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Certain Properties of a New Subclass of Harmonic Univalent Bounded Turning Functions

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Abstract: ***In the present paper, the author investigates the 4^th^-order differential inequality of harmonic univalent functions with bounded turning in the unit disk. As consequences, coefficient bound, growth formulae, extreme points, convolution property, convex combinations and closure under an integral operator are gained.***

Keywords: Univalent function, Harmonic function, bounded turning, growth bound

1. Introduction

Harmonic functions play a significant role in a variety of problems in engineering, physics and applied mathematics. In geometric function theory (GFT), harmonic univalent functions have raised the interest of numerous complex analysts since the mid-1980s, which are a generalization of the holomorphic (analytic) functions. The first study of the theory of harmonic univalent functions was by Clunie and Sheil-Small [1] in 1984. In their studies, they discussed every harmonic function φ in a simply connected domain can be expressed in the form $\varphi=\mu+\bar{\nu}$, where the functions μ and ν are holomorphic in $\Delta=\{z\in C:|z|<1\}$. The function μ is called the holomorphic part while ν is the co-holomorphic part of φ. A necessary and sufficient condition [1] for φ to be locally univalent and sense preserving in $\Delta=\{z\in C:|z|<1\}$ is for $|\mu'(z)|>|\nu'(z)|$ in Δ.

Clunie and Sheil-Small [1] also introduced a class H of harmonic functions $\varphi=\mu+\bar{\nu}$ that are univalent, sense-preserving in the unit disk Δ, and normalized by the conditions $\varphi(0)=\varphi'(0)-1=0$, where the functions μ and ν are of the form:

$$\mu(z)=z+\sum_{n=2}^{\infty}\rho_n z^n \text{ and } \nu(z)=z+\sum_{n=2}^{\infty}\sigma_n z^n,\ |\sigma_n|<1. \tag{1}$$

Moreover, they investigates its geometric properties, that includes coefficient bounds, growth and distortion theorems. Note that for the co-holomorphic part $\nu=0$, the class H reduces to the class S of normalized holomorphic univalent functions for which φ can be expressed as: [1]

$$\mu(z)=z+\sum_{n=2}^{\infty}\rho_n z^n,\quad (z\in\Delta). \tag{2}$$

Further, let A denote the class of functions φ of the form (2) which are holomorphic in the unit disk Δ. For $0\le\xi<1$, let $\mathrm{B}(\zeta)$ be the subclass of A consisting of all functions that satisfy

$$\Re\{\varphi'(z)\}>\zeta,\quad (z\in\Delta). \tag{3}$$

The functions in $\mathrm{B}(\zeta)$ are called functions of bounded turning (also called functions whose derivatives has positive real parts). It is known that functions of bounded turning are univalent in the unit disk Δ, [2].

In 2003, Yalcin et al. [3] introduced a more general subclass of harmonic univalent functions (including $\mathrm{B}(\zeta)$ as a special case). This is the subclass $\mathrm{HP}(\zeta)$ consisting of functions $\varphi\in\mathrm{H}$ that satisfy first-order differential inequality:

$$\Re\{\mu'(z)+\nu'(z)\}>\zeta,\quad (0\le\xi<1, z\in C). \tag{4}$$

Also, let $\mathrm{HP}^*(\zeta)=\mathrm{HP}(\zeta)\cap\mathrm{N_H}$, where $\mathrm{N_H}$ [4] is the subclass of H such that the functions μ and ν in $\varphi=\mu+\bar{\nu}$ are of the form:

$$\mu(z)=z-\sum_{n=2}^{\infty}\rho_n z^n \text{ and } \nu(z)=z-\sum_{n=2}^{\infty}\sigma_n z^n,\quad |\sigma_n|<1. \tag{5}$$

Furthermore, they studied a sufficient condition

$$\sum_{n=2}^{\infty}n(|\rho_n|+|\sigma_n|)\le 2-\zeta, \tag{6}$$

where $\sigma_1=1$, for functions to be in the subclass $\mathrm{HP}(\zeta)$. This condition is necessary when the coefficients are negative. Growth theorem and extreme points are also derived.

Note that the condition (4) slightly modifies the one given originally in (3). Note also that the subclass $\mathrm{B}(\zeta)$ corresponds to $\nu=0$, [3]. Since then, various studies were conducted on a subclass $\mathrm{HP}(\zeta)$. Pursuing this line of study will be presented here.

In 2004, Yalcin and Ozturk [4] considered a subclass $\Phi(\lambda)$ consists of functions $\varphi\in\mathrm{H}$ satisfying second-order differential inequality as:

$$\Re\{\lambda z(\mu''(z)+\nu''(z))+\mu'(z)+\nu'(z)\}>0, \tag{7}$$

where $0\le\lambda$. Further, they discussed a sufficient condition

$$\sum_{n=2}^{\infty}n(1+\lambda(n-1))(|\rho_n|+|\sigma_n|)\le 2, \tag{8}$$

where $\sigma_1=1$, for functions involving to an aforementioned subclass $\Phi(\lambda)$, which is shown to be necessary when the coefficients are negative. They analyzed growth bounds and extreme points as well.

In 2010, based on the study of Yalcin and Ozturk [4], Chandrashekar et al. [5] introduced a subclass $\Phi(\lambda,\zeta)$ consists of functions $\varphi \in \mathrm{H}$ satisfying the following condition

$$\Re\{\lambda z(\mu''(z)+\nu''(z))+\mu'(z)+\nu'(z)\}>\zeta, \qquad (9)$$

where, $0 \le \lambda$ and $0 \le \zeta < 1$.. They also studied a sufficient condition

$$\sum_{n=2}^{\infty} n(1+\lambda(n-1))(|\rho_n|+|\sigma_n|) \le 2-\zeta, \qquad (10)$$

where $\sigma_1 = 1$, for functions including to above subclass $\Phi(\lambda,\zeta)$, which is shown to be necessary when the coefficients are negative. Note that $\Phi(0,\zeta)=\mathrm{HP}(\zeta)$. Further, the subclass $\Phi(\lambda,\zeta)$ reduces to $\mathrm{B}(\zeta)$ if $\lambda=\nu=0$.

In 2015, Sokol et al. [6] imposed a subclass $\Psi(\lambda,\zeta)$ consists of functions $\varphi \in \mathrm{H}$ satisfying third-order differential equation as:

$$\Re\{\lambda z^2(\mu'''(z)+\nu'''(z))+3\lambda z(\mu''(z)+\nu''(z))+\mu'(z)+\nu'(z)\}>\zeta, \qquad (11)$$

where, $0 \le \lambda$ and $0 \le \zeta < 1$. They examined a sufficient condition

$$\sum_{n=2}^{\infty} n(1+\lambda(n^2-1))(|\rho_n|+|\sigma_n|) \le 2-\zeta, \qquad (12)$$

where $\sigma_1 = 1$, for functions belonging to above subclass $\Psi(\lambda,\zeta)$, which is shown to be necessary when the coefficients are negative. Moreover, growth bounds, extreme points, convolution and convex combinations are studied. Note that $\Psi(0,\zeta)=\mathrm{HP}(\zeta)$. Also, the subclass $\Psi(\lambda,\zeta)$ reduces to $\mathrm{B}(\zeta)$ if $\lambda=\nu=0$.

Motivated by previous works on harmonic functions, we establish a new subclass $\Sigma(\lambda,\zeta)$ of functions $\varphi \in \mathrm{H}$ satisfying fourth-order differential inequality. In addition, coefficient bound, growth bound, extreme points, convolution, convex combinations, and closure under an integral operator are also discussed for harmonic functions satisfying the subclass $\Sigma(\lambda,\zeta)$. Harmonic functions with negative coefficients are also considered in this investigation.

2. Geometric Results

This section is composed of two subsections. Subsection 2.1 presents a new subclass $\Sigma(\lambda,\zeta)$ of harmonic univalent functions with bounded turning in Δ. Subsection 2.2 provides some geometric properties involving coefficient condition, growth theorem, extreme points, convolution, convex combinations, and closure under an integral operator for this considered subclass.

2.1 Subclasses $\Sigma(\lambda,\zeta)$

The part is devoted to define new subclasses $\Sigma(\lambda,\zeta)$ and $\Sigma^*(\lambda,\zeta)$ of harmonic univalent functions with positive and negative coefficients respectively in the open unit disk that satisfy fourth-order differential inequality.

Definition 2.1.1 A function $\varphi \in \mathrm{H}$ is said to be in subclass $\Sigma(\lambda,\zeta)$ [the subclass of harmonic univalent bounded turning functions] if it satisfies the following inequality:

$$\Re\left\{\begin{array}{l}\lambda z^3(\mu''''(z)+\nu''''(z))+6\lambda z^2(\mu'''(z)+\nu'''(z))\\ +7\lambda z(\mu''(z)+\nu''(z))+(\mu'(z)+\nu'(z))\end{array}\right\}>\zeta, \qquad (13)$$

where, $\sigma_1 = 1$, $0 \le \lambda$, $0 \le \zeta < 1$, and $z \in \Delta$.

Also denote by $\Sigma^*(\lambda,\zeta)=\Sigma(\lambda,\zeta)\cap \mathrm{N_H}$ where $\mathrm{N_H}$ is the subclass of harmonic univalent functions φ with negative coefficients given in (5).

Remark 2.1.1 We note that

1) For $\lambda = 0$ in (13), the subclass $\Sigma(\lambda,\zeta)$ reduces to the subclass $\mathrm{HP}(\zeta)$ of harmonic univalent bounded turning functions defined in (4).
2) For $\lambda=\nu=0$ in (13), the subclass $\Sigma(\lambda,\zeta)$ reduces to the earlier subclass $\mathrm{B}(\zeta)$ of bounded turning functions introduced in (3).

2.2 Basic Properties of Subclass $\Sigma(\lambda,\zeta)$

In this subsection, a sufficient coefficient condition for functions included in the subclass $\Sigma(\lambda,\zeta)$ is determined. This condition is also shown to be necessary when the coefficients are negative, which leads to growth formulae, extreme points, convolution, convex combinations, and closure under an integral operator.

The first theorem gives a necessary and sufficient condition for a function φ to be in the subclass $\Sigma(\lambda,\zeta)$.

Theorem 2.2.1 Let $\varphi=\mu+\bar{\nu}$ be of the form (1). If

$$\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))|\rho_n| + \sum_{n=1}^{\infty} n(1+\lambda(n^3-1))|\sigma_n| \le 1-\zeta, \qquad (14)$$

or

$$\sum_{n=1}^{\infty} n(1+\lambda(n^3-1))\cdot(|\rho_n|+|\sigma_n|) \le 2-\zeta, \qquad (15)$$

where, $\sigma_1 = 1$, $0 \le \lambda$, $0 \le \zeta < 1$, and $z \in \Delta$, then φ is harmonic univalent, sense-preserving in Δ, and $\varphi \in \Sigma(\lambda,\zeta)$.

Proof. Suppose $z_1, z_2 \in \Delta$ such that $z_1 \ne z_2$, then

$$\left|\frac{\varphi(z_1)-\varphi(z_2)}{\mu(z_1)-\mu(z_2)}\right| \ge 1-\left|\frac{\mu(z_1)-\mu(z_2)}{\nu(z_1)-\nu(z_2)}\right| = 1-\left|\frac{\sum_{n=1}^{\infty}\sigma_n(z_1^n-z_2^n)}{(z_1-z_2)-\sum_{n=2}^{\infty}\rho_n(z_1^n-z_2^n)}\right|$$

$$>1-\frac{\sum_{n=1}^{\infty} n|\sigma_n|}{1-\sum_{n=2}^{\infty} n|\rho_n|} \ge 1-\frac{\sum_{n=1}^{\infty}\frac{n(1+\lambda(n^3-1))}{1-\zeta}|\sigma_n|}{1-\sum_{n=2}^{\infty}\frac{n(1+\lambda(n^3-1))}{1-\zeta}|\rho_n|} \ge 0. \qquad (16)$$

Hence, $|\varphi(z_1)-\varphi(z_2)|>0$ and φ is univalent in Δ. To prove φ locally univalent and sense-preserving in Δ, it is enough to show that $|\mu'(z)|>|\nu'(z)|$.

$$
\begin{aligned}
|\mu'(z)| &\ge 1-\sum_{n=2}^{\infty} n|\rho_n||z|^{n-1} \\
&> 1-\sum_{n=2}^{\infty} n|\rho_n| \ge 1-\zeta-\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))|\rho_n| \\
&\ge \sum_{n=2}^{\infty} n(1+\lambda(n^3-1))|\sigma_n| \\
&> \sum_{n=1}^{\infty} n|\sigma_n||z|^{n-1}=|\nu'(z)|.
\end{aligned} \tag{17}
$$

Utilizing the fact $\Re\{\omega\}>\zeta$ if and only if $|1-\zeta+\omega|>|1+\zeta-\omega|$, it is sufficient to show that

$$
\left|(1-\zeta)+\lambda z^3(\mu''''+\nu'''')+6\lambda z^2(\mu'''+\nu''')+7\lambda z(\mu''+\nu'')+(\mu'+\nu')\right|-\left|(1+\zeta)-\lambda z^3(\mu''''+\nu'''')-6\lambda z^2(\mu'''+\nu''')-7\lambda z(\mu''+\nu'')+(\mu'+\nu')\right|\ge 0. \tag{18}
$$

in proving $\varphi\in\Sigma(\lambda,\zeta)$. Substituting for $\mu(z)$ and $\nu(z)$ in (18) yields,

$$
\begin{aligned}
&\left|(2-\zeta)+\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))\rho_n z^{n-1}+\sum_{n=1}^{\infty} n(1+\lambda(n^3-1))\sigma_n z^{n-1}\right| \\
&-\left|\zeta-\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))\rho_n z^{n-1}-\sum_{n=1}^{\infty} n(1+\lambda(n^3-1))\sigma_n z^{n-1}\right| \\
&\ge 2\left[(1-\zeta)-|z|\left[\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))|\rho_n|+\sum_{n=1}^{\infty} n(1+\lambda(n^3-1))|\sigma_n|\right]\right] \\
&=2(1-\zeta)(1-|z|)>0,
\end{aligned} \tag{19}
$$

by the condition (14). So the proof is complete..

Remark 2.2.1 The harmonic function

$$
\varphi(z)=z+\sum_{n=2}^{\infty}\frac{1}{n(1+\lambda(n^3-1))}A_n z_n+\sum_{n=1}^{\infty}\frac{1}{n(1+\lambda(n^3-1))}B_n z_n, \tag{20}
$$

where $\sum_{n=2}^{\infty}|A_n|+\sum_{n=1}^{\infty}|B_n|=1$, shows that the coefficient bound given by (14) is sharp. The functions of from (20) are in subclass $\Sigma(\lambda,\zeta)$ because condition (14) can be satisfied as follows:

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\left[\frac{n(1+\lambda(n^3-1))}{1-\zeta}\right]|\rho_n|+\sum_{n=1}^{\infty}\left[\frac{n(1+\lambda(n^3-1))}{1-\zeta}\right]|\sigma_n| \\
&=\sum_{n=2}^{\infty}|A_n|+\sum_{n=1}^{\infty}|B_n|=1.
\end{aligned} \tag{21}
$$

If we $\lambda=0$ in Theorem 2.2.1, then we yield the following result introduced by Yalcin et al. [3].

Corollary 2.2.1 Let $\varphi=\mu+\bar{\nu}$ be given by (1). If $\sum_{n=2}^{\infty} n\cdot(|\rho_n|+|\sigma_n|)\le 2-\zeta$ where, $\sigma_1=1$, $0\le\lambda$, $0\le\zeta<1$, and $z\in\Delta$, then φ is harmonic univalent, sense-preserving in Δ, and $HP(\zeta)$.

In the following outcome, it is shown that the condition (14) is also necessary for functions $\varphi=\mu+\bar{\nu}$ where μ and ν of the form (5).

Theorem 2.2.2 Let $\varphi=\mu+\bar{\nu}$ be of the form (5). Then $\varphi\in\Sigma^*(\lambda,\zeta)$ if and only if condition (14) is achieved and it is as follows:

$$
\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))|\rho_n|+\sum_{n=1}^{\infty} n(1+\lambda(n^3-1))|\sigma_n|\le 1-\zeta,
$$

where, $\sigma_1=1$, $0\le\lambda$, $0\le\zeta<1$.

Proof. Considering that $\Sigma^*(\lambda,\zeta)\subset\Sigma(\lambda,\zeta)$, we only need to prove the necessary part of the theorem. Assume that $\varphi\in\Sigma^*(\lambda,\zeta)$, then by virtue of (13), we gain

$$
\Re\left\{(1-\zeta)-\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))|\rho_n|-\sum_{n=1}^{\infty} n(1+\lambda(n^3-1))|\sigma_n|\right\}\ge 0. \tag{22}
$$

The above required condition (22) must hold for all values of z in Δ. Upon choosing the values of z on the positive real axis where $0<z=r<1$, we must have

$$
\begin{aligned}
&(1-\zeta)-\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))|\rho_n|r^{n-1} \\
&-\sum_{n=1}^{\infty} n(1+\lambda(n^3-1))|\sigma_n|r^{n-1}\ge 0.
\end{aligned} \tag{23}
$$

Letting $r\to 1^-$ through real values, it follows that

$$
(1-\zeta)-\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))|\rho_n|-\sum_{n=1}^{\infty} n(1+\lambda(n^3-1))|\sigma_n|\ge 0. \tag{24}
$$

Therefore, we yield

$$
\sum_{n=2}^{\infty} n(1+\lambda(n^3-1))|\rho_n|+\sum_{n=1}^{\infty} n(1+\lambda(n^3-1))|\sigma_n|\le 1-\zeta, \tag{25}
$$

which is the required condition.

Remark 2.2.1 The harmonic function

$$
\varphi(z)=z-\sum_{n=2}^{\infty}\frac{1}{n(1+\lambda(n^3-1))}A_n z_n-\sum_{n=1}^{\infty}\frac{1}{n(1+\lambda(n^3-1))}B_n z_n, \tag{26}
$$

where $0\le\lambda$, $0\le\zeta<1$ and $\sum_{n=2}^{\infty}|A_n|+\sum_{n=1}^{\infty}|B_n|\le 1$ belongs to the subclass $\Sigma^*(\lambda,\zeta)$.

The following result gives the upper and lower bound formula (growth formulae) for functions in $\Sigma^*(\lambda,\zeta)$.

Theorem 2.2.3 Let $\varphi \in \Sigma^*(\lambda,\zeta)$. Then $r=|z|<1$

$$|\varphi(z)| \le (1+|\sigma_1|)r + \left(\frac{1-\zeta}{2(1+7\lambda)}\right)\left[1-\frac{|\sigma_1|}{1-\zeta}\right]r^2, \qquad (27)$$

and

$$|\varphi(z)| \ge (1+|\sigma_1|)r - \left(\frac{1-\zeta}{2(1+7\lambda)}\right)\left[1-\frac{|\sigma_1|}{1-\zeta}\right]r^2 \qquad (28)$$

where $0 \le \lambda$, $0 \le \zeta < 1$.

Proof. Let $\varphi \in \Sigma^*(\lambda,\zeta)$. Taking the absolute value of φ, we have

$$|\varphi(z)| \le (1+|\sigma_1|)r + \sum_{n=2}^{\infty}(|\rho_n|+|\sigma_n|)r^n$$

$$\le (1+|\sigma_1|)r + r^2\sum_{n=2}^{\infty}\left(\frac{1-\zeta}{2(1+7\lambda)}\right)\cdot\left(\frac{2(1+7\lambda)}{1-\zeta}(|\rho_n|+|\sigma_n|)\right)$$

$$\le (1+|\sigma_1|)r + r^2\sum_{n=2}^{\infty}\left(\frac{1-\zeta}{2(1+7\lambda)}\right)\cdot\left(\frac{n(1+\lambda(n^3-1))}{1-\zeta}(|\rho_n|+|\sigma_n|)\right)$$

$$\le (1+|\sigma_1|)r + \left(\frac{1-\zeta}{2(1+7\lambda)}\right)\left[1-\frac{|\sigma_1|}{1-\zeta}\right]r^2.$$

(29)

And

$$|\varphi(z)| \ge (1+|\sigma_1|)r - \sum_{n=2}^{\infty}(|\rho_n|+|\sigma_n|)r^n$$

$$\ge (1+|\sigma_1|)r - r^2\sum_{n=2}^{\infty}\left(\frac{1-\zeta}{2(1+7\lambda)}\right)\cdot\left(\frac{2(1+7\lambda)}{1-\zeta}(|\rho_n|+|\sigma_n|)\right)$$

$$\ge (1+|\sigma_1|)r - r^2\sum_{n=2}^{\infty}\left(\frac{1-\zeta}{2(1+7\lambda)}\right)\cdot\left(\frac{n(1+\lambda(n^3-1))}{1-\zeta}(|\rho_n|+|\sigma_n|)\right)$$

$$\ge (1+|\sigma_1|)r - \left(\frac{1-\zeta}{2(1+7\lambda)}\right)\left[1-\frac{|\sigma_1|}{1-\zeta}\right]r^2.$$

(30)

Next we determine the extreme points of closed convex hulls of $\Sigma^*(\lambda,\zeta)$ denoted by $co\overline{\Sigma}^*(\lambda,\zeta)$.

Theorem 2.2.4 A function $\varphi \in co\overline{\Sigma}^*(\lambda,\zeta)$ if and only if

$$\varphi(z) = \sum_{n=1}^{\infty}(X_n\rho_n(z) + Y_n\sigma_n(z)), \qquad (31)$$

where, $\varphi_1(z) = z$,

$$\rho_n(z) = z - \frac{1-\zeta}{n(1+\lambda(n^3-1))}z^n, \quad (n=2,3,4\ldots),$$

$$\sigma_n(z) = z - \frac{1-\zeta}{n(1+\lambda(n^3-1))}\overline{z}^n, \quad (n=1,2,3\ldots),$$

and $\sum_{n=1}^{\infty}(X_n+Y_n)=1$, $X_n \ge 0$ and $Y_n \ge 0$.

Proof. For functions $\varphi = \mu + \overline{\nu}$, where μ and ν are given by (5), we have

$$\varphi(z) = \sum_{n=1}^{\infty}(X_n\rho_n(z) + Y_n\sigma_n(z))$$

$$= \sum_{n=1}^{\infty}(X_n+Y_n)z - \sum_{n=2}^{\infty}\frac{1-\zeta}{n(1+\lambda(n^3-1))}X_n z^n$$

$$- \sum_{n=1}^{\infty}\frac{1-\zeta}{n(1+\lambda(n^3-1))}Y_n\overline{z}^n$$

$$= z - \sum_{n=2}^{\infty}\frac{1-\zeta}{n(1+\lambda(n^3-1))}X_n z^n - \sum_{n=1}^{\infty}\frac{1-\zeta}{n(1+\lambda(n^3-1))}Y_n\overline{z}^n.$$

(32)

Therefore, in view of Theorem 2.2.2, we acquire

$$\sum_{n=2}^{\infty}\frac{n(1+\lambda(n^3-1))}{1-\zeta}|\rho_n| + \sum_{n=1}^{\infty}\frac{n(1+\lambda(n^3-1))}{1-\zeta}|\sigma_n|$$

$$\le \sum_{n=2}^{\infty}X_n + \sum_{n=1}^{\infty}Y_n = 1 - X_1 \le 1.$$

(33)

Thus, $\varphi \in co\overline{\Sigma}^*(\lambda,\zeta)$.

Conversely, suppose that $\varphi \in co\overline{\Sigma}^*(\lambda,\zeta)$. Setting

$$X_n(z) = \frac{n(1+\lambda(n^3-1))}{1-\zeta}|A_n|, \quad 0 \le X_n \le 1 \quad (n=2,3,4\ldots),$$

and

$$Y_n(z) = \frac{n(1+\lambda(n^3-1))}{1-\zeta}|B_n|, \quad 0 \le Y_n \le 1 \quad (n=1,2,3\ldots),$$

and $X_1 = 1 - \sum_{n=2}^{\infty}X_n - \sum_{n=1}^{\infty}Y_n$. Hence, φ can be written as

$$\varphi(z) = z - \sum_{n=2}^{\infty}|A_n|z^n - \sum_{n=1}^{\infty}|B_n|\overline{z}^n$$

$$= z - \sum_{n=2}^{\infty}\frac{1-\zeta}{n(1+\lambda(n^3-1))}X_n z^n - \sum_{n=1}^{\infty}\frac{1-\zeta}{n(1+\lambda(n^3-1))}Y_n\overline{z}^n$$

$$= \sum_{n=1}^{\infty}(X_n\rho_n(z) + Y_n\sigma_n(z)),$$

(36)

as required.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

$$\varphi(z) = z - \sum_{n=2}^{\infty}|\rho_n|z^n - \sum_{n=2}^{\infty}|\sigma_n|\overline{z}^n, \qquad (37)$$

and

$$\psi(z) = z - \sum_{n=2}^{\infty}|A_n|z^n - \sum_{n=2}^{\infty}|B_n|\overline{z}^n, \qquad (38)$$

the convolution of $\varphi(z)$ and $\psi(z)$ is given by

$$(\varphi * \psi)(z) = z - \sum_{n=2}^{\infty}|\rho_n||A_n|z^n - \sum_{n=2}^{\infty}|\sigma_n||B_n|\overline{z}^n.$$

(39)

Utilizing this definition, we show that the subclass $\Sigma^*(\lambda,\zeta)$ is closed under convolution.

Theorem 2.2.5 For $0 \le \beta \le \zeta < 1$, let $\varphi \in \Sigma^*(\lambda,\zeta)$ and $\psi \in \Sigma^*(\lambda,\beta)$. Then $\varphi * \psi \in \Sigma^*(\lambda,\zeta) \subset \Sigma^*(\lambda,\beta)$.

Proof. Since $\varphi \in \Sigma^*(\lambda,\zeta)$ and $\psi \in \Sigma^*(\lambda,\beta)$, the coefficient of $\varphi * \psi$ must satisfy the required condition given in Theorem 2.2.2. For $\psi \in \Sigma^*(\lambda,\beta)$ we note that $|A_n| \le 1$ and $|B_n| \le 1$. Now, for the convolution function $\varphi * \psi$, we yield

$$\sum_{n=2}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\beta} |\rho_n||A_n| + \sum_{n=1}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\beta} |\sigma_n||B_n|$$
$$\le \sum_{n=1}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\beta} |\rho_n| + \sum_{n=1}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\beta} |\sigma_n|$$
$$\le \sum_{n=2}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} |\rho_n| + \sum_{n=1}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} |\sigma_n|$$
$$\le 1, \tag{40}$$

Since $0 \le \beta \le \zeta < 1$ and $\varphi \in \Sigma^*(\lambda,\zeta)$. Therefore $\varphi * \psi \in \Sigma^*(\lambda,\zeta) \subset \Sigma^*(\lambda,\beta)$.

Now we show that $\Sigma^*(\lambda,\zeta)$ is closed under convex combinations of its member.

Theorem 2.2.6 The subclass $\Sigma^*(\lambda,\zeta)$ is closed under convex combinations.

Proof. For $i = 1,2,\ldots.,$ suppose that $\varphi_i \in \Sigma^*(\lambda,\zeta)$ where

$$\varphi_i(z) = z - \sum_{n=2}^{\infty} |\rho_{n,i}| z^n - \sum_{n=2}^{\infty} |\sigma_{n,i}| \bar{z}^n . \tag{41}$$

Then, by Theorem 2.2.2, we gain

$$\sum_{n=2}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} |\rho_{n,i}| + \sum_{n=1}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} |\sigma_{n,i}| \le 1. \tag{42}$$

For $\sum_{i=1}^{\infty} k_i = 1$, $0 \le k_i \le 1$, the convex combination of φ_i may be written as

$$\sum_{i=1}^{\infty} k_i \varphi_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} k_i |\rho_{n,i}| \right) z^n - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} k_i |\sigma_{n,i}| \right) \bar{z}^n. \tag{43}$$

Utilizing the Theorem 2.2.2, it follows that

$$\sum_{n=2}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} \left(\sum_{i=1}^{\infty} k_i |\rho_{n,i}| \right) + \sum_{n=1}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} \left(\sum_{i=1}^{\infty} k_i |\sigma_{n,i}| \right)$$
$$\le \sum_{i=1}^{\infty} k_i \left(\sum_{n=2}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} |\rho_{n,i}| + \sum_{n=1}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} |\sigma_{n,i}| \right)$$
$$\le \sum_{i=1}^{\infty} k_i = 1 \tag{44}$$

and therefore $\sum_{i=1}^{\infty} k_i \varphi_i \in \Sigma^*(\lambda,\zeta)$.

Now, we examine a closure property of the subclass $\Sigma^*(\lambda,\beta)$ under the generalized Bernardi-Libera-Livingston integral operator $F(z)$

$$F(z) = (\eta+1)\int_0^1 t^{\eta-1}\varphi(tz)dt, \quad (\eta > -1) \tag{45}$$

which was defined by Bernardi [7].

Theorem 2.2.7 Let $\varphi \in \Sigma^*(\lambda,\zeta)$. Then $F \in \Sigma^*(\lambda,\zeta)$.

Proof. Let

$$\varphi(z) = z - \sum_{n=2}^{\infty} |\rho_n| z^n - \sum_{n=2}^{\infty} |\sigma_n| \bar{z}^n . \tag{46}$$

Then

$$F(z) = (\eta+1)\int_0^1 t^{\eta-1}\varphi(tz)dt$$

$$F(z) = (\eta+1)\int_0^1 t^{\eta-1}\left((tz) - \sum_{n=2}^{\infty} |\rho_n| (tz)^n - \overline{\sum_{n=2}^{\infty} |\sigma_n| (tz)^n} \right) dt$$
$$= z - \sum_{n=2}^{\infty} A_n z^n - \overline{\sum_{n=1}^{\infty} B_n z^n} \tag{47}$$

where $A_n = \frac{\eta+1}{\eta+n} |\rho_n|$ and $B_n = \frac{\eta+1}{\eta+n} |\sigma_n|$.

Therefore, since $\varphi \in \Sigma^*(\lambda,\zeta)$,

$$\sum_{n=2}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} \left(\frac{\eta+1}{\eta+n} |\rho_n| \right) + \sum_{n=1}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} \left(\frac{\eta+1}{\eta+n} |\sigma_n| \right)$$
$$\le \sum_{t=1}^{\infty} k_t \left(\sum_{n=2}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} |\rho_n| + \sum_{n=1}^{\infty} \frac{n(1+\lambda(n^3-1))}{1-\zeta} |\sigma_n| \right) \le 1. \tag{48}$$

Thus by Theorem 2.2.2, $F \in \Sigma^*(\lambda,\zeta)$.

3. Conclusion

In this study, we have discussed new subclass of harmonic univalent bounded turning functions in the open unit. Differential inequality of 4th-order is suggested in this work. This inequality has been shown the coefficient condition , growth bounds, extreme points, convolution property, convex linear combination and a class-preserving integral operator.

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