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Factorial Numbers and Patterns in their Prime Factors

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Abstract: ***Factorial numbers play a fundamental role in number theory, combinatorics, and mathematical analysis. Factorial of a natural number n is the product of the first n natural numbers, factorials exhibit rich arithmetic structures, particularly in the distribution and behavior of their prime factors. This paper presents a detailed study of factorial numbers with emphasis on patterns arising in their prime factorization. Classical results such as Legendre's formula and Wilson's theorem are discussed, along with observations on the growth of prime powers in factorials.***

Keywords: Factorial numbers, prime factorization, Legendre's formula, prime powers, number theory

1. Introduction

Factorial numbers are among the most elementary powerful objects in mathematics. For a positive integer 'n', the factorial of n, denoted by n!, is defined as n! = 1×2×3×….×n. Factorials arise naturally in counting problems, permutations, combinations, series expansions, and probability theory. Despite their simple definition, factorial numbers grow very rapidly and possess deep number-theoretic properties.

One important aspect of factorial numbers is their prime factorization. Since n! is the product of all integers from 1 to n, every prime number less than or equal to n divides n!. The manner in which these primes and their powers appear reveals interesting and structured patterns. The aim of this paper is to study these patterns systematically and present classical results related to the prime factors of factorial numbers.

2. Preliminaries

Definition (Factorial Number)
For a natural number n, the factorial of n is defined by n! = 1×2×3×….×n. By convention,
0! = 1.

Definition (Prime Factorization)
Every integer greater than 1 can be expressed uniquely (up to order) as a product of prime numbers: $n = p_1 \times p_2 \times \ldots \times p_r$, where $p_1, p_2, \ldots, p_r$ are primes.

Prime Factors of Factorial Numbers

Proposition
Every prime number p ≤ n divides n!. every prime number between 2 and n must be a prime factor of $n!$.

3. Legendre's Formula and Prime Power Patterns

Theorem (Legendre's Formula)
Let p be a prime number. The exponent of p in the prime factorization of n! is given by

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor,$$

where └x┘ denotes the greatest integer less than or equal to x.

Example
For instance, consider 10!:
$\nu_2(10!) = \lfloor 10/2 \rfloor + \lfloor 10/4 \rfloor + \lfloor 10/8 \rfloor = 5 + 2 + 1 = 8,$
$\nu_3(10!) = \lfloor 10/3 \rfloor + \lfloor 10/9 \rfloor = 3 + 1 = 4.$
$\nu_5(10!) = \lfloor 10/5 \rfloor = 2.$
$\nu_7(10!) = \lfloor 10/7 \rfloor = 1.$

Thus,

$$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$$

Let $n = 20$. The p-adic valuations are:

Prime p	$\nu_p(20!)$
2	18
3	8
5	4
7	2
11	1
13	1
17	1
19	1

This illustrates that every prime $p \le n$ divides $n!$, while larger primes occur only with exponent one.

Observed Pattern
- Smaller primes occur with much higher powers in (n!).
- As the prime p increases, the exponent $\nu_p(n!)$ decreases.
- For primes (p > n/2), we have $\nu_p(n!) = 1$.
- Using Legendre's formula, we compute the prime factorizations of $n!$ for small values of n.

n	$n!$	Prime Factorization
5	120	$2^3 \cdot 3 \cdot 5$
6	720	$2^4 \cdot 3^2 \cdot 5$
7	5040	$2^4 \cdot 3^2 \cdot 5 \cdot 7$
8	40320	$2^7 \cdot 3^2 \cdot 5 \cdot 7$
10	3628800	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$

This table clearly demonstrates the rapid growth of powers of small primes, especially 2.

4. Patterns in Prime Factors of Factorials

Several interesting patterns emerge in the prime factorization of factorial numbers:

1) **Growth of Prime Powers**

Smaller primes such as 2 and 3 appear with much higher powers compared to larger primes. For example,

$$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$$

One of the most striking features of factorial prime factorizations is the dominance of small primes. For fixed n, the exponent $v_p(n!)$decreases rapidly as p increases. In particular,

$$v_2(n!) > v_3(n!) > v_5(n!) > \cdots$$

for sufficiently large n.

Growth Pattern of $v_2(n!)$

n	$v_2(n!)$
10	8
20	18
50	47
100	97

This supports the asymptotic estimate

$$v_2(n!) \sim n.$$

2) Monotonic Increase

The exponent of any prime p in $n!$ is a non-decreasing function of n. For any fixed prime p, the function $v_p(n!)$is monotonically increasing in n. That is,

$$v_p((n+1)!) \geq v_p(n!).$$

This follows immediately from the multiplicative definition of factorials.

3) Distribution of Primes

Every prime number less than or equal to n divides $n!$, which connects factorials closely with the distribution of prime numbers.

4) Trailing Zeros

The number of trailing zeros in $n!$depends on the exponent of 5 in its prime factorization, since factors of 2 are more abundant than factors of 5.

These patterns make factorial numbers highly composite and explain why n! is divisible by many integers.

Connections with Classical Theorems

Wilson's Theorem

A prime number p divides (p-1)! +1.

This result highlights a special relationship between primes and factorials.

Factorials and Binomial Coefficients

Binomial coefficients are defined using factorials: The integrality of binomial coefficients is closely related to the cancellation of prime powers in factorial factorizations.

5. Applications of Factorial

There are various applications of the factorial. Some of the applications of factorials are listed below:

1) **Combinatorics:** Factorials are essential in combinatorics, which is the study of counting, arrangement, and combination of objects. They are used to calculate:
   - Permutations
   - Combinations

2) **Probability:** In probability, factorials are used to determine the number of possible outcomes in experiments. For example:
   - The probability of drawing a specific hand of cards from a deck can be calculated using combinations, which involve factorials.
   - Factorials are also used in calculating probabilities in binomial distributions, where the number of ways to achieve a certain number of successes in trials is calculated using factorials.

3) **Statistics:** Factorials are used in various statistical formulas, including:
   - Binomial Coefficient
   - Poisson Distribution

4) **Mathematical Series:** Factorials are utilized in the expansion of power series, such as the Taylor and Maclaurin series. These series represent functions as infinite sums of terms calculated from the derivatives of functions at a single point, where factorials appear in the denominators.

5) **Games and Puzzles:** Factorials are used in games that involve arranging items or characters in specific orders, such as in board games and card games. The number of possible arrangements can often be calculated using factorials.

6) **Computer Programming:** Factorials are commonly implemented in programming for various applications such as:
   - Generating permutations and combinations.
   - Solving mathematical problems that require combinatorial logic.

6. Conclusion

Factorial numbers, though simple in definition, reveal rich and structured behavior in their prime factorizations. Patterns described by Legendre's formula show how prime powers are distributed within factorials, with smaller primes playing a dominant role. These insights are fundamental in number theory and have significant applications in

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combinatorics and computational mathematics. Further research may explore asymptotic behavior and connections with advanced topics such as prime number theory and analytic number theory.

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