International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2016): 79.57 | Impact Factor (2015): 6.391

Stochastic Differential Equation on Extinction Probability

Agba, M. E[1], Ugbebor, O. O[2]

Department of Mathematics, F.C.E, Obudu, Nigeria

Department of Mathematics, University of Ibadan, Ibadan, Nigeria

Abstract: ***This study was designed to develop and apply a stochastic differential equation that proves and computes the probability of extinction of a giving family using factorization method. This method was used on the stochastic birth-death process equation to determine the probability generating function. The limiting values of the generating function G(t, z) as t→ ∞, z = 0 were taken to determine the extinction probability. Forty (40) families were studied and purposefully administered (by one-on-one interaction) to heads of families using the ten (10) wards in Bekwarra Local Government Area, Cross River State to ascertain the birth rate, date rate and extinction probability.***

Keywords: Extinction probability, Stochastic differential equation, Birth-death process

1. Introduction

The probability of extinction is an important phenomenon in population dynamics and the study of evolution. The major factors affecting extinction are the birth and death respectively (Dzaan, Onah, and Kimbir, 2009; Scott, 2006; and Omotosho 2014).

Researchers have studied birth-death processes using certain mathematical models without employing the factorization method which is tractable in analysis the stochastic process in extinction probability.

2. Methodology of stochastic differential equation for birth-death process

The factorization method is used on the stochastic birth-death process equation to determine the probability generating function.

Consider the stochastic birth-death process equation given as;

$$\frac{\partial}{\partial t}p_n(t) = \lambda(n-1)p_{n-1}(t) + \mu(n+1)p_{n+1}(t) - (\lambda+\mu)np_n(t) \quad (1)$$

[Where, n is an integer and the probability for negative n is assumed to be zero, ($p_n(t) = 0$ for $n < 0$ for non-negative initial population size $n(0) > 0$), (Fugo, 2009).

The probability generating function for a probability distribution $p_n(t)$ is defined:

$$G(t, z) = \sum_{n=o}^{\infty} p_n(t)z^n \quad (2)$$

We have (2) as; $\frac{\partial}{\partial t}G(t, z) = \sum_{n=0}^{\infty} z^n \frac{\partial}{\partial t}p_n(t) \quad (3)$

But,

$$p'_n(t) = \lambda(n-1)p_{n-1}(t) + \mu(n+1)p_{n+1}(t) - (\lambda+\mu)np_n(t)$$

Then,

$$\frac{\partial G}{\partial t}(t; z) = \sum_{n=0}^{\infty} z^n p'_n(t) = \lambda \sum_{n=0}^{\infty}(n-1)p_{n-1}(t)z^n + \mu\sum(n+1)p_{n+1}(t)z^n - (\lambda+\mu)\sum np_n(t)z^n$$

$$=[(p_1(t)z^2 + 2p_2(t)z^3 + 3p_3(t)z^4) + \ldots +]$$
$$+ \mu[p_1(t)z^0 + 2p_2(t)z + 3p_3(t)z^2 + 4p_4(t)z^3 +\ldots +]$$
$$+ (\lambda+\mu)[-p_1(t)z - 2p_2(t)z^2 - 3p_3(t)z^3 - \ldots] +$$

$$= [\lambda p_1(t)z^2 + \mu p_1(t)z^0 - (\lambda+\mu)p_1(t)z]$$
$$+ [2\lambda p_2(t)z^3 + 2\mu p2(t)z - (\lambda+\mu)2p_2(t)z^2]$$
$$+ [3\lambda p_3(t)z^4 + 3\mu p_3(t)z^2 - (\lambda+\mu)3p_3(t)z^3]$$
$$= [\lambda p_1(t)z^2 + \mu p_1(t)z^0 - \lambda p_1(t)z - \mu p_1(t)z]$$
$$+ [2\lambda p_2(t)z^3 + 2\mu p_2(t)z - 2\lambda p_2(t)z^2 - 2\mu p_2(t)z^2]$$
$$+ [3\lambda p_3(t)z^4 + 3\mu p_3(t)z^2 - 3\lambda p_3(t)z^3 - 3\mu p_3(t)z^3] + \ldots \quad (4)$$

Rearrange the summation and factorise we now have:

$$\lambda_1 p_1(t)z^2 + \mu p_1(t)z^0 - \lambda p_1(t)z - \mu p_1(t)z = \lambda_1 p_1(t)z^2 + \mu p_1(t)z^0 - \lambda p_1(t)z - \mu p_1(t)z$$
$$= \lambda p_1(t)z^2 - \lambda p_1(t)z + \mu p_1(t)z^0 - \mu p_1(t)z$$
$$= \lambda p_1(t)z(z-1) + \mu p_1(t)(1-z) \quad (5)$$

Also,

$$2\lambda p_2(t)z^3 + 2\mu p_2(t)z - 2\lambda p_2(t)z^2 - 2\mu p_2(t)z^2$$
$$= 2\lambda p_2(t)z^3 - 2\lambda p_2(t)z^2 + 2\mu p_2(t)z - 2\mu p_2(t)z^2$$
$$= 2\lambda p_2(t)z^2[z-1] + 2\mu p_2(t)z[1-z] \quad (6)$$

similarly,

$$3\lambda p_3z^4 + 3\mu p_3(t)z^2 - 3\lambda p_3(t)z^3 - 3\mu p_3(t)z^3$$
$$= 3\lambda p_3(t)z^3[z-1] + 3\mu p_3(t)z^2[1-z] \quad (7)$$

Sum equations (5), (6) and (7), generated from equation (4)

$$[\lambda p_1(t)z(z-1) + \mu p_1(t)(1-z)]$$
$$+[2\lambda p_2(t)z^2(z-1) + 2\mu p_2(t)z(1-z)]$$
$$+[3\lambda p_3(t)z^3(z-1) + 3\mu p_3(t)z^2(1-z)]$$
$$= \lambda z(z-1)[p_1(t) + 2p_2(t)z + 3p_3(t)z^2] +$$
$$\mu(1-z)[p_1(t) + 2p_2(t)z + 3p_3(t)z^2] + \ldots$$
$$= [\lambda z(z-1) + \mu(1-z)][p_1(t) + 2p_2(t)z + 3p_3(t)z^2 + \ldots +]$$
$$= [\lambda z(z-1) + \mu(1-z)] \sum_{n=0}^{\infty} np_n(t)z^{n-1}$$
$$= (\lambda z^2 - \lambda z - \mu z + \mu)\sum_{n=0}^{\infty} np_n(t)z^{n-1}$$
$$\frac{\partial G}{\partial t}(t, z) = [\lambda z^2 - (\lambda z^2 + \mu)z + \mu]\sum_{n=0}^{\infty} np_n(t)z^{n-1} = \sum_{n=0}^{\infty} z_n\frac{\partial}{\partial t}p_n(t)$$

Therefore,

$$\sum_{n=0}^{\infty} z^n p'_n(t) = \{\lambda z^2 - (\lambda+\mu)z + \mu\}\sum_{n=0}^{\infty} np_n(t)z^{n-1}$$

We write,

$$[\lambda z^2 - (\lambda+\mu)z + \mu]\sum_{n=0}^{\infty} np_n(t)z^{n-1} = \frac{\partial G}{\partial t}(t, z)$$

Or

$$\frac{\partial G}{\partial t}(t, z) = [\lambda z^2 - (\lambda+\mu)z + \mu]\sum_{n=0}^{\infty} np_n(t)z^{n-1} \quad (8)$$

Recall that,

$\frac{\partial G}{\partial z}(t, z) = \sum_{n=0}^{\infty} np_n(t)z^{n-1}$, for $G(t, z) = \sum_{n=0}^{\infty} np_n(t)z^n$

Then, for the linear birth and death process in equation (8) we have;

$$\frac{\partial G}{\partial t}(t, z) = [\lambda z^2 - (\mu + \lambda)z + \mu]\frac{\partial G}{\partial z}(t, z) \qquad (9)$$

Equating to zero :

$$\frac{\partial G}{\partial t}(t, z) - [\lambda z^2 - (\mu + \lambda)z + \mu]\frac{\partial G}{\partial z}(t, z) = 0$$

Let

$$a(z,t,G) = -[\lambda z^2 - (\mu + \lambda)z + \mu]\frac{\partial G}{\partial z}(t, z)$$

$$b(z,t,G) = 1$$

$$c(z,t,G) = 0$$

The partial differential equation is of the form;

$$b\frac{\partial G}{\partial t} + a\frac{\partial G}{\partial z} = c$$

Written as,

$$b\frac{\partial G}{\partial t} + a\frac{\partial G}{\partial z} = 0$$

With the initial and boundary conditions for N(0) at time t = 0, the initial condition is $P_n(0)$ for n$\neq N(0)$ and $P_{N(0)} = 1$, that is

$$G(t, z) = \sum_n P_n(t)z^n = z^{n(0)}, \qquad (10)$$

The boundary condition determines G(t, 0) and G(t, 1) is expressed as;

$$G(t, z) = P_0 \qquad (11)$$

$$G(t, 1) = \sum_n P_n(t)z = 1 \qquad (12)$$

p_o is undermined at this point. Solving $p_n(t)$ (t) is now reduced to solving the partial differential equation of G(t, z) in (9), with condition (10), (11) and (12).

Consider the partial differential equation in G(t, z) involving $\frac{\partial G}{\partial t}(t, z)$ and $\frac{\partial G}{\partial t}(t, z)$, (Fugo, 2009):

The auxiliary equation is given as;

$$\frac{\partial x}{1} = \frac{\partial y}{1} = \frac{\partial i}{1},$$

Adopting similar pattern, we have the auxiliary equation as;

$$\frac{-\partial z}{\lambda z^{2+\mu-(\lambda+\mu)z}} = \frac{\partial t}{1} = \frac{\partial G}{0}$$

$$\frac{-\partial z}{(1-z)(\mu-\lambda z)} = \frac{\partial t}{1} = \frac{\partial G}{0}$$

Integrate the equation

$$\int \frac{1}{(1-z)(\mu-\lambda z)}\partial z = -\int 1\partial t \qquad (13)$$

First, resolve LHS of (13)

$$\frac{1}{(1-z)(\mu-\lambda z)} = \frac{A}{(1-Z)} + \frac{B}{(\mu-\lambda)}$$

Multiply all through by $(1-z)(\mu-\lambda z)$

$$1 = A(\mu-\lambda z) + B(1-z) \qquad (14)$$

putting z = 1 in (14), we have,

$$1 = A(\mu-\lambda)$$

$$A = \frac{1}{\mu-\lambda}$$

Putting z = 0,

$$1 = A\mu + B$$

$$B = 1 - A\mu$$

$$= 1 - \frac{\mu}{\mu-\lambda}$$

$$= \frac{-\lambda}{\mu-\lambda}$$

Therefore,

$$\frac{1}{(1-z)(\mu-\lambda z)} \equiv \frac{1}{\mu-\lambda}\left[\frac{1}{1-z} - \frac{\lambda}{\mu-\lambda z}\right]$$

we have (13) as,

$$\int \frac{1}{\mu-\lambda}\left(\frac{1}{1-z} - \frac{\lambda}{\mu-\lambda z}\right)\partial z = -\int \partial t$$

$$\frac{1}{\mu-\lambda}[-In(1-z) + In(\mu-\lambda z] = -t$$

$$In\left(\frac{\mu-\lambda z}{1-z}\right) = e^{-(\mu-\lambda)t}$$

$$\frac{\mu-\lambda z}{(1-z)e^{(\lambda-\mu)t}} = K_1$$

$$K_1 = \frac{(\mu-\lambda z)e^{-(\lambda-\mu)t}}{(1-z)}$$

$$\frac{\partial t}{1} = \frac{\partial G}{0}$$

$$\int 0\partial t = \int \partial G$$

$$K_2 = X$$

The general solution is,

$$G(t, z) = X(V)$$

X can be any function of,

$$V = e^{-(\lambda-\mu)t}\left(\frac{\mu-\lambda z}{1-z}\right), \qquad (15)$$

for time (t) = 0 and $\lambda \neq \mu, \sum_{n=1}^{\infty} p_n(0)$

we solve $G(t,z)$ *in the general solution*

$$G(t,z) = e^{-(\lambda-\mu)t}\left(\frac{\mu-\lambda z}{1-z}\right),$$

$$G(0,z) = x\left(\frac{\mu-\lambda z}{1-z}\right)$$

$$G(0,z) = \sum_{n=0}^{\infty} p_n(o)z^n = z^n$$

for t = 0 (15) becomes,

make z the subject in (16)

$$V(1-z) = \mu - \lambda z$$

$$V - Vz = \mu - \lambda z$$

$$\lambda z - Vz = \mu - V$$

$$z(\lambda - V) = \mu - V \qquad (17)$$

$$z = \frac{\mu-V}{\lambda-V},$$

substitute (16) into (17) for,

$$G(0, z) = X(V|_{t=0}) = z^{N(0)}$$

Yields

$$G(0, z) = X(V|_{t=0}) = \left(\frac{\mu-V|_{t=0}}{\lambda-V|_{t=0}}\right)^{N(o)} \qquad (18)$$

X is determined in it functional form by, G(t, z) = X(V) for G(t, 1) = X(0), z = 1 and t ≥ 0.

Applying the uniqueness of solution of partial differential equation, G(t, z) is evaluated, hence

$$G(t, z) = z^{N(0)}$$

$$= \left(\frac{\mu-V}{\lambda-V}\right)^{N(o)} \qquad (19)$$

Substitute (16) in (19),

$$= \frac{\mu - \frac{\mu - \lambda z}{1-z} e^{-(\lambda-\mu)t}}{\lambda - \frac{\mu - \lambda z}{1-z} e^{-(\lambda-\mu)t}}$$

$$\therefore G(t, z) = \left[\frac{\mu-(1-z)-(\mu-\lambda z)e^{-(\lambda-\mu)t}}{\lambda(1-z)-(\mu-\lambda z)e^{-(\lambda-\mu)t}}\right]^{N(o)} \qquad (20)$$

Equation (20) is the probability generating function, Van Doorn (2004).

The extinction probability is derived from (20) by letting population size n = 0 at time $p_0(t)$ and is given by G(t, 0).

In the Birth – Death process, equation (20) yields:

$$p_0(t) = G(t, 0) = \left[\frac{\mu-\mu e^{-(\lambda-\mu)t}}{\lambda-\mu e^{-(\lambda-\mu)t}}\right]^{N(o)}$$

for, $\lambda > \mu$

$$p_0(t\to\infty) = \lim_{t=\infty} G(0,t) = \left(\frac{\mu}{\lambda}\right)^{N(0)} < 1,$$

$\lambda < \mu$

$$p_0(t\to\infty) = \lim_{t=\infty} G(0,t) = 1,$$

If the birth rate is less than the death rate i.e, $\lambda < \mu$ the populations go extinct and if the birth rate is larger than the death rate $(\lambda > \mu)$ the populations can go extinct with non-zero probability $\left(\frac{\mu}{\lambda}\right)^{N(0)}$.

For stochastic process, the probability of extinction is always positive and population is never free from extinction in the stochastic world

Table 1: Invaluable information arising from Locality

S/N	WARD	WIFE	N_D	N_A	T_B	T_D	T_A	T_E	$\bar{X}$	λ	μ	P_E
1	Otukpuru	1	1	2	5	1	6	4	2.500	833	167	0.200
2	Ibiaragidi	3	0	4	16	2	18	14	1.333	889	111	0.125
3	Ugboro	2	1	2	10	1	11	9	1.430	909	91	0.100
4	Ibiaragidi	2	1	3	11	2	13	9	1.570	846	154	0.182
5	Ukpah	1	0	5	9	3	12	6	2.250	750	250	0.333
6	Nyanya	2	0	1	3	0	3	3	1.500	1000	0	0.000
7	Ukpah	3	1	3	18	2	20	16	1.286	900	100	0.111
8	Gakem	1	0	3	6	2	8	4	2.000	750	250	0.333
9	Ibiaragidi	4	3	8	17	7	24	10	2.800	708	292	0.412
10	Ukpah	3	1	3	6	1	7	5	3.00	857	143	0.170
11	Gakem	3	1	4	8	3	11	5	2.670	727	273	0.376
12	Abouchiche	4	3	8	17	7	24	10	2.800	708	292	0.412
13	Ibiaragidi	2	0	7	13	3	16	10	2.170	813	188	0.231
14	Abuochiche	3	1	11	21	3	24	18	2.100	875	125	0.143
15	Afrike 1	1	0	4	15	2	17	13	1.364	882	118	0.134
16	Nyanya	2	1	7	13	2	15	11	2.600	867	133	0.153
17	Ugboro	3	3	7	38	11	42	27	1.36	776	224	0.031
18	Abuochiche	4	0	3	12	1	13	11	1.330	923	77	0.083
19	Otukpuru	4	2	5	12	2	14	10	2.400	857	143	0.167
20	Beten	7	2	10	62	3	65	59	1.240	954	46	0.048
21	Ukpah	3	2	4	10	3	13	7	2.500	769	231	0.300
22	Nyanya	4	0	3	13	1	14	12	1.300	929	71	0.076
23	Otukpuru	3	2	4	12	5	17	7	2.400	706	294	0.416
24	Beten	2	0	0	3	0	3	3	1.000	1000	0	0.000
25	Ibiaragidi	8	0	4	12	3	15	9	1.50	800	200	0.250
26	Afrike II	3	1	3	7	2	9	5	2.330	778	222	0.285
27	Otukpuru	4	4	8	20	7	27	13	2.500	741	259	0.350
28	Beten	5	2	12	25	6	31	19	2.27	806	194	0.241
29	Afrike II	4	4	12	30	8	38	22	2.140	789	211	0.267
30	Ukpah	5	5	20	36	8	44	26	4.000	818	182	0.222
31	Afrike I	3	1	5	11	4	15	7	2.200	733	267	0.364
32	Afrike II	3	1	10	22	9	31	13	2.000	710	290	0.408
33	Nyanya	7	4	18	40	14	54	36	2.220	741	259	0.350
34	Abouchiche	4	1	7	14	4	18	10	2.300	778	222	0.129
35	Ugboro	8	3	54	70	8	78	62	5.380	897	103	0.145
36	Beten	41	1	4	10	2	12	8	2.000	833	167	0.200
37	Gakem	9	6	25	65	9	85	45	1.910	765	235	0.307
38	Afrike I	7	4	15	31	11	42	20	2.580	738	262	0.355
39	Afrike II	8	3	28	44	4	48	40	3.40	917	83	0.091
40	Beten	6	3	14	30	12	42	18	2.310	909	286	0.315

Where,
N_D = number of individuals who died without offspring, N_A = number of individuals who are alive without offspring, T_B = total number of births, T_D = total number of death, T_E = total number of events, T_A = total number of individual who are alive, $\bar{x}$ = mean number of individuals, λ= birth rate per thousands, μ = death rate per thousands and p_k = extinction probability.

Evaluating extinction probability in the first family with number of birth 5 and death 1.

Let

$$\mu = \frac{d}{b+d}.(1000)$$

Where. λ = birth rate, μ = death rate, b = number of births in the family,
d = number of death in the family.

Problem 1
Let b = 5 and d = 1 in the first family, then;

$$\mu = \frac{d}{b+d}.(1000)$$
$$= \frac{5}{5+1}.(1000)$$
$$= \frac{5}{6}.(1000)$$
$$= \frac{5000}{6}$$
$$= 833$$
$$\mu = \frac{d}{b+d}.(1000)$$
$$= \frac{1}{5+1}.(1000)$$

$$= \frac{1000}{6} = 167$$

The probability of extinction (3.3.2) for the family is given by:

$$P_E = \left(\frac{\mu}{\lambda}\right)^{n_o}$$
$$= \left(\frac{107}{833}\right)^1$$
$$= 0.200$$

$\bar{X}$= The mean number of offspring $= \frac{Total\ number\ \ of\ births}{Number\ \ of\ fertile\ \ parents}$

$$= \frac{T_B}{T_B-(N_D+N_A)}$$
$$= \frac{5}{5-3}$$
$$= 2.500$$

Problem 2
We have the second family as;

b = 16, d = 2:

Then, $\lambda = \frac{d}{b+d}.(1000)$

$$= \frac{16}{16+2}.(1000)$$
$$= \frac{16}{18}.(1000)$$
$$= \frac{16000}{18}$$
$$= 889$$
$$\mu = \frac{2}{16+2}.(1000)$$
$$= \frac{2}{18}.(1000)$$
$$= \frac{2000}{18}$$
$$= 111$$

Again the probability of extinction is given by

$$P_E = \left(\frac{\mu}{\lambda}\right)^{n_o}$$
$$= \left(\frac{111}{889}\right)^1$$
$$= 0.125$$

From the results, number of individuals that died without offspring = 0.

Number of individuals alive without offspring = 4

To evaluate the mean number, we first know the number of individuals who died without offspring (N_D) and those who are alive without offspring (N_A). let,
$N_d = 0$, $N_a = 4$.

Mean number of offspring ($\bar{x}$) = $\frac{Total\ number\ \ of\ births}{Number\ \ of\ fertile\ \ parents}$

$$= \frac{T_B}{T_B-(N_D+N_A)}$$
$$= \frac{16}{16-4}$$
$$= \frac{16}{12}$$
$$= 1.333$$

The mean number of offspring ($\bar{x}$) = 1.33

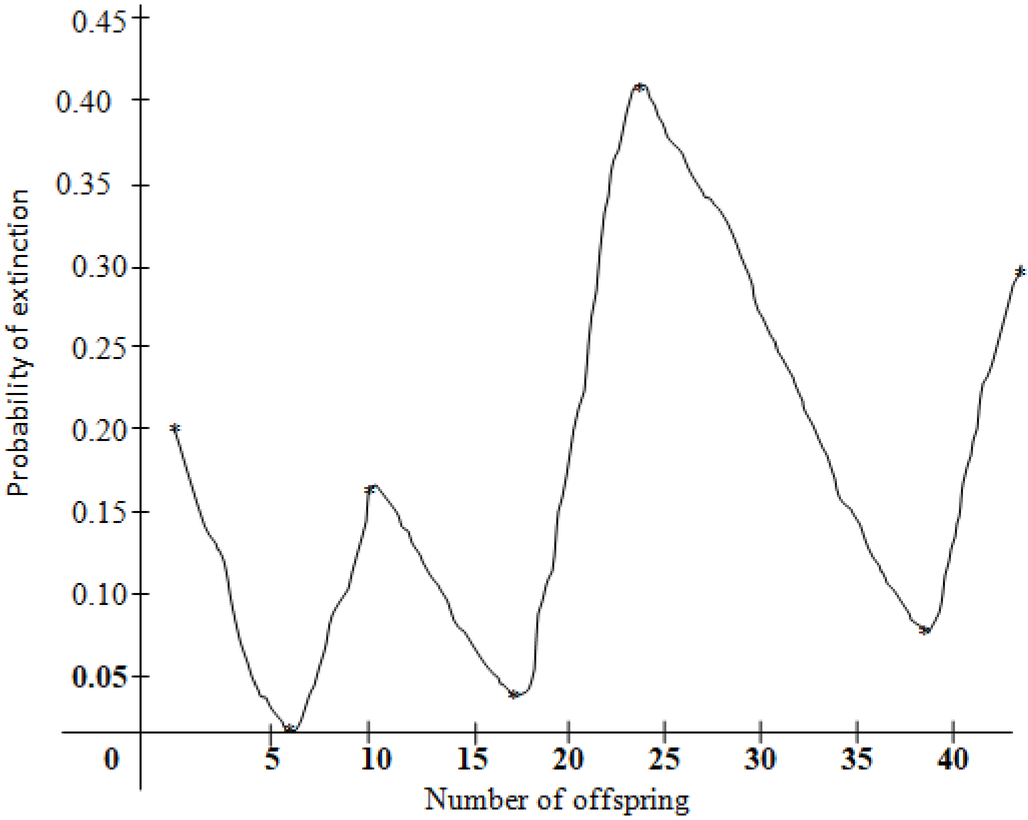

Figure 1: Graph of Extinction Probability against Number of offspring

3. Conclusion

We have developed a method on the stochastic differential equation for birth-death process to generate and solving extinction probability. The family with $p_{k_{17}} = 0.013$ has low extinction probability rate as shown in fig 1. $p_{k_{23}}$ has the highest extinction probability and killing rate. If the killing continue at this rate, death will overtake birth while $p_{k_6} = 0 = p_{24}$ *has no* killing effect since death is impossible.

References

[1] Dzaan, S., Onah, E.S and Kimbir, A. R., 2009. On the extinction probability of a family name. *Science world Journal. Www. Scienceworldjournal.org.* 4: 1597-6343.
[2] Omotosho, M., 2014. College and University Statistics. Ibadan : Yosod Book Publisher Limited.
[3] Scott. E.C., 2006. Extinction (Biology). Microsoft@ Student 2007 [DVD].Redmond, WA : Microsoft Corporation ,2006.
[4] Fugo, T., 2013. Birth- Death models. J. Appl. Prob 6, 205-218
[5] Van Doorn, E., 2004. Representation for the rate of convergence of Birth - Death process. Theory of probability. Math statistic. 65, 37-43.