B, \Lambda_1, \Lambda_2)$; named, the F.W. indiscrete bitopology, where *i =1, 2*.

(d) The F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$ is defined to be a F.W. set over B with F.W. bitopology.

(e) We consider the bitopological product $B \times T$, for every bitopological space T, like a F.W. bitopological spaces over B by the first projection.

Definition 2.3. The F.W. function $\varphi : M \rightarrow N$ where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$ are said to be:

(a) i −continuous if for every point $m \in M_b$; $b \in B$, the inverse image of every σ_i −open set of $\varphi(m)$ is τ_i −open set of m. φ is named continuous if it is i −continuous for every $i = 1,2$.

(b) i −open if for every point $m \in M_b$; $b \in B$, the direct image of every τ_i −open set of m is σ_i −open set of $\varphi(m)$. φ is named open if it is i −open for every $i = 1,2$.

(c) i −clsed if for every point $m \in M_b$; $b \in B$, the direct image of every τ_i −closed set of m is σ_i −closed set of $\varphi(m)$. φ is named closed if it is i −closed for every $i = 1,2$.

If $\varphi : M \rightarrow N$ is a F.W. function where M is a F.W. set and (N, σ_1, σ_2) is a F.W. bitopoligical space over $(B, \Lambda_1, \Lambda_2)$. We can give M the induced bitopology, in the ordinary sense and this is necessarily a F.W. bitopology. We may refer to it, therefore, like the induced F.W. bitopology and note the next characterizations.

Proposition 2.4. Let $\varphi : M \rightarrow N$ be a F.W. function, where (N, σ_1, σ_2) a F.W. bitopoligical space over $(B, \Lambda_1, \Lambda_2)$ and M has the induced F.W. bitopology. Then for every F.W. bitopoligical space (Q, δ_1, δ_2) a F.W. function $\psi: Q \rightarrow (M, \tau_1, \tau_2)$ is continuous iff the composition $\varphi o \psi: Q \rightarrow N$ is continuous.

Proof. $(\Rightarrow)$ Suppose that ψ is continuous. Let $q \in Q_b$; $b \in B$ and let V be σ_i-open set of $(\varphi \, o \, \psi)(q) = n \in N_b$ in N. Since φ is continuous then, $\varphi^{-1}(V)$ is τ_i −open set containing $\psi(q) = m \in M_b$ in M. Since ψ is continuous, then $\psi^{-1}(\varphi^{-1}(V))$ is a δ_i −open set containing $q \in Q_b$ in Q and $\psi^{-1}(\varphi^{-1}(V)) = (\varphi \, o \, \psi)^{-1}(V)$ is a δ_i −open set containing $q \in Q_b$ in Q, where $i = 1,2$.

$(\Leftarrow)$ Suppose that $\varphi \, o \, \psi$ is continuous. Let $q \in Q_b$; $b \in B$ and U be a τ_i −open set of $\psi(q) = m \in M_b$ in M. Since φ is open then, $\varphi(U)$ is a σ_i −open set containing $\varphi(m) = \varphi(\psi(q)) = n \in N_b$ in N. Since $\varphi o \psi$ is continuous, then $(\varphi \, o \, \psi)^{-1}(\varphi(U)) = \psi^{-1}(U)$ is a δ_i −open set containing $q \in Q_b$ in Q, where $i = 1,2$.

Proposition 2.5. Let $\varphi : M \rightarrow N$ be a F.W. function where, (N, σ_1, σ_2) a F.W. bitopoligical space over $(B, \Lambda_1, \Lambda_2)$ and M has the induced F.W. bitopology. Then for every F.W. bitopoligical space (Q, δ_1, δ_2) surjectve F.W. function $\psi: Q \rightarrow (M, \tau_1, \tau_2)$ is open iff the composition $\varphi o \psi: Q \rightarrow N$ is open.

Proof. $(\Rightarrow)$ Suppose that ψ is open. Let $q \in Q_b$; $b \in B$ and let U be a δ_i-open set of q in Q. Since ψ is open $\psi(U)$ is τ_i open set containing $\psi(q) = m \in M_b$ in M $where \ i = 1,2$. Since φ is open $\varphi(\psi(U))$ is σ_i-open set containing $\varphi(m) = n \in N_b$ in N. And $\varphi(\psi(U)) = \varphi o \psi(U)$. $where \ i = 1,2$.

$(\Leftarrow)$ Suppose, $\varphi \, o \, \psi$ is open. Let $q \in Q_b$; $b \in B$. Let U be a δ_i − open set of q in Q, since $\varphi o \psi$ is open. $\varphi \, o \psi(U)$ is σ_i-open set containing $\varphi \, o \, \psi(q) = n \in N_b$. Since φ is continuous $\varphi^{-1}(\varphi \, o \, \psi(U))$ is τ_i-open set of $\psi(q) = m \in M_b$ in M. But $\varphi^{-1}(\varphi \, o \, \psi(U)) = \psi(U)$, where $i = 1,2$.

Let us pass of general cases of propositions (2.4) and (2.5) as follows:

Similarly in case of families $\{\varphi_r\}$ of F.W. functions, where $\varphi_r: M \rightarrow N_r$ with $(N_r, \sigma_{r1}, \sigma_{r2})$ F.W. bitopological space over B for every r. Specially, given a family $\{(M_r, \tau_{r1}, \tau_{r2})\}$ of F.W. bitopological space over B, the F.W. bitopological product $\prod_B M_r$ is defined to be the F.W. product with the F.W. bitopology generated by the family of projections $\pi_r: \prod_B M_r \rightarrow M_r$. Then for every F.W. bitopological space (Q, δ_1, δ_2) over B a F.W. function $\theta: Q \rightarrow \prod_B M_r$ is continuous (resp. open). For example when $M_r = M$ for every index r we see that the diagonal $\Delta: M \rightarrow \prod_B M$ is continuous (resp. open) iff the composition $\pi_r o \Delta = id_M$ is continuous (resp. open).

Again if $\{(M_r, \tau_{r1}, \tau_{r2})\}$ is a family of F.W. bitopological spaces over B and $\psi: \coprod_B M_r \rightarrow M$ is a F.W. function where (M, τ_1, τ_2) a F.W. bitopology over B and $\coprod_B M_r$ is F.W. bitopological coproduct at the set-theoretic level with the ordinary coproduct bitopology, also for every F.W. bitopology $(M_r, \tau_{r1}, \tau_{r2})$ with the family of F.W. insertions $\sigma_r: M_r \rightarrow \coprod_B M_r$ is continuous (resp. open) iff the composition $\psi_r = \psi \, o \, \sigma_r: M_r \rightarrow M$ is continuous (resp. open). For example when $M_r = M$ for every index r we see that the codiagonal $\Delta : \coprod_B M_r \rightarrow M$ is continuous (resp. open).

3. Fibrewise Closed and Fibrewise Open Bitopological Spaces

We present the ideas of F.W. closed and F.W. open bitopological spaces over B, several topological property on the obtained concepts are studies.

Definition 3.1. The F.W. bitopoligical space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named F.W. closed if the projection p is closed.

For example, trivial F.W. bitopological space with compact fibre is F.W. closed.

Proposition 3.2. Let $\varphi : M \to N$ be closed F.W. function where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then M is F.W. closed if N is F.W. closed.

Proof. Assume that $\varphi : M \to N$ is closed F.W. function and N is F.W. closed i.e. the projection $p_N : N \to B$ is closed. To prove that M is F.W. closed i.e. $p_M : M \to B$ is closed. Now, let $m \in M_b$; $b \in B$, let F be τ_i –closed set of m where $i = 1, 2$. Since φ is closed so that $\varphi(F)$ is σ_i – closed set of $\varphi(m) = n \in N_b$ in N. Since p_N is closed so $p_N(\varphi(F))$ is Λ_i – closed set in B. But, $p_N \, o \, \varphi(F) = p_M(F)$ is σ_i –closed set of F. Thus, p_M is closed and M is F.W. closed where i = 1, 2.

Proposition 3.3. If (M, τ_1, τ_2) is a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that M_j is F.W. closed for every member M_j of a finite covering of M. Then M is F.W. closed.

Proof. Assume that M is a F.W. bitopological space over B, then the projection $p_M : M \to B$ exist. To prove that p is closed. Since M_j is F.W. closed, then the projection $p_{M_j} : M_j \to B$ is closed for every member M_j of a finite covering of M. Let F be τ_i-closed subset of M . Then $p(F) = \cup \, p_j(M_j \cap F)$ which is finite union of closed sets and so p is closed, so that M is F.W. closed where $i = 1, 2$.

Proposition 3.4. Let (M, τ_1, τ_2) be a F.W. bitopoligical space over $(B, \Lambda_1, \Lambda_2)$. Then (M, τ_1, τ_2) is a F.W. closed iff for every fibre M_b of M and every τ_i – open set U of M_b in M, there is a Λ_i –open set O of b where $M_O \subset U$, $i = 1, 2$.

Proof. (⇒) Assume that M is closed i.e. $p : M \to B$ is closed. Now, let $b \in B$ and U be τ_i –open set of M_b where $i = 1, 2$ so we have $M - U$ is τ_i –closed set and $p(M - U)$ is Λ_i –closed set. Let $O = B - p(M - U)$ is Λ_i –open set of b. Hence, $M_O = p^{-1}(B - p(M - U))$ wich is a subset of U. Thus $M_O \subset U$, where $i = 1,2$.

(⇐) Suppose that the assumption is hold, to show that M is closed. Let F be τ_i –closed set in M where i=1, 2. Let $b \in B - p(F)$ and every τ_i_open set U of M_b in M. By assumption there is Λ_i –open set O of b such that $M_O \subset U$. It's easy to show that $O \subset B - p(F)$. So that $B - p(F)$ is Λ_i –open set in B. Hence $p(F)$ is a Λ_i –closed in B, p is closed and M is F.W. closed bitopological, where i=1, 2.

Definition 3.5. The F.W. bitopoligical space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named F.W. open if the projection p is open.

For example, trivial F.W. bitopological spaces are always F.W. open.

Proposition 3.6. Let $\varphi : M \longrightarrow N$ be an open F.W. function where (M, τ_1, τ_2), (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If N is F.W. open, then M is F.W. open.

Proof. Since N is F.W. open, we have $p_N : N \to B$ is open. To prove that p_M is open, i.e. $p_M : M \to B$ is open. Let $m \in M_b$; $b \in B$, and let U be τ_i – open set of m where $i = 1, 2$, $\varphi(U)$ is σ_i – open set of $\varphi(m) = n \in N_b$ in N, since φ is open. Also, since N is F.W. open then the projection p_N (φ(U)) is Λ_i – open set in B, but $p_N \, o \, \varphi(U) = p_M(U)$. So that p_M is open and M is F.W. open, where $i = 1,2$.

Proposition 3.7. Let $\varphi : M \longrightarrow N$ be a F.W. function where (M, τ_1, τ_2), (N, σ_1, σ_2) are F.W. bitopological spaces over$(B, \Lambda_1, \Lambda_2)$. Assume that the product: $id_M \times \varphi : (M \times_B M, \tau_1 \times \tau_1, \tau_2 \times \tau_2) \longrightarrow (M \times_B N, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is open and M is F.W. open . Then φ itself open.

Proof. Consider the following figure:

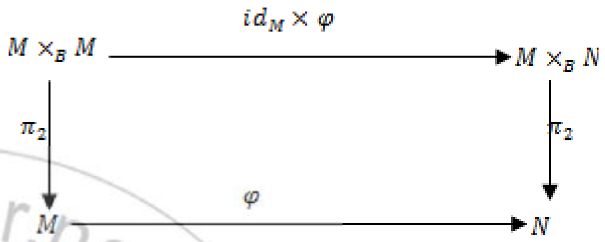

Figure 3.1: Diag. of Prop. 3.7.

The projection on the right is surjective. While The projection on the left is open since M is F.W. open bitopological space. Thus $\pi_2 o id_M \times \varphi = \varphi o \pi_2$ is open and so φ is open.

Our next three results apply equally to F.W. closed and F.W. open bitopological spaces.

Proposition 3.8. Let $\varphi : M \longrightarrow N$ be a surjection F.W. continuous where (M, τ_1, τ_2), (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then N is F.W. closed (resp. open) if M is F.W. closed (resp. open).

Proof. Suppose that M is F.W. closed (resp. open). Then $p_M : M \longrightarrow B$ is closed (resp. open). To prove that N is F.W. closed (resp. open) bitopological space over B i.e. the projection p_N : $(N, \sigma_1, \sigma_2) \to (B, \Lambda_1, \Lambda_2)$ is closed (resp. open). Suppose that $n \in N_b$; $b \in B$. Let V be σ_i – closed (resp. open) set of n where $i = 1, 2$. Since φ is continuous so $\varphi^{-1}(V)$ is τ_i –closed (resp. τ_i-open) set of $\varphi^{-1}(n) = m \in M_b$ in M where $i = 1, 2$. Since p_M is closed (resp. open) then, the projection $p_M(\varphi^{-1}(V))$ is closed (resp. open) set in B, but $p_M(\varphi^{-1}(V)) = p_N(V)$. Thus p_N is closed (resp. open), and N is F.W. closed (resp. open).

Proposition 3.9. If (M, τ_1, τ_2) is a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that M is F.W. closed (resp. open) over B. Then M_{B^*} is a F.W. closed (resp. open) over B^* for every subspace B^* of B.

Proof. Assume that M is F.W. closed (resp. open), so that the projection $p : M \to B$ is closed (resp. open). To prove that M_{B^*} is closed (resp. open). i.e. the projection $p_{B^*} : M_{B^*} \longrightarrow B^*$ is closed (resp. open). Let $m \in M \mid B^*$, G be τ_i_ closed (resp. τ_i-open) set of m where $i = 1, 2$, $G \cap M_{B^*}$ is τ_{iB^*} – closed (resp. τ_{iB^*} – open) set of M_{B^*}.

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$p_{B^*}(G \cap M_{B^*}) = p(G \cap M_{B^*}) = p(G) \cap p(M_{B^*}) = p(G) \cap B^*$ which is Λ_{iB^*} −closed (resp. Λ_{iB^*}-open) set in B^*. p_{B^*} is closed (resp. open). So that M_{B^*} is F.W. closed (resp. open), where $i = 1,2$.

Proposition 3.10. Let (M, τ_1, τ_2) be a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that $(M_{B_j}, \tau_{1B_j}, \tau_{2B_j})$ is a F.W. closed (resp. open) bitopoligical spaces over $(B_j, \Lambda_{1B_j}, \Lambda_{2B_j})$ for every member of a Λ_{iB_j}-open covering of B. So M is F.W. closed (resp. open) bitopoligical space over B , where $i = 1, 2$.

Proof. Assume that M is F.W. bitopological space over B so, the projection $p: M \to B$ is exist .To prove that p is closed (resp. open). Since M_{Bj} is closed (resp. open) over B_j for every member Λ_i −open covered of B where $i = 1, 2$, then the projection $p_{Bj} : M_{Bj} \to B_j$ is closed (resp. open). Now, let F be τ_i-closed (resp. τ_i-open) set of M_b ; $b \epsilon B$, $p(F) = \cup P_{Bj}(F \cap M_{Bj})$ wich is finite union of Λ_i −closed (resp. open) sets of B. Thus p is closed (resp. open) and M is closed (resp. open) F.W. bitopological space over B, where $i = 1,2$.

Actually, the past proposition is true to locally finite closed covering by using theorem (1.1.11) and corollary (1.1.12) in [3].

There are several subclasses of the class of F.W. open bitopological spaces which induced many important examples and have interesting properties.

4. Fibrewise Locally Sliceable and Fibrewise Locally Sectionable Bitopological Spaces

We present the ideas of F.W. locally sliceable and F.W. locally sectionable bitopological spaces over $(B, \Lambda_1, \Lambda_2)$, several topological properties on the obtained concepts are studied.

Definition 4.1. The F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named locally sliceable if for every point $m \in M_b$; $b \in B$, there exist a Λ_i −open set W of b and a section $s: W \to M_W$ and $s(b) = m$, where $i = 1, 2$.

The condition lead to p is open for if U is a τ_i −open set of m in M, then $s^{-1}(M_W \cap U) \subset p(U)$ is a Λ_i −open set of b in W and hence in B, where $i = 1, 2$. The class of locally sliceable bitopological space is finitely multiplicative stated in.

Proposition 4.2. Let $\{ (M_r, \tau_{r1}, \tau_{r2}) \}_{r=1}^{k}$ be a finite family of locally sliceable bitopological space over $(B, \Lambda_1, \Lambda_2)$. The F.W. bitopological product $M = \prod_B M_r$ is locally sliceable.

Proof. Let $m = (m_r)$ be a point of M_b ; $b \in B$, so that $m_r = \pi_r(m)$ for every index r. Since M_r is locally sliceable bitopological space there is a Λ_i −open set W_r of b and a section $s_r: W_r \to M_r \mid W_r$ where $s_r(b) = m_r$. Then the intersection $W = W_1 \cap \ldots \cap W_n$ is a Λ_i −open set of b and a section $s: W \to M_W$ is given by $(\pi_r \circ s)(w) = s_r(w)$ for every index r and every point $w \in W$, where $i = 1, 2$

Proposition 4.3. Let $\varphi: M \to N$ be continuous F.W. surjection, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M is locally sliceable, then N is so.

Proof. Let $n \in N_b$; $b \in B$. Then $n = \varphi(m)$, for some $m \in M_b$. If M is locally sliceable then, there is a Λ_i −open set W of b and a section $s: W \to M_W$ where $s(b) = m$. Then $\varphi o s: W \to N_W$ is a section such that $s(b) = n$, where $i = 1, 2$ as required.

Definition 4.4. The F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named F.W. discrete if the projection p is a local homeomorphism.

This means, we recall, that for every point b of B and every point m of M_b there is a τ_i − open set V of m in M and a Λ_i −open set W of b in B where p maps V homeomorphically onto W, in that case we say that W is evenly covered by V, where $i = 1, 2$. It is clear that F.W. discrete bitopological spaces are locally sliceable there for F.W. open.

The class of F.W. discrete bitopological spaces are finitely multiplicative.

Proposition 4.5. Let $\{(M_r, \tau_{r1}, \tau_{r2})\}_{r=1}^{k}$ be a finite family of F.W. discrete bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then the F.W. bitopological product $(M = \prod_B M_r, \tau_1, \tau_2)$ is F.W. discrete.

Proof. Given a point $m \in M_b$; $b \in B$, there is for every index r a τ_i − open set U_r of $\pi_r(m)$ in M_r, where the projection $p_r = p \circ \pi_r^{-1}$ maps U_r homeomorphically onto the Λ_i −open $p_r(U_r) = W_r$ of b. Then the τ_i − open $\prod_B U_r$ of m is mapped homeomorphically onto the intersection $W = \cap W_r$ which is a Λ_i −open of b, where $i = 1, 2$.

An attractive characterization of F.W. discrete bitopological spaces are given by the following:

Proposition 4.6. If (M, τ_1, τ_2) is F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then, M is F.W. discrete iff:
(a) M is F.W. open
(b) The diagonal embedding $\Delta: M \to M \times_B M$ is open

Proof. (⇐) Suppose that (a) and (b) are satisfied. Let $m \in M_b$; $b \in B$, then $\Delta(m) = (m, m)$ admits a $\tau_i \times \tau_i$ −open set in $M \times_B M$ which is entirely contained in $\Delta(M)$. Without real lacking in general we may suppose the $\tau_i \times \tau_i$ −open set is of the form $U \times_B U$, where U is a τ_i − open set of m in M. Then $p|U$ is a homeomorphism. Therefore, M is F.W. discrete where $i = 1, 2$.

(⇒) Assume that M is F.W. discrete. We have already seen that M is F.W. open. To prove that Δ is open it is sufficient to prove that $\Delta(M)$ is $\tau_i \times \tau_i$-open in $M \times_B M$. So let $m \in M_b$; $b \in B$, and let U be a τ_i − open set of m in M, where

$W = p(U)$ is a Λ_i −open set of b in B and p maps U homeomorphically onto W. Then $U \times_B U$ is contained in $\Delta(M)$ since if not then there exist distinct $\xi, \xi^* \in M_W$, where $w \in W$ and $\xi, \xi^* \in U$, which is absurd.

Open subset of F.W. discrete bitopological spaces are also F.W. discrete, Actually, we have.

Proposition 4.7. Assume that $\varphi : M \to N$ is a continuous F.W. injection, where (M, τ_1, τ_2), (N, σ_1, σ_2) are F.W. open bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If N is F.W. discrete then M is so.

Proof. Consider the diagram shown below.

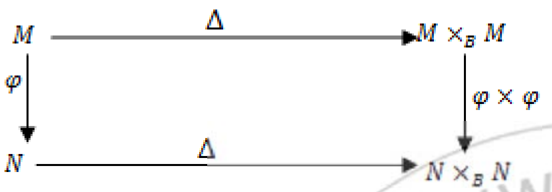

Figure 4.1: Diag. of Prop. 4.7

Since φ is continuous so is $\varphi \times \varphi$. Now $\Delta(N)$ is $\sigma_i \times \sigma_i$ −open in $N \times_B N$, by proposition (4.6). Since N is F.W. discrete, then $\Delta(M) = \Delta\left(\left(\varphi^{-1}(N)\right)\right) = (\varphi \times \varphi)^{-1}\left(\Delta(N)\right)$ is $\tau_i \times \tau_i$ −open in $M \times_B M$. Thus, proposition (4.7) follows from proposition (4.6) where $i = 1, 2$.

Proposition 4.8. Assume that $\varphi : M \to N$ be an open F.W. surjection, where (M, τ_1, τ_2), (N, σ_1, σ_2) are F.W. open bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M is F.W. discrete then N is so.

Proof. In the above figure, with these fresh hypotheses on φ, if M is F.W. discrete then $\Delta(M)$ is $\tau_i \times \tau_i$ −open in $M \times_B M$, by proposition (4.6), so $\Delta(N) = \Delta\left(\left(\varphi(M)\right)\right) = (\varphi \times \varphi)\left(\Delta(M)\right)$ is $\sigma_i \times \sigma_i$ −open in $N \times_B N$. Thus, proposition (4.8) follows from proposition (4.6) again ,where $i = 1, 2$.

Proposition 4.9. If $\varphi, \psi : M \to N$ is a continuous F.W. functions, where (M, τ_1, τ_2) is F.W. bitopological and (N, σ_1, σ_2) is F.W. discrete bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then the coincidence set $K(\varphi, \psi)$ of φ and ψ is open in M.

Proof. The coincidence set is precisely $\Delta^{-1}(\varphi \times \psi)^{-1}\left(\Delta(N)\right)$, where:

Figure 4.2: Diag. of Prop. 4.9.

Hence Proposition.(4.9) follows at once from Proposition(4.6). In particular take $M = N$, take $\varphi = id_M$ and take $\psi = sop$ where s is a section, we conclude that s is an open embedding when M is F.W. discrete.

Proposition 4.10. If $\varphi : M \to N$ is a continuous F.W. functions, where (M, τ_1, τ_2) is F.W. open and (N, σ_1, σ_2) is F.W. discrete bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then the F.W. graph:

$$\Gamma : M \to M \times_B N$$

of φ is an open embedding.

Proof. The F.W. graph is defined in the same way as the ordinary graph, but with values in the F.W. bitopological product, therefore the diagram shown below is commutative.

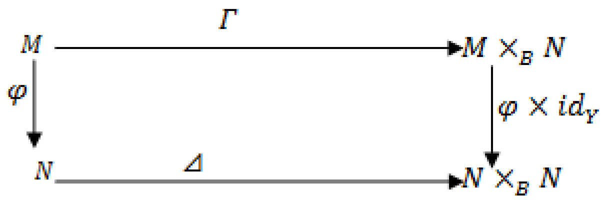

Figure 4.3: Diag. of Prop. 4.10

Since $\Delta(N)$ is $\sigma_i \times \sigma_i$ −open in $N \times_B N$, by Proposition (4.6), so $\Gamma(M) = (\varphi \times id_N)^{-1}\left(\Delta(N)\right)$ is $\tau_i \times \sigma_i$ −open in $M \times_B N$, where $i = 1, 2$, as asserted.

Remark 4.11. If (M, τ_1, τ_2) is F.W. discrete bitopological space over $(B, \Lambda_1, \Lambda_2)$ then for every point $m \in M_b$; $b \in B$, there is a Λ_i −open set W of b a unique section $s : W \to M_W$ exist satisfying $s(b) = m$, we may refer to s as the section through m.

Definition 4.12. The F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named locally sectionable if for every point $b \in B$, admits an open set W and a section $s : W \to M_W$.

Remark 4.13. The F.W. non-empty locally sliceable bitopological spaces are locally sectionable, but the converse is false. In fact, locally sectionable bitopological spaces are not necessarily F.W. open, for example take $M = (-1, 1] \subset \mathbb{R}$ with (M, τ_1, τ_2) : $\tau_1 = \tau_2$, the natural projection onto $B = \mathbb{R} \mid \mathbb{Z}$; $(B, \Lambda_1, \Lambda_2) : \Lambda_1 = \Lambda_2$.

The class of locally sectionable bitopological spaces is finitely multiplicative.

Proposition 4.14. If $\{(M_r, \tau_{r1}, \tau_{r2})\}$ is a finite family of locally sectionable bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. The F.W. bitopological product $M = \prod_B M_r$ is locally sectionable.

Proof. Given a point b of B, there exist Λ_i −open set W_r of b and a section $s_r : W_r \to M_r \mid W_r$ for every index r. Since there are finite number of indices the intersection W of the Λ_i −open sets W_r is also a Λ_i −open set of b, and a section $s : W \to (\prod_B M_r)_W$ is given by $\pi_r \, o \, s(w) = s_r(w)$, for $w \in W$, where $i = 1, 2$.

Our last two result apply equally well to every of the above three propositions.

Proposition 4.15. If (M, τ_1, τ_2) is a F.W. discrete bitopological space over $(B, \Lambda_1, \Lambda_2)$. Suppose that (M, τ_1, τ_2) is locally sliceable, F.W. discrete or locally sectionable over $(B, \Lambda_1, \Lambda_2)$. Then so is M_{B^*} over B^* for every Λ_i −open set B^* of B, where $i = 1, 2$.

Proposition 4.16. Let (M, τ_1, τ_2) be F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that M_{B_j} is locally sliceable F.W. discrete or locally sectionable over B_j for every member B_j of a Λ_i −open covering of B. So is M over B, such that, $i = 1, 2$.

Remark 4.17. It is not difficult to give example of different F.W. discrete bitopologies on the same F.W. set which are in equivalent, as F.W. bitopologies. For this reason we must be careful not to say the F.W. discrete bitopology.

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