

Some Results on Invariant Submanifolds in an Indefinite Trans-Sasakian Manifold-II

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Abstract: The purpose of this paper is to study invariant submanifolds in a indefinite trans-Sasakian manifold. Necessary and sufficient conditions are given on a submanifold of a indefinite trans-Sasakian manifold to be invariant and invariant case is considered. In this case further properties and some theorems are given related to an invariant submanifolds in a indefinite trans-Sasakian manifold.

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1. Introduction

In general, the geometry an invariant submanifold inherits almost all properties of the ambient manifold. In 1976 K.Yano and M.Kon introduced invariant and anti-invariant sub-manifolds in [1]. J.A.Oubina [2] introduced the notion of a trans sasakian manifold of type (α, β) . Trans sasakian manifold is an important kind of sasakian manifold such that $\alpha = 1$ and $\beta = 1$. In 1998, Xu.Xufeng and Chao Xiadi proved that every - sasakian manifold is a hyper surfaces of an indefinite Khalerian manifold and established a necessary and sufficient condition for an odd dimensional Riemannian manifold to be an - sasakian manifold [3]. In [4]U.C.De and Avijit Sarkar introduced and studied the notion of - Kenmotsu manifolds with indefinite metric with an example.

In 2010 S.S.Shukla and D.D.Singh [5]Studies -Trans sasakian manifolds. In this paper they have obtained some results on -Trans sasakian manifolds. Conditions for Indefinite trans sasakian manifolds to be D-totally geodesics, D \perp -totally geodesics, mixed totally geodesic is given by Arindam Bhattacharya and Bandana Das in [7].

Recently some basic results on locally ϕ -recurrent, locally ϕ symmetric and ϕ quasi confor- maly symmetric Indefinite tran sasakian manifolds have been obtained by Venkatesha and sumangala.B.[6]. In 2015 Shyam kishor and Abhishek Singh [8]has studied on weekly con- circular symmetric and weakly concircular Ricci symmetric 3-dimensional - tran sasakian manifold. Avijit Sarkara and Matilal Sen[10] has obtained invariant submanifolds with the second fundamental form satisfying some conditions and difference between the conditions for submanifolds of α - Sasakian and

β -Kenmotsu manifolds to be totally geodesic is shown. In this paper necessary and sufficient conditions are given on a submanifold of an indefinite trans-Sasakian manifold to be invariant and further properties and some theorems are given related to an invariant submanifolds in a indefinite trans-Sasakian manifold.

2. Preliminaries

Let M be an $(2n+1)$ -dimensional indefinite almost contact metric manifold with indefinite almost contact metric structure (ϕ, ξ, η, g) then they satisfies

$$\phi^2 = -I + \eta \otimes \xi \quad (2.1)$$

$$\eta(\xi) = 1, \phi\xi = 0$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X) \eta(Y), \quad (2.2)$$

$$g(\phi X, Y) = -g(X, \phi Y) \quad (2.3)$$

$$\epsilon g(X, \xi) = \eta(X)$$

Where X, Y are vector fields on M and $\epsilon = \pm 1$

An indefinite almost contact metric structure (ϕ, ξ, η, g) on M is called indefinite trans- Sasakian if

$$(\nabla X \phi)(Y) = \alpha[g(X, Y)\xi - \epsilon \eta(Y)X] + \beta[g(\phi X, Y)\xi - \epsilon \eta(Y)\phi X] \quad (2.4)$$

Where α and β are non zero scalar functions on \bar{M} of type $(\alpha, \beta v)$ is a Levi-civita connection on \bar{M} . In particular, an indefinite trans-Sasakian manifold is normal.

From above formula, one easily obtains

$$\nabla_X \xi = -\alpha \epsilon \phi X + \beta X - \epsilon \eta(X)\xi = \epsilon [-\alpha \phi X + \phi^2 X], \quad (2.5)$$

$$\nabla_X \eta(Y) = -\alpha g(\phi X, Y) + \beta[g(X, Y) - \epsilon \eta(X)\eta(Y)], \quad (2.6)$$

Further in an indefinite trans sasakian manifold, the following holds true,

$$R(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y]$$

Let M be an $(2m + 1)$ dimensional $(n > m)$ manifold imbedded in M. The induced metric g of M is given $g(X, Y) = g(X, Y)$ for any vector fields X, Y on M.

Let $T_X(M)$ and $T_X(M)^\perp$ denote that tangent and normal bundles of M and $x \in M$. Let ∇_X denote the Riemannian connection on M determined by the induced metric g and R denote the Riemannian curvature tensor of M. Then Gauss-Weingarten formula is given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad (2.8)$$

$$\bar{\nabla}_X N = -A_N(X) + D_X N \quad (2.9)$$

for any vector fields X, Y tangent to M and any vector field N normal to M , where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T_X(M)^\perp$. Both A and B are called the second fundamental forms of they satisfy

$$g(B(X, Y), N) = g(A_N(X, Y)).$$

3. Invariant Submanifolds of an indefinite trans-Sasakian manifold

Let M be an immersed submanifold of an indefinite trans-Sasakian manifold \bar{M} . If $\phi(T_X M) \subset T_X M$, for any point $x \in M$, then M is called an invariant submanifold of \bar{M} .

In this case, we have

$$\phi B X = B \phi X, \quad (3.1)$$

$$\phi N_i = \sum_{l=1}^t \lambda_{li} N_s, \quad (3.2)$$

$$\xi = B V + \sum_{l=1}^t \alpha_l N_s. \quad (3.3)$$

Let ∇ be the Levi-civita connection of M with respect to the induced metric g . Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X, Y), \quad (3.4)$$

$$\nabla_X N = \nabla^\perp N - A_N X. \quad (3.5)$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^\perp$. ∇^\perp is the connection in the normal bundle, h is the second fundamental form of M and A_N is the weigarten endomorphism associated with N . The second fundamental form h and the shape operator A related by,

$$g(h(X, Y), N) = g(A_N X, Y). \quad (3.6)$$

Lemma 3.1. Let M be an invariant submanifold of an indefinite trans sasakian manifold \bar{M} , then we have,

$$h(X, \xi) = 0 \quad (3.7)$$

for any $X \in T\bar{M}$.

Proof: For any indefinite trans sasakian manifold using (3.4) we get

$$\nabla_X \xi = \varepsilon[-\alpha \phi X + \beta(X - \eta(X)\xi)],$$

on the other hand, we get

$$\nabla_X \xi + h(X, \xi) = \varepsilon[-\alpha \phi X + \beta(X - \eta(X)\xi)],$$

from the gauss formula then the equation is implied that $h(X, \xi) = 0$.

Theorem 3.1. Let M be an invariant submanifolds of an indefinite trans sasakian manifold \bar{M} then we have

$$1(\phi X, Y) = \bar{\phi}(h(X, Y)) = h(X, \phi Y). \quad (3.8)$$

for any $X, Y \in \Gamma(TM)$.

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi(\bar{\nabla}_X Y),$$

$$= \nabla_X \phi Y + B(X, \phi Y) - \phi(\nabla_X Y) - \phi(B(X, Y)),$$

$$= (\nabla_X \phi)Y + B(X, \phi Y) - \phi(B(X, Y)).$$

Then we have

$$\begin{aligned} & \alpha(g(X, Y)\xi - \varepsilon \eta(Y)X) + \beta(g(\phi X, Y)\xi - \varepsilon \eta(Y)\phi X) \\ &= \alpha(g(X, Y)\xi - \varepsilon \eta(Y)X) + \beta(g(\phi X, Y)\xi - \varepsilon \eta(Y)\phi X) + B(X, \phi Y) - \phi(B(X, Y)) \end{aligned}$$

thus we get

$$B(X, \phi Y) = \phi(B(X, Y))$$

on the other hand, it follows that

$$(\nabla_Y \phi)X = (\nabla_Y \phi)X + B(Y, \phi X) - \phi(B(Y, X))$$

Hence we have $h(\phi X, Y) = \bar{\phi}(h(X, Y)) = h(X, \phi Y)$

Lemma 3.2. Let M be an invariant submanifolds of an indefinite trans sasakian manifold \bar{M} then we have

$$h(\phi X, \phi Y) = -h(X, Y) \quad (3.9)$$

Proof: From (3.8) and (2.1) we get

$$\begin{aligned} h(\phi X, \phi Y) &= \phi(h(X, \phi Y)) = \phi^2(h(X, Y)), \\ &= -h(X, Y) + \eta(h(X, Y))\xi = -h(X, Y) \end{aligned}$$

Lemma 3.3. If X is an arbitrary tangent vector of an invariant submanifold of an indefinite trans sasakian, then $\nabla_X \xi$ is also an arbitrary tangent vector different from a vector spanned by ξ .

Proof: from 2.5 we get,

$$\nabla_X \xi = \beta \varepsilon X - \alpha \varepsilon \phi X - \beta \varepsilon \eta(X)\xi \quad (3.10)$$

From the above equation we see that $\nabla_X \xi$ is a linear combination of X , ϕX , and ξ . Since an invariant submanifold of an indefinite trans sasakian manifold is also an indefinite trans sasakian manifold, the dimension of the submanifold is odd. Hence we can consider that an orthonormal ϕ -basis [9]. $\{e_i, \phi e_i, \xi\}$, $i=1, 2, 3, \dots, n$ is the dimension of the submanifold, and we can write

$$X = \sum_{i=1}^n a_i b_i + \sum_{i=1}^n b_i \phi e_i + c \xi. \quad (3.11)$$

$$\phi X = -\sum_{i=1}^n b_i e_i + \sum_{i=1}^n a_i \phi e_i. \quad (3.12)$$

Here a_i, b_i and c are scalars. From (3.10), (3.11) and (3.12) we obtain

$$\nabla_X \xi = \varepsilon \left[\sum_{i=1}^n (\beta a_i + \alpha b_i) e_i + \sum_{i=1}^n (\beta b_i - \alpha a_i) \phi e_i \right]$$

If we write $TM = D \oplus \langle \xi \rangle$, we see that $\nabla_X \xi \in D$ and, if X is arbitrary, $\nabla_X \xi$ is also arbitrary, because a_i, b_i and c are arbitrary.

If we write $TM = D \oplus \langle \xi \rangle$, we see that $\nabla_X \xi \in D$ and, if X is arbitrary, $\nabla_X \xi$ is also arbitrary, because a_i, b_i and c are arbitrary.

Theorem 3.2: Let M be an immersed submanifolds of an indefinite trans sasakian manifold \bar{M} then we have the second fundamental form h is η -parallel if

$$(\nabla_X h)(Y, Z) = \alpha[\varepsilon \eta(Y) \phi h(X, Z) + \varepsilon \eta(Z) \phi h(X, Y)] - \beta[\varepsilon \eta(Y) h(X, Z) + \varepsilon \eta(Z) h(X, Y)]$$

Proof: We know that

$$(\nabla_X h)(\phi Y, \phi Z) = (\nabla_X h)(\phi Y, \phi Z) - h(\nabla_X \phi Y, \phi Z) - h(\phi Y, \nabla_X \phi Z),$$

Then

$$(\nabla_X h)(\phi Y, \phi Z) = h(\nabla_X \phi Y, \phi Z) + h(\phi Y, \nabla_X \phi Z)$$

Using (3.9)

$$-(\nabla_X h)(Y, Z) = h[(\nabla_X \phi)Y + \phi(\nabla_X Y), \phi Z] + h[\phi Y, (\nabla_X \phi)Z + \phi(\nabla_X Z)],$$

$$-(\nabla_X h)(Y, Z) = h((\nabla_X \phi)Y, \phi Z) - h(\nabla_X Y, Z) + h(\phi Y, (\nabla_X \phi)Z) - h(Y, \nabla_X Z),$$

Thus by using (2.4) we have

$$-(\nabla_X h)(Y, Z) = -\alpha \varepsilon \eta(Y) h(X, \phi Z) - \beta \varepsilon \eta(Y) h(\phi X, \phi Z) - \alpha \varepsilon \eta(Z) h(\phi Y, X) - \beta \varepsilon \eta(Z) h(\phi Y, \phi X),$$

$$(\nabla_X h)(Y, Z) = \alpha[\varepsilon \eta(Y) \phi h(X, Z) + \varepsilon \eta(Z) \phi h(X, Y)] - \beta[\varepsilon \eta(Y) h(X, Z) + \varepsilon \eta(Z) h(X, Y)]$$

Hence the proof.

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