Composite Operators on Atomic Orlicz Spaces

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Abstract: In this paper we study composition operators on atomic measure spaces. The behaviour of composition operators on atomic measure spaces differs from that of composition operators on non-atomic measures spaces. Characterization for compact, Fredholm, invertible and composition operators with closed ranges are studied in the paper.

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1. Introduction

Let \( \varphi: [0, \infty) \rightarrow [0, \infty) \) be a continuous convex function such that
(i) \( \varphi(x) = 0 \) if and only if \( x = 0 \)
(ii) \( \lim_{x \to \infty} \varphi(x) = \infty \)

Such a function \( \varphi \) is known as young function. Let \((X, S, \mu)\) be a \(\sigma\)-finite measure space and let \( L^\varphi(\mu) = \{ f : X \rightarrow C \) is measurable:
\[
\int \varphi(|f|)d\mu < \infty \quad \text{for some } \epsilon > 0 \}
\]

If we set
\[
||f||_\varphi = \inf \{ \varepsilon > 0 : \int \varphi(|f|)d\mu \leq 1 \},
\]
then \( L^\varphi(\mu) \) is a Banach space under the norm \( ||.||_\varphi \).

If \( \varphi(x) = x^p, 1 \leq p < \infty \), then \( L^\varphi(\mu) = L^p(\mu) \), the well-known Banach space of \( p \)-integrable functions on \( X \).

A young function \( \varphi: R \rightarrow R^+ \) is said to satisfy the \( \Delta_2 \) condition (globally) if \( \varphi(2x) < k\varphi(x), x \geq x_0 \) for some absolute constant \( k > 0 \). If \( \mu(x) = \infty \) then \( \varphi \) is called \( \Delta_2 \) regular. With each young function \( \varphi \) we can associate another convex function \( \psi: R \rightarrow R^+ \) defined by \( \psi(y) = \sup\{\varepsilon|\varepsilon - \varphi(x): x > x_0, y \in R \} \) which have similar properties.

The function \( \psi \) is called the complementary function to \( \varphi \). In general, simple functions are not dense in \( L^\varphi(\mu) \), but in case \( \varphi \) satisfy the \( \Delta_2 \) condition, then the class of simple functions become dense in \( L^\varphi(\mu) \). For more literature concerning orlicz space we refer to Roo [4] Kufner [3] and Hudzik [7]. Throughout our paper we assume that \( \varphi \) satisfy \( \Delta_2 \) condition.

A bounded linear operator \( A: E \rightarrow E \) (where \( E \) is a Banach space) is called compact if \( A(B_E) \) has compact closure, where \( B_E \) denotes the closed unit ball of \( E \). A bounded linear operator \( A: E \rightarrow E \) is called Fredholm if \( A \) has closed range, \( \dim \ker A \) and \( \text{co dim Ran}(A) \) are finite. The support of a function \( f \in L^\varphi(\mu) \) is denoted by \( \text{supp} f \) and the Randon Nidohym derivative of the measure \( d\mu T^{-1} \) with respect to \( \mu \) is denoted by \( f_\mu \). In this paper we study composite operators on atomic Orlicz spaces. The compact, Fredholm and invertible composite operators are also characterized.

2. Bounded Composite Operators on Atomic Orlicz Space

Theorem 2.1 Suppose \( \varphi_2 < \varphi_1 \). Then \( C_T, L^\varphi(\lambda) \rightarrow L^{\varphi_2}(\lambda) \) is a bounded operator if and only if \( f_0 \) is a bounded function.

Proof: We first assume that \( f_0 \) is a bounded function. Then there exist \( M \geq 1 \) such that \( ||f_0(x)|| \leq M\varphi_2(x) \in \Omega \). Since \( \varphi_2 < \varphi_1 \), there exists \( \alpha > 0 \) such that \( \varphi_1(x) \leq \varphi_1(\alpha x) \forall x \). Therefore
\[
||f_0(x)||_\varphi \leq \frac{M\varphi_2(x)}{\varphi_1(\alpha x)} \leq \varphi_1(\alpha x)
\]

Hence
\[
||C_T f_0||_{\varphi_2} \leq aM||f_0||_{\varphi_1},
\]

which proves that \( C_T \) is a bounded operator.

Conversely, if \( f_0 \) is not a bounded function, then for every \( n \in \mathbb{N} \) there exists \( x_n \in \Omega \) such that \( f_0(x_n) > n \).

Hence
\[
n\varphi_1(\alpha y) \leq f_0(x_n)\varphi_1(\alpha y) \leq f_0(x_n)\varphi_1(\alpha y) \forall y \in C.
\]

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n||\chi_{\{n\}}||_{\phi} \leq \alpha||C_T\chi_{\{n\}}||_{\phi_2} \forall n \\

which shows that $C_T$ is an unbounded operator. Hence $f_0$ must be a bounded function.

3. Compact and Invertible Composite Operators on atomic Orlicz Space

This section characterizes compact and invertible composite operators on atomic Orlicz space.

**Theorem 3.1:** Let $C_T \in B(L^\phi(\lambda))$. Then $C_T$ is compact if and only if the set $ME = \{x: \lambda T^{-1}(x) > E\lambda(x)\}$ is a finite set for each $E > 0$.

**Proof:** Let $E > 0$ be given. Let $\{f_n\}$ be a sequence in $L^\phi(\lambda)$ converges weakly to zero. Suppose $M_n$ contains k-elements say $m_1, m_2, ..., m_k$. Since $\lambda T^{-1}(x) < E\lambda(x)$ for every $x \in M_k$, we have

$$||C_T \frac{f_n}{||f_n||_\phi}||_\phi = \int_\Omega \phi(\frac{f_n(x)}{||f_n||_\phi}) d\lambda T^{-1}(x)$$

$$= \int_{M_k} \phi(\frac{f_n(x)}{||f_n||_\phi}) d\lambda T^{-1}(x) + \int_{M_k'} \phi(\frac{f_n(x)}{||f_n||_\phi}) d\lambda T^{-1}(x)$$

$$\leq M \int_{M_k} \phi(\frac{f_n(x)}{||f_n||_\phi}) d\lambda(x) + \int_{M_k'} \phi(\frac{f_n(x)}{||f_n||_\phi}) d\lambda(x)$$

$$= Mk\phi(f_n(x_T), \lambda(x_p) + \epsilon \int_\Omega \phi(\frac{f_n(x)}{||f_n||_\phi}) d\lambda(x)$$

where

$$\phi(f_n(x_T)) = \max_{x \in M_k} \phi(f_n(x))$$

$$\lambda(x_p) = \max_{x \in M_k} \lambda(x)$$

and $M > 0$ is such that $\lambda T^{-1}(x) \leq M\lambda(x)$. Now the sequence $\{f_n\}$ converges to zero pointwise. Thus for some $E_k > 0$ there exists $n_k$ such that

$$||C_T \frac{f_n}{||f_n||_\phi}||_\phi \geq \epsilon_k + \epsilon$$

Hence from (1)

$$||C_T \frac{f_n}{||f_n||_\phi}||_\phi \geq \epsilon_k + \epsilon$$

since $E_k$ and $E$ are arbitrary, we have

$$\lim_{n \to \infty} ||C_T f_n||_\phi = 0$$

This shows that $C_T$ is compact.

Conversely if for some $E > 0$, $M_k$ is an infinite set, then for each $x_{n_k} \in M_k$, set

$$f_{n_k} = y_{n_k} \chi_{\{n_k\}}$$

Where

$$y_{n_k} = \phi^{-1}(\frac{1}{\lambda_{n_k}})$$

So that $||f_{n_k}||_\phi = 1$. Now $f_{n_k} \to 0$. Weakly, but

$$\int_\Omega \phi(\frac{|f_{n_k}|}{||C_T f_{n_k}||_\phi}) d\lambda \leq \int_\Omega \phi(\frac{|f_{n_k}|}{||C_T f_{n_k}||_\phi}) d\lambda T^{-1}$$

$$\leq \int_\Omega \phi(\frac{|C_T f_{n_k}|}{||C_T f_{n_k}||_\phi}) d\lambda \leq 1$$

Thus

$$||Ef_{n_k}||_\phi \leq ||C_T f_{n_k}||_\phi$$

or

$$||C_T f_{n_k}||_\phi \geq E$$

Hence $\{C_T f_{n_k}\}$ cannot converge strongly to zero which contradicts the compactness of $C_T$.

**Theorem 3.2:** Let $\varphi_1 \approx \varphi_2$. Then $C_T: L^{\varphi_1}(\lambda) \rightarrow L^{\varphi_2}(\lambda)$ is invertible if and only if

(I) $T$ is invertible

(II) There exists $\delta > 0$ such that $f_0(x) \geq \delta$ for all $x \in \Omega$.

**Proof:** If $C_T$ is invertible, then $T$ is invertible. We need only to prove that condition (II). For if it is false, then for every $n \in N$ there exists $x_n \in \Omega$ such that $f_0(x_n) < \frac{1}{n}$. Choose $a > \frac{1}{n}$ then

$$\int_\Omega \phi_2(\frac{nB}{a}||\chi_{\{x_n\}}||_\phi_1) d\lambda = \int_\Omega \phi_2(\frac{nB}{a}||\chi_{\{x_n\}}||_\phi_1) d\lambda$$

$$\leq \int_\Omega \phi_2(\frac{nB}{a}||\chi_{\{x_n\}}||_\phi_1) d\lambda$$

and

$$\int_\Omega \phi_2(\frac{nB}{a}||\chi_{\{x_n\}}||_\phi_1) d\lambda \leq \int_\Omega \phi_2(\frac{nB}{a}||\chi_{\{x_n\}}||_\phi_1) d\lambda$$

Hence

$$||C_T||_{\phi_1} \leq \frac{1}{n} ||\chi_{\{x_n\}}||_\phi_1$$

for every $n$.

which shows that $C_T$ is not bounded away from zero so that $C_T$ is not invertible. Hence condition (II) must hold. Conversely if $T$ is invertible then $T^{-1}(S) = S$ a.e. so that $C_T$ has dense range. Suppose the condition (II) holds. We can assume that $0 < \delta < 1$.

Let $f \in L^{\phi_1}(\lambda)$. Consider
\[ \int_{\Omega} \phi_1 \left( \frac{\delta |f|}{\|C_T f\|_{\phi_2}} \right) d\lambda \leq \int_{\Omega} \phi_2 \left( \frac{\delta |f|}{\|C_T f\|_{\phi_2}} \right) d\lambda \]

\[ \leq \int_{\Omega} \phi_2 \left( \frac{|f|}{\|C_T f\|_{\phi_2}} \right) d\lambda \]

\[ \leq \int_{\Omega} f_0 \phi_2 \left( \frac{|f|}{\|C_T f\|_{\phi_2}} \right) d\lambda \]

\[ = \int_{\Omega} \phi_2 \left( \frac{|C_T f|}{\|C_T f\|_{\phi_2}} \right) d\lambda \]

Hence

\[ \|C_T f\|_{\phi_2} > \frac{\delta}{\alpha} \|f\|_{\phi_2} \]

for every \( f \in L^q (\Omega) \)

This proves that \( C_T \) is bounded away from zero. Now \( C_T \) is bounded away from zero and has dense range. Therefore \( C_T \) is invertible.

**Theorem 3.3:** Let \( C_T \in B(L^q (\lambda)) \). Then \( C_T \) has closed range if and only if \( f_0 \) is bounded away from zero on \( \text{supp} f_0 = s \).

**Proof:** Suppose \( f_0 \) is bounded away from zero on \( s \). Then there exists \( 1 \geq \epsilon > 0 \) such that

\[ f_0(x) \geq \epsilon \quad \forall x \in s \quad (1) \]

For \( f \in L^q (s) \) consider,

\[ \int_s \phi \left( \frac{|f|}{\|C_T f\|_{\phi}} \right) d\lambda \leq \int_s \phi \left( \frac{|f|}{\|C_T f\|_{\phi}} \right) d\lambda \]

\[ \leq \int_s f_0 \phi \left( \frac{|C_T f|}{\|C_T f\|_{\phi}} \right) d\lambda \]

\[ \leq \int_s \phi \left( \frac{|C_T f|}{\|C_T f\|_{\phi}} \right) d\lambda \]

This shows that

\[ |C_T f\|_{\phi} \geq \epsilon \|f\|_{\phi} \quad \forall f \in L^q (s) \]

But \( \ker C_T \) is \( L^q (\Omega) \). It follows that \( C_T \) has closed range. Conversely suppose \( C_T \) has closed range. Then there exists \( \delta > 0 \) such that

\[ |C_T f\|_{\phi} \geq \delta \|f\|_{\phi} \quad \text{for every} \quad f \in L^q (s) \quad (2) \]

For each \( n \in H \), define

\[ H_n = \left\{ x \in s : \frac{1}{(n+1)^2} \leq f_0(x) \leq \frac{1}{n^2} \right\} \]

Set

\[ H = \left\{ n : \lambda (H_n) > 0 \right\} \]

Let

\[ f = \sum_{n \in H} \phi^{-1} \left( \frac{1}{\lambda (H_n)} \right) C_T \chi_{H_n} \]

Then

\[ \int_{\Omega} \phi (f) d\lambda = \sum_{n \in H} \int \phi \left( \frac{1}{\lambda (H_n)} \right) C_T \chi_{H_n} d\lambda \]

\[ = \sum_{n \in H} \int \frac{1}{\lambda (H_n)} C_T \chi_{H_n} d\lambda \]

\[ = \sum_{n \in H} \frac{1}{\lambda (H_n)} \int_{H_n} f_0 d\lambda \]

\[ \leq \sum_{n \in H} \frac{1}{n^2} < \infty \]

Using (2) we see that \( f_1 \in L^q (S) \), where

\[ f_1 = \sum_{n \in H} \phi^{-1} \left( \frac{1}{\lambda (H_n)} \right) \chi_{H_n} \]

But

\[ \int_{\Omega} \phi (f_1) d\mu = \sum_{n \in H} \int \phi \left( \frac{1}{\lambda (H_n)} \right) x_{H_n} d\lambda \]

\[ = \sum_{n \in H} \int \frac{1}{\lambda (H_n)} x_{H_n} d\lambda \]

\[ \sum_{n \in H} \int_{H_n} f_0 d\lambda = \sum_{n \in H} 1 = \infty \]

If \( H \) is an infinite set. Hence \( H \) must be a finite set. Therefore there exists \( n_0 \) such that \( n \geq n_0 \) implies \( \lambda (H_n) = 0 \). Thus

\[ f_0(x) > 0 \quad \exists \delta \quad \forall x \in s \]

**4. Fredholm Composite Operators on Atomic Orlicz Space**

**Theorem 4.1:** Let \( C_T \in B(L^q (\lambda)) \). Then \( C_T \) is fredholm if and only if

(I) \( \Omega \cap \Omega (\Omega) \) is a finite set.

(II) \( \{ x : p^T (x) > 2 \} \) is a finite set.

(III) There exists \( \delta > 0 \) such that \( f_0(x) > \delta \quad \forall x \in s \).

**Proof:** Since \( \ker C_T = L^q (\Omega) \cap \Omega (\Omega) \), it follows that \( \ker C_T \) is finite dimensional if and only if \( \Omega \) is a finite set. Then \( C_T \) has dense range if and only if the condition (III) holds. We need only to prove that \( L^q (\Omega) \cap \Omega (\Omega) \) is finite dimensional if and only if the condition (II) holds. Suppose \( T^{-1} (\lambda) \cap \lambda (\Omega) \) holds. Suppose \( T^{-1} (x) \cap \lambda (\Omega) \) holds. Then \( \lambda (\Omega) \cap \lambda (\Omega) \) is finite dimensional if and only if the condition (III) holds. Hence it shows that \( L^q (\Omega) \cap \lambda (\Omega) \) is finite dimensional if and only if the set \( \{ x \} \) is finite set.

**Example 4.2:** Let \( a \in N, \Omega = N, S = P (N) \)

\[ \lambda ([a]) = \alpha^{2a} < \alpha < 1 \]

Define \( T^a : N \rightarrow N \) by...
where $E = \{\alpha n: n \in N \setminus \{\alpha\}\}$. Then

$$T^{-1}_\alpha(F) = \begin{cases} m & \text{if } m = n\alpha \in E \setminus \{0\} \\ m+1 & \text{if } m \notin E \end{cases}$$

But $N \setminus E$ is an infinite set. Therefore $\lambda T^{-1}_\alpha(F)$ is a finite set. Hence $C^\alpha_T$ is not compact.

**Example 4.3** Let $\Omega = N, \alpha > 1, a > 1$

\[
\lambda \{n\} = a^{an}
\]

Define $T_\alpha : N \to N$ by $T_\alpha(n) = n\alpha$

\[
T^{-1}_\alpha(F) = \begin{cases} m & \text{if } m \notin n\alpha \\ n & \text{if } m = n\alpha \end{cases}
\]

\[
\lambda T^{-1}_\alpha(F) = \frac{a^{am}}{a^am}
\]

\[
= \frac{1}{a^{(a-1)m}} \to 0 \text{ as } m \to \infty
\]

Hence the set $\{m: \lambda T^{-1}_\alpha(F) > \lambda(m)\}$ is a finite set. Therefore $C^\alpha_T$ is a compact operator for any integer $\alpha > 1$.

**Example 4.4** Let $\alpha \in N, a > 1$ be an integer, Define $\lambda(n) = a^n$ for each $n \in N$ Let $T_\alpha : N \to N$ as

\[
T_\alpha(m) = \begin{cases} \alpha & \text{if } m \leq \alpha \\ m+1 & \text{if } m > \alpha \end{cases}
\]

Then

\[
f_0(m) = \frac{\lambda T^{-1}_\alpha(F)}{\lambda(F)} = \frac{a^{am}}{a^am} = \frac{1}{a^\alpha}
\]

for all $m > \alpha$. Hence $C^\alpha_T$ has closed range. Clearly $\dim \ker C^\alpha_T = a$ and $\dim L^q(\Omega) = \dim C^\alpha_T = a - 1$. Hence $C^\alpha_T$ is Fredholm for each $\alpha$. Clearly $C^\alpha_T$ is not invertible.

**References**


