Mathematical Model for Diabetes Use Glucose Tolerance Test (GTT)

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Abstract: ***Diabetes is disease of metabolism whose essential feature is excessive sugar in the blood and urine. Second order differential equation has been formulated to describe the performance of BGRS during GTT and its solution has been shown over here. Intersting points in that the sociological factors play an important role in the Blood Glucose Rogulatory system.***

Keywords: Mathematical Modeing, Diabetes, Second order differential equation, and Glucose Tolerance Test

1. Introduction

Diabetes

Most of the food we eat is turned into glucose (sugar) for our bodies to use for energy. The pancreas, an organ near the stomach, makeser hormone called insulin to help glucose get into our body cells. When you have diabetes, your body either doesn't make enoυugh insulin or can't use its own insulin very well. This problem causes glucose to build up in your blood.

Diabetes means that a peperson's blood sugar is too high. Your blood always has some sugar in it because the body needs sugar for energy to keep you going. But too much sugar in the blood can cause serious damage to the eyes, Kidneys, nerves and heart.

Signs and symptoms of Diabetes

You may recall having some of these signs before you found out you had diabetes:
- Being very thirsty.
- Urinating a lot – often at night
- Having blurry vision from time to time.
- Feeling very tired much of the time
- Losing weight without trying
- Having very dry skin
- Having sores that are slow to heal.

Types of Diabetes

There are two main types of diabetes
- Type 1 Diabetes
- Type 2 Diabetes

Another type of diabetes appears during pregnancy in some women. It's called gestational diabetes.

Type 1 Diabetes

One out of 10 people with diabetes has type 1 diabetes. These people usually find out they have diabetes when they are children or young adults. People with type 1 diabetes must inject insulin every day to live. The pancreas of a person with type 1 make little (or) no insulin.

Type 2 Diabetes

Most people with diabetes 9 out of 10 have type 2 diabetes. The pancreas of people with type 2 diabetes keeps making insuling for some time, but the bdy can't use it very well. Most people with type 2 find out about their diabetes after age 30 or 40.

Certain risk factors make people more likely to develop type 2 diabetes. Some of these are.
- A family history of diabetes.
- Lack of exercise.
- Weighing too much
- Being of African American.
- Being of African American, American India, Asian / pacific Island heritage
- Gestational diabetes history.

Glucose Tolerance Lest (GTT)

To diagnose diabetes is by the Glucose Tolerance test (GTT) in which the patient is called the hospital after overnight fasting. On his arrival he is given a large dose of glucose (the form of sugar in which it occurs in the blood streem) and then several measurements of the concentration of glucose in the patient's blood is taken during next 3 to 5 hours. Based on his measurements, the physician makes the diagnosis of diabetes which obviously depends on his interpretation of the results.

Formulation of Mathematical Model:

We formulate an appropriate model in two steps.

Step 1 Background Information on the model

We assume that the following two concentrations adequately describe the performance of blood glucose regulatory system (BGRS).

(i) Concentration of glucose in the Blood (g).
(ii) Net Hormonal concentration (h).

Since, the variable g and h change with time, we consider g and has dependent variables while t(time) as the independent variable. From the elementary concentration of the biological facts, stated above, the logistic law govering the performance of BGRs may be written as

$$\frac{dg}{dt} = E_1(g,h) + F(t) \quad (1)$$

$$\frac{dh}{dt} = E_2(g,h) \quad (2)$$

Where E_1 and E_2 are the same functions of g and h, while F(t) is the external rate at which the Blood Glucose concentration (BGC) is being increased.

2. Mathematical Model use Glucose Tolerence Test

Formulate a second order differential equations model to describe the performance of BGRS during a GTT.
Let g_0 and h_0 be the optimal value of g and h respectively.
We set $g = G - G_0$ and $h = H - H_0$

Substituting these values of g and h in equations (1) and (2) and using Taylor's Expansion, we get.

$$\frac{dg}{dt} = \left[E_1(G_0,H_o) + G\left(\frac{\partial E1}{\partial g}\right)_0 + H\left(\frac{\partial E1}{\partial g}\right)_0 + d_1\right] + \text{F(t)} \quad (3)$$

$$\frac{dh}{dt} = \left[E_2(G_0,H_0) + G\left(\frac{\partial E2}{\partial g}\right)_0 + h\left(\frac{\partial E2}{\partial g}\right)_0 + d_2\right] \quad (4)$$

Where $\left(\frac{\partial E_1}{\partial g}\right)_0$, and $\left(\frac{\partial E_2}{\partial g}\right)$ denotes $g = G_0$ and $h = H_0$ and d_1, d_2 contains terms of second and higher powers in g and h.

(i) Assume that if $E_1(G_0, H_0) = 0$ and $E_2(G_0, H_0) = 0$, because it is assumed that g and h have their optimal values G_0 and H_0 respectively by the time the fasting patient arrives at the hospital and (ii) E_1 and E_2 being small quantities may be neglected and g and h are very small.

Substituting these two conditions in equation (3) and (4) we get

$$\frac{dg}{dt} = \left[G\left(\frac{\partial E_1}{\partial g}\right)_0 + H\left(\frac{\partial E_1}{\partial h}\right)_0\right] + \text{F(t)} \quad (5)$$

$$\frac{dh}{dt} = \left[G\left(\frac{\partial E_2}{\partial g}\right)_0 + H\left(\frac{\partial E_2}{\partial h}\right)_0\right] \quad (6)$$

To find the value of

$$\left(\frac{\partial E_1}{\partial g}\right)_0, \left(\frac{\partial E_1}{\partial h}\right)_0, \left(\frac{\partial E_2}{\partial g}\right)_0 \text{ and } \left(\frac{\partial E_2}{\partial h}\right)_0.$$

Case (i)

We consider g >0, h =0 (excessive glucose) ie $\frac{dg}{dt} < 0$.

The equation (5) implies that $\left(\frac{\partial E_1}{\partial h}\right)_0$ may be negative.

Case (ii)

If h >0, g =0 (excessive insulin) ie., $\frac{dh}{dt} < 0$.

Then the equation (5) implies that $\left(\frac{\partial E_1}{\partial h}\right)_0$ may be negative.

Similarly, $\left(\frac{\partial E_1}{\partial h}\right)_0$ and $\left(\frac{\partial E_2}{\partial h}\right)$ also be negative.

Therefore equation (5) and (6) can be written as

$$\frac{dg}{dt} = -A_1 g - A_2 h + F(t) \quad (7)$$

$$\frac{dh}{dt} = A_3 g - A_4 h \quad (8)$$

Where are all positive constants. A_1, A_2, A_3 and A_4.

Since it is the BGS that can be measured easily therefore we attempt to eliminate h if possible, between equations (7) and (8).

Differenciate equation (7), (8) with respect to t.

$$\frac{d^2g}{dt^2} = -A_1\frac{dg}{dt} - A_2\frac{dh}{dt} + \frac{dF}{dt} \quad (9)$$

$$\frac{d^2h}{dt^2} = -A_3\frac{dg}{dt} - A_4\frac{dh}{dt} \quad (10)$$

Substitute - equation (7) the value of $\frac{dh}{dt}$ in equation (9) we get

$$\frac{d^2g}{dt^2} = -A_1\frac{dg}{dt} - A_2(A_3g - A_4h) + \frac{dF}{dt}$$

$$\frac{d^2g}{dt^2} = -A_1\frac{dg}{dt} - A_2A_3g + A_2A_4h + dF/dt \quad (11)$$

Substitute – the value of A_2h in (7) in (11) we get

$$\frac{d^2g}{dt^2} = -A_1\frac{dg}{dt} - A_2A_3g + (-A_1g + F(t) - \frac{dg}{dt})A_4 + dF/dt$$

$$\frac{d^2g}{dt^2} = -\frac{dg}{dt}(A_1 + A_4) - g(A_2A_3 + A_1A_4)F(t)A_4 + dF/dt$$

$$\frac{d^2g}{dt^2} + 2\beta + w_0^2 = \text{m(t)} \quad (12)$$

Where $2\beta = (A_1 + A_4)$, $w_0^2 = (A_2A_3 + A_1A_4)$ and m(t) = F(t)

$$A_4 + dF/dt \cdot \frac{d^2g}{dt^2} + 2\beta\frac{dg}{dt} + w_0^2 g = 0 = \text{m(t)} \quad (13)$$

Where m(t) is identically zero. Equation (13) is known as second order differential equation with constant coefficient which governs, the BGRS after a heavy load of glucose is ingested.

3. Analysis of the Model

The mathematical model have been analyzed the following step.

The auxiliary equation of (13) is written as

$$m^2 + 2\beta m + w_0^2 = 0$$

Whose roots are given by

$$m = \beta \pm \sqrt{\beta^2 - w_0^2}$$

Three cases can be considered.

$$\beta^2 - w_0^2 \le 0, \quad \beta^2 - w_0^2 \ge 0.$$

And it is fact that equation (13) approaches to 0 as $t \to \infty$ and so our model confirme to reality in predicting that the BGC tends to return ultimately to its optimal concentration. So it passes the test of consistency.

For the case $\beta^2 - w_0^2 < 0$

$$g(t) = Ae^{-\beta t} \text{Cos}(wt - \delta) \quad (14)$$

where $w = w_0^2 - \beta_0^2$

Particular integral

$$\text{P.I} = \frac{1}{D^2 + 2\beta D + w_0^2} e^{-at}$$

$$= \frac{1}{a^2 + w_0^2 - 2\beta a} e^{-at}$$

$$g(t) = g_0 + Ae^{-\beta t}\cos(wt - \delta) + \frac{1}{a^2 + w_0^2 - 2\beta a} e^{-at} \quad (15)$$

Equation (15) have unknown g_0, β, w_0, δ and a.

g_0, being Blood Glucose Concentration before the glucose load is ingested can be determined by measuring the patients Blood glucose concentration immediately upon this arrival at the hospital.

$$g_i(t) = g_0 + Ae^{-\beta t_i}\cos(wt_i - \delta) + \frac{1}{a^2 + w_0^2 - 2\beta a} e^{-at_i},$$

$i = 1, 2, 3, 4, 5, 6, \ldots n$

By taking n measurements of $g_1, g_2, \ldots g_n$ of the patients BGC at time $t_1, t_2, \ldots t_n$ respectively.

If we take n = 7 or 8 measurements of $g_1, g_2, g_3, g_4, g_5, g_6, g_7$ we find the optimal values of g_0, β, w_0, a and δ. Such that the least square error given by

$$e = \sum_{j=1}^{n} g_i - g_0 - Ae^{-\beta t}\cos(wt_i - \delta)^2 \quad (16)$$

4. Conclusion

The aim of study can be concluded that the optimal value of g_0, β, w_0, a, δ can be calculated in Blood Glucose regulatory system to a Glucose Tolerance Test. If w_0 may be regarded has been defined by $T_0 = \frac{2\pi}{w_0}$, w_0 is the natural frequency of the system and here it is considered "T_0" as a suitable parameter for diagnosis of diabetes. If has been concluded that a value of less than four hours for T_0 indicated normally suppose T_0 is more than four hours implied mild diabetes.

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