Composite Operators on Atomic Orlicz Spaces

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Abstract: ***In this paper we study composition operators on atomic measure spaces. The behaviour of composition operators on atomic measure spaces differs from that of composition operators on non-atomic measures spaces. Characterization for compact, fredholm, invertible and composition operators with closed ranges are studied in the paper.***

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1. Introduction

Let φ: [0, ∞) → [0, ∞) be a continuous convex function such that
(i) φ(x)=0 if and only if x = 0
(ii) $\lim_{x\to\infty} \phi(x) = \infty$
Such a function φ is known as young function. Let (X, S, μ) be a σ-finite measure space and let $L^{\varphi}(\mu)$= {f: X → C is measurable:
$\int \phi(\in |f|)d\mu < \infty$ for some $\in > 0\}$
If we set
$||f||_{\phi} = inf\{\epsilon > 0 : \int \phi(|f|)d\mu \leq 1\},$
then $L^{\varphi}(\mu)$ is a Banach space under the norm ||.||φ.

If $\phi(x) = x^p, 1 \leq p < \infty$, then $L^{\varphi}(\mu)$= $L^{p}(\mu)$, the well-known Banach space of p-integrable functions on X.
A young function φ: R→R^+ is said to satisfy the Δ_2-condition (globally) if φ(2x) < kφ(x),x ≥ x_0 ≥ 0(x_0 = 0) for some absolute constant k> 0. If μ(x)= ∞ then φ is called Δ_2-regular. With each young function φ we can associate another convex function ψ: R→R^+ defined by ψ(y)= sup{x|y|− φ(x): x> 0},y ∈R which have similar properties.

The function ψ is called the complementary function to φ. In general, simple functions are not dense in $L^{\varphi}(\mu)$, but in case φ satisfy the Δ_2-condition, then the class of simple functions become dense in $L^{\varphi}(\mu)$. For more literature concerning orlicz space we refer to Roo [4] Kufner [3] and Hudzik [7]. Throughout our paper we assume that φ satisfy Δ_2-condition.

A bounded linear operator A: E → E (where E is a Banach space) is called compact if A(B_1) has compact closure, where B_1 denotes the closed unit ball of E. A bounded linear operator A: E → E is called fredholm if A has closed range, dim ker A and co dim Ran(A) are finite. The support of a function f ∈ $L^{\varphi}(\mu)$ is denoted by supp f and the Randon Nidohym derivative of the measure $d\mu T^{-1}$ with respect to μ is denoted by f_0. In this paper we study composite operators on atomic Orlicz spaces. The compact, Fredholm and invertible composite operators are also characterized.

2. Bounded Composite Operators on Atomic Orlicz Space

Theorem 2.1 *Suppose $\varphi_2 < \varphi_1$. Then C_T: $L^{\varphi}(\lambda) \to L^{\varphi}(\lambda)$ is a bounded operator if and only if f_0 is a bounded function.*

Proof: We first assume that f_0 is a bounded function. Then there exist M ≥ 1 such that $|f_0(x)| \leq M \forall x \in \Omega$. Since $\varphi_2 < \varphi_1$, there exists α> 0 such that $\varphi_2(x) \leq \varphi_1(\alpha x) \forall x$. Therefore

$$f_0(x)\varphi_2(x) \leq M\varphi_1(\alpha x)$$
$$\leq \varphi_1(M\alpha x)$$

Hence

$$||C_T f||_{\varphi 2} \leq \alpha M ||f||_{\varphi 1},$$

which proves that C_T is a bounded operator.

Conversely, if f0 is not a bounded function, then for every n ∈ N there exists $x_n \in \Omega$ such that $f_0(x_n) > n$.

Hence
$n\varphi_1(|y|) \leq f_0(x_n)\varphi_1(y)$
$\leq f_0(x)\varphi_2(|\alpha y|)\ \forall\ y \in C.$

Now

$$\int_\Omega n\phi_1(\frac{\chi_{\{x_n\}}}{\alpha||C_T\chi_{\{x_n\}}||_{\phi_2}})d\lambda \leq \int_\Omega f_0(x)\phi_2(\frac{\chi_{\{x_n\}}}{||C_T\chi_{\{x_n\}}||_{\phi_2}})d\lambda$$
$$= \int_\Omega \phi_2(\frac{C_T\chi_{\{x_n\}}}{||C_T\chi_{\{x_n\}}||_{\phi}})d\lambda \leq 1$$

Hence

$n\|\chi_{\{x_n\}}\|_{\varphi 1} \le \alpha\|C_T\chi_{\{x_n\}}\|_{\varphi 2}\ \forall n$

which shows that C_T is an unbounded operator. Hence f_0 must be a bounded function.

3. Compact and Invertible Composite Operators on atomic Orlicz Space

This section characterizes compact and invertible composite operators on atomic Orlicz space.

Theorem 3.1: *Let $C_T \in B(L^{\varphi}(\lambda))$. Then C_T is compact if and only if the set $ME = \{x: \lambda T^{-1}(x) > E\lambda(x)\}$ is a finite set for each $E > 0$.*

Proof: Let $E > 0$ be given. Let $\{f_n\}$ be a sequence in $L^{\varphi}(\lambda)$ converges weakly to zero. Suppose M_E contains k-elements say $m_1 m_2 \dots m_k$. Since $\lambda T^{-1}(x) < \in \lambda(x)$ for every $x \in M'_\epsilon$, we have

$$\begin{aligned}
\|C_T\frac{f_n}{\|f_n\|_\phi}\| &= \int_\Omega \phi(\frac{|f_n(x)|}{\|f_n\|_\phi})d\lambda T^{-1}(x)\\
&= \int_{M_\epsilon}\phi(\frac{|f_n(x)|}{\|f_n\|_\phi})d\lambda T^{-1}(x) + \int_{M'_\epsilon}\phi(\frac{|f_n(x)|}{\|f_n\|_\phi})d\lambda T^{-1}(x)\\
&\le M\int_{M_\epsilon}\phi(\frac{|f_n(x)|}{\|f_n\|_\phi})d\lambda(x) + \epsilon\int_{M'_\epsilon}\phi(\frac{|f_n(x)|}{\|f_n\|_\phi})d\lambda(x)\\
&\le Mk\phi(f_n(x_r)\lambda(x_p) + \epsilon\int_\Omega \phi(\frac{|f_n(x)|}{\|f\|_\phi})d\lambda(x)\\
&\le mk\phi(f_n(x_r)\lambda(x_p) + \epsilon \qquad (1)
\end{aligned}$$

where

$$\begin{aligned}
\phi(f_n(x_r)) &= \max_{i\in M_\epsilon}\phi(f_n(x_i))\\
\lambda(x_p) &= \max_{i\in M_\epsilon}\lambda(x)
\end{aligned}$$

and $M > 0$ is such that $\lambda T^{-1}(x) \le M\lambda(x)$. Now the sequence $\{f_n\}$ converges to zero pointwise. Thus for some $E_1 > 0$ there exists n_0 such that

$$|\phi(f_n(x_r))| < \frac{\epsilon_1}{mk\lambda(x_p)}\forall n \ge n_0.$$

Hence from (1)

$$\|C_T\frac{f_n}{\|f_n\|_\phi}\|_\phi \ge \epsilon_1 + \epsilon$$

since E_1 and E are arbitrary, we have

$$\lim_{n\to\infty}\|C_T f_n\|_\phi = 0$$

This shows that C_T is compact.

Conversely if for some $E > 0$, M_E is an infinite set, then for each $x_{n_k} \in M_\epsilon$, set

$$f_{n_k} = y_{n_k}\chi\{n_k\}$$

Where

$$y_{n_k} = \phi^{-1}(\frac{1}{\lambda_{n_k}})$$

So that $\|f_{nk}\|_\varphi = 1$. Now $f_{nk} \to 0$. Weakly,

but

$$\begin{aligned}
\int_\Omega \phi(\frac{|\epsilon f_{n_k}|}{\|C_T f_{n_k}\|_\phi})d\lambda &< \int_\Omega \epsilon\phi(\frac{|f_{n_k}|}{\|C_T f_{n_k}\|_\phi})d\lambda\\
&< \int_\Omega \phi(\frac{|f_{n_k}|}{\|C_T f_{n_k}\|_\phi})d\lambda T^{-1}\\
&< \int \phi(\frac{|C_T f_{n_k}|}{\|C_T f_{n_k}\|_\phi})d\lambda \le 1
\end{aligned}$$

Thus

$\|Ef_{nk}\|_\varphi \le \|C_T f_{nk}\|_\varphi$

or

$\|C_T f_{nk}\|_\varphi \ge E$

Hence $\{C_T f_{nk}\}$ cannot converge strongly to zero which contradicts the compactness of C_T.

Theorem 3.2: *Let $\varphi_1 \approx \varphi_2$. Then $C_T: L^{\varphi 1}(\lambda) \to L^{\varphi 2}(\lambda)$ is invertible if and only if*
(I) T is invertible
(II) There exists $\delta > 0$ such that $f_0(x) \ge \delta$ for all $x \in \Omega$.

Proof: If C_T is invertible, then T is invertible. We need only to prove that condition (II). For if it is false, then for every $n \in N$ there exists $x_n \in \Omega$ such that $f_0(x_n) < \frac{1}{n}$. Choose $\alpha > n$. Then

$$\begin{aligned}
\int_\Omega \phi_2(\frac{nbC_T\chi_{\{x_n\}}}{a\|\chi_{\{x_n\}}\|_{\phi_1}})d\lambda &= \int_\Omega f_0\phi_2(\frac{nb\chi_{\{x_n\}}}{a.\|\chi_{\{x_n\}}\|_{\phi_1}})\lambda\\
&= \int_\omega f_0(x_n)\phi_2(\frac{nb}{a\|\chi_{\{x_n\}}\|_{\phi_1}})\lambda(x_n)\\
&\le \frac{1}{n}\frac{n}{a}\int_\Omega \phi_2(\frac{b}{\|\chi_{\{x_n\}}\|_{\phi_1}})\lambda(x_n)\\
&\le \int_\Omega \phi_2(\frac{b}{\|\chi_{\{x_n\}}\|_{\phi_1}})\lambda(x_n)\\
&\le \int_\Omega \phi_1(\frac{1}{\|\chi_{\{x_n\}}\|_{\phi_1}})\lambda(x_n) \text{ for some b}\\
&= \int_\Omega \phi_1(\frac{\chi_{\{x_n\}}}{\|\chi_{\{x_n\}}\|_{\phi_1}})d\lambda \le 1
\end{aligned}$$

Hence $\|C_T\|\chi_{\{x_n\}}\|_{\phi_2} \le \frac{a}{nb}\|\chi_{\{x_n\}}\|_{\phi_1}$ for every n.

which shows that C_T is not bounded away from zero so that C_T is not invertible. Hence condition (II) must hold. Conversely if T is invertible then $T^{-1}(S) = S$ a.e. so that C_T has dense range. Suppose the condition (II) holds. We can assume that $0 < \delta < 1$.

Let $f \in L^{\phi_1}(\lambda)$. Consider

$$\int_\Omega \phi_1(\frac{\delta|f|}{a||C_T f||_{\phi_2}})d\lambda \leq \int_\Omega \phi_2(\frac{\delta|f|}{||C_T f||_{\phi_2}})d\lambda$$
$$\leq \int_\Omega \delta.\phi_2(\frac{|f|}{||C_T f||_{\phi_2}})d\lambda$$
$$\leq \int_\Omega f_0\phi_2(\frac{|f|}{||C_T f||_{\phi_2}})d\lambda$$
$$= \int_\Omega \phi_2(\frac{|C_T f|}{||C_T f||_{\phi_2}})d\lambda \leq 1$$

Hence

$$||C_T f||_{\phi_2} > \frac{\delta}{a}||f||_{\phi_1}$$

for every $f \in L^{\varphi 1}(\lambda)$

This proves that C_T is bounded away from zero. Now C_T is bounded away from zero and has dense range. Therefore C_T is invertible.

Theorem 3.3: *Let $C_T \in B(L^\varphi(\lambda))$. Then C_T has closed range if and only if f_0 is bounded away from zero on $\text{supp} f_0 = s$.*

Proof: Suppose f_o is bounded away from zero on s. Then there exists $1 \geq \epsilon > o$ such that
$f_0(x) \geq \epsilon \ \forall x \in s$ (1)

For $f \in L^\varphi(s)$ consider.

$$\int_s \phi(\frac{\epsilon|f|}{||C_T f||_\phi})d\lambda \leq \int_s \epsilon\phi(\frac{|f|}{||C_T f||_\phi})d\lambda$$
$$\leq \int_s f_0\phi(\frac{|f|}{||C_T f||_\phi})d\lambda$$
$$\leq \int_s \phi(\frac{|C_T f|}{||C_T f||_\phi})d\lambda \leq 1$$

This shows that

$$|C_T f||_\phi \geq \epsilon||f||_\phi \forall f \epsilon L^\phi(s)$$

But $\ker C_T = L^\varphi(\Omega|s)$.It follows that C_T has closed range. Conversely suppose C_T has closed range. Then there exists $\delta > 0$ such that

$$||C_T f||_\phi \geq \delta||f||_\phi \text{ for every } f \epsilon L^\phi(s) \quad (2)$$

For each $n \in H$, define

$$H_n = \{x \in s : \frac{1}{(n+1)^2} \leq f_0(x) \leq \frac{1}{n^2}\}$$

Set

$$H = \{n : \lambda(H_n) > 0\}$$

Let

$$f = \sum_{n\in H} \phi^{-1}(\frac{1}{\lambda(H_n)})C_T\chi_{H_n}$$

Then

$$\int_\Omega \phi(f)d\lambda = \sum_{n\in H}\int \phi(\phi^{-1}(\frac{1}{\lambda(H_n)})C_T\chi_{H_n})d\lambda$$
$$= \sum_{n\in H}\int \frac{1}{\lambda(H_n)}C_T\chi_{H_n}d\lambda$$
$$= \sum_{n\in H} \frac{1}{\lambda(H_n)}\int_{H_n} f_0 d\lambda$$
$$\leq \sum_{n\in H}\frac{1}{n^2} < \infty$$

Using (2) we see that $f_1 \in L^{\varphi 1}(S)$, where

$$f_1 = \sum_{n\in H} \phi^{-1}\frac{1}{\lambda(H_n)}\chi_{H_n}$$

But

$$\int_\Omega \phi(f_1)d\mu = \sum_{n\in H}\int \phi(\phi^{-1}(\frac{1}{\lambda(H_n)})\chi_{H_n})d\lambda$$
$$= \sum_{n\in H}\int \frac{1}{\lambda(H_n)}\chi_{H_n}d\lambda$$
$$= \sum_{n\in H}\int_{H_n}\frac{1}{\lambda(H_n)}d\lambda = \sum_{n\in H} 1 = \infty$$

If H is an infinite set. Hence H must be a finite set. Therefore there exists n_0 such that $n \geq n_0$ implies $\lambda(H_n) = 0$. Thus

$$f_0(x) > \frac{1}{n_0^2} = \delta \text{ (say) } \forall x \in s$$

4. Fredholm Composite Operators on Atomic Orlicz Space

Theorem 4.1: *Let $C_T \in B(L^\varphi(\lambda))$. Then C_T is fredholm if and only if*
(I) $\Omega \backslash T(\Omega)$ is a finite set.
(II) $\{x: pT^{-1}(x) > 2\}$ is a finite set.
(III)There exists $\delta > 0$ such that $f_0(x) \geq \delta \ \forall x in S$.

Proof: Since $\ker C_T = L^\varphi(\Omega \backslash T(\Omega))$, it follows that $\text{ke}C_T$ is finite dimensional if and only if Ω is a finite set. Then C_T has dense range if and only if the condition (III) holds. We need only to prove that $L^\varphi(\Omega|ranC_T$ is finite dimensional if and only if the condition (II) holds. Suppose $T^{-1}(\{x\}) - \{x_1, x_2, ..., x_n\} \in ranC_T$ for some $x \in \Omega$. Clearly $C_T\chi_{\{n\}} = \chi_{\{x_1, x_2, ..., x_n\}}$. It can be easily seen that $E = span\{\chi_{\{x_i\}}; 1 \leq i \leq n\}$ is n-dimensional. Moreover there are n -1 vectors $g_1, g_2 ... g_{n-1}$ which are linearly independent as well as linearly independental of the vectors $\chi_{\{x_1, x_2 ... x_n\}}$ in E. It can be easily seen that $\{f_1 + ranC_T, ... f_{n-1} + ranC_T\}$ are linerly independental vectors in $L^\varphi(\lambda)|ranC_T$. Hence it shows that $L^\varphi(\lambda)|ranC_T$ is finite dimensional if and only if the set $\{x|| pT^{-1}(x) \geq 2\}$ is a finite set.

Example 4.2: Let $\alpha \in N$, $\Omega = N$, $S = P(N)$
$\lambda(\{n\}) = \alpha^{2n} 0 < \alpha < 1$

Define $T_\alpha: N \to N$ by

$$T_\alpha(m) = \begin{cases} (n-1)\alpha & if\ m = n\alpha \in E \\ m & if\ m \notin E \end{cases}$$

where E = {αn: n ∈ N\{α}}Then

$$T_\alpha^{-1}(\{m\}) = \begin{cases} m & if\ n \notin E \\ (n+1)\alpha & if\ m = n\alpha \in E for n \neq 1 \end{cases}$$

But N|E is an infinite set. Therefore

$$\{n : \lambda T_\alpha^{-1}(\{n\}) > \tfrac{1}{2}\lambda(n)\}$$

is an infinite set. Hence C_{T_α} cannot be compact.

Example 4.3 Let Ω= N,α > 1,a > 1

$$\lambda(\{n\}) = a^{\alpha n}$$

$$Define\ T_\alpha : N \to N\ by\ T_\alpha(n) = n\alpha$$

$$T_\alpha^{-1}(\{m\}) = \begin{cases} \phi & if\ m \neq n\alpha \\ \frac{n}{\alpha} & if\ m = n\alpha \end{cases}$$

$$\lambda T_\alpha^{-1}(\{m\}) = 0 or \frac{a^m}{a^{\alpha m}}$$

$$= \frac{1}{a^{(\alpha-1)m}} \to 0\ as\ m \to \infty$$

Hence the set {m: $\lambda T^{-1}_{\alpha}(\{m\})$ >∈ λ(m)} is a finite set. Therefore C_{T_α} is a compact operator for any integer α> 1.

Example 4.4 Let α ∈ N, a > 1 be an integer, Define λ({n})= a^{an} for each n ∈ N Let T_α: N → N as

$$T_\alpha(m) = \begin{cases} \alpha & if\ m \le \alpha \\ m+1 & if\ m > \alpha \end{cases}$$

Then

$$f_0(m) = \frac{\lambda T^{-1}(\{m\})}{\lambda(\{m\})} = \frac{a^{\alpha(n-1)}}{a^{\alpha n}} = \frac{1}{a^\alpha}$$

for all m>α. Hence C_T has closed range. Clearly dim kerC_T = α and dim $L^\varphi(\Omega)$—ran C_T = α − 1 Hence C_{T_α} is fredholm for each α . Clearly C_{T_α} is not invertible.

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