On \((3, k)\)-Regular Fuzzy Graphs

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Abstract: In this paper, we defined \(d_3\)-degree of a vertex in fuzzy graphs, total \(d_3\)-degree of a vertex in fuzzy graphs, \((3, k)\)-regular fuzzy graphs and totally \((3, k)\) - regular fuzzy graphs. \((3, k)\)-regular fuzzy graphs and totally \((3, k)\)-regular fuzzy graphs are compared through various examples. A necessary and sufficient condition under which they are equivalent is provided. Also, \((3, k)\)-regularity on some fuzzy graphs whose underlying crisp graphs are a path on six vertices \(P_6\), a Corona graph \(C_n \ast K_1(n \geq 5)\), a Wagner graph and a cycle \(C_n(n \geq 5)\) is studied with some membership function. Some properties of \((3, k)\)-regular fuzzy graphs studied and they are examined for totally \((3, k)\)-regular fuzzy graphs.

Keywords: \(d_3\), degree of a vertex in fuzzy graphs, total \(d_1\)- degree of a vertex in fuzzy graphs, \((3, k)\)-regular fuzzy graphs, totally \((3, k)\)-regular fuzzy graphs.

1. Introduction

Azriel Rosenfeld introduced fuzzy graphs in 1975. It has been growing fast and has numerous applications in various fields. Nagoor Gani and Radha introduced regular fuzzy graphs, total degree and totally regular fuzzy graphs, we call it as \((3, k)\)-regular graphs and studied some properties on \((3, k)\)-regular graphs. N.R. Santhi Maheswari and C. Sekar introduced \(d_3\) of a vertex in graph and also discussed some properties on \(d_3\) of a vertex in graphs. In this paper, we define \(d_3\) -degree of a vertex in fuzzy graphs and totally \(d_1\)- degree of a vertex in fuzzy graphs and introduce \((3, k)\)-regular fuzzy graphs, totally \((3, k)\)-regular fuzzy graphs. We make comparative study between \((3, k)\)-regular fuzzy graphs and totally \((3, k)\)-regular fuzzy graphs. We provide a necessary and sufficient condition under which they become equivalent. A characterization of \((3, k)\)-regular graphs on a path on six vertices \(P_6\), a Corona graph \(C_n \ast K_1(n \geq 5)\), a Wagner graph and a cycle \(C_n(n \geq 5)\) is provided.

We present some known definitions and results for a ready reference to go through the work presented in this paper.

2. Preliminaries

Definition 2.1: For a given graph \(G\), the \(d_1\)-degree of a vertex \(v\) in \(G\), denoted by \(d_1(v)\) means number of vertices at a distance three away from \(v\).

Definition 2.2: A graph \(G\) is said to be \((3, k)\)-regular (\(d_1\) - regular) if \(d_1(v) = k\), for all \(v \in G\).

We observe that \((3, k)\)-regular and \(d_1\)-regular graphs are same.

Definition 2.3: A fuzzy graph \(G\) is a pair of functions \(G: (\sigma, \mu)\) where \(\sigma: V \rightarrow [0, 1]\) is a fuzzy subset of a non empty set \(V\) and \(\mu: V \times V \rightarrow [0, 1]\) is symmetric fuzzy relation on \(\sigma\) such that for all \(u, v \in V\) the relation \(\mu(uv) \leq \mu(u)+\mu(v)\) and \(\sigma(u) \land \sigma(v)\) is satisfied.

A fuzzy graph \(G\) is complete if \(\mu(uv) = \sigma(u) \land \sigma(v)\) for all \(u, v \in V\) where \(uv\) denotes the edge between \(u\) and \(v\).

Definition 2.4: Let \(G: (\sigma, \mu)\) be fuzzy graph. The degree of a vertex \(u\) is \(d_0(u) = \sum_{uv} \mu(uv)\) for \(uv \in E\) and \(\mu(uv)=0\) for \(uv\) not in \(E\), this equivalent to \(d_0(u) = \sum_{uv \in E} \mu(uv)\)

The minimum degree of \(G\) is \(\delta(G) = \Lambda \{d(v); v \in V\}\). The maximum degree of \(G\) is \(\Delta(G) = \vee \{d(v); v \in V\}\). The order of a fuzzy graph is \(O(G) = \sum \sigma(u)\).

Definition 2.5: The strength of connectedness between two vertices \(u\) and \(v\) is \(\mu^*(uv) = \text{sup}\{\mu(uv) / k = 1, 2, \ldots\} \text{ where } (uv) = \text{sup}\{\mu(u_u_1)\Lambda \mu(u_u_2)\Lambda \ldots \Lambda \mu(u_\_n,v)|u_1,u_2,\ldots,u_n \in V\}\).

Definition 2.6: Let \(G: (\sigma, \mu)\) be fuzzy graph on \(G^*(V, E)\). If \(d(v) = k\) for all \(v \in V\), then \(G\) is said to be regular fuzzy graph of degree \(k\).

Definition 2.7: Let \(G: (\sigma, \mu)\) be fuzzy graph on \(G^*(V, E)\). If \(d(v) \neq k\) for all \(v \in V\), then \(G\) is said to be irregular fuzzy graph.

Definition 2.8: Let \(G: (\sigma, \mu)\) be fuzzy graph on \(G^*(V, E)\). The total degree of a vertex \(u\) is defined as \(td(u) = \sum \mu(uv) + \sigma(u) = d(u) + \sigma(u), u \in E\).

If each vertex of \(G\) has the same total degree \(k\), then \(G\) is said to be totally regular fuzzy graph of degree \(k\) or \(k\)-totally regular fuzzy graph.

If each vertex of \(G\) has not the same total degree \(k\), then \(G\) is said to be totally irregular fuzzy graph.

Definition 2.9: The Wagner graph is a \(3\)-regular with \(8\) vertices and \(12\) edges.

Definition 2.10: The Corona of two graphs \(G_1\) and \(G_2\) is the graph \(G = G_1 \circ G_2\) formed from one copy of \(G_1\) and \(|v(G_1)|\) copies of \(G_2\) where the \(i^{th}\) vertex of \(G_1\) is adjacent to every vertex in the \(i^{th}\) copy of \(G_2\).

[Note that \(G_1\) is a cycle of length \(\geq 5\) and \(G_2\) is \(K_1\).]

Example 2.11:
Here, \( \mu(v) = 0.3 \) and \( \sigma(v) = 0.4 \) for \( v \in V \).

\[ \text{Example 3.3: Consider } G^* : (V, E) \text{ where } V = \{ v_1, v_2, v_3, v_4, v_5 \} \text{ and } E = \{ v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1 \}. \]

\( G^* \) is a fuzzy graph. The total degree of a vertex \( u \in V \) is defined as \( t \delta_1 (G) = \sum_{v \in V} t \delta_1 (uv) \).

\[ \text{Example 3.2: Consider } G^* : (V, E) \text{ where } V = \{ v_1, v_2, v_3, v_4, v_5 \} \text{ and } E = \{ v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1 \}. \]

Let \( G : (\sigma, \mu) \) be a fuzzy graph. The total d\(_3\)-degree of a vertex \( u \in V \) is defined as \( t \delta_3 (G) = \sum_{v \in V} t \delta_3 (uv) + \sigma(u) \).

\[ \text{Example 3.5: Consider } G^* : (V, E) \text{ where } V = \{ v_1, v_2, v_3, v_4, v_5 \} \text{ and } E = \{ v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1 \}. \]

Let \( G : (\sigma, \mu) \) be a fuzzy graph. The total degree of a vertex \( u \in V \) is defined as \( \delta_3 (G) = \sum_{v \in V} \delta_3 (uv) + \sigma(u) \).

\[ \text{Total degree of a vertex } u \text{ in } G \text{ is } \delta_3 (u) = \sum_{v \in V} \delta_3 (uv) + \sigma(u). \]
Here, \(d(v_i) = \{0.2 \land 0.4 \land 0.4\} = 0.2 + 0.4 = 0.6\).

\[
d_i(v_j) = \{0.2 \land 0.2 \land 0.4\},
\]

\[
d_i(v_j) = \{0.3 \land 0.2 \land 0.4\},
\]

\[
d_i(v_j) = \{0.4 \land 0.2 \land 0.2\},
\]

\[
d_i(v_j) = \{0.4 \land 0.4 \land 0.4\} = 0.2 + 0.2 = 0.4.
\]

\[
d_t d_i(v_i) = d_i(v_i) + \sigma(v_i) = 0.5 + 0.5 = 1.0.
\]

\[
t d_i(v_i) = d_i(v_i) + \sigma(v_i) = 0.4 + 0.6 = 1.0.
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t d_i(v_i) = d_i(v_i) + \sigma(v_i) = 0.4 + 0.6 = 1.0.
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t d_i(v_i) = d_i(v_i) + \sigma(v_i) = 0.5 + 0.5 = 1.0.
\]

\[
\sigma(v_i) = 0.9 \text{ and } \mu(v_i v_j) = 0.2, \mu(v_i v_k) = 0.3, \mu(v_j v_i) = 0.2, \mu(v_j v_k) = 0.3, \mu(v_k v_i) = 0.3, \mu(v_k v_j) = 0.2, \mu(v_j v_k) = 0.3, \mu(v_k v_k) = 0.2, \mu(v_k v_k) = 0.2.
\]

\[
\text{Example 4.3: Consider } G^*:(V, E), \text{ where } V = \{v_1, v_2, v_3, v_4, v_5\} \text{ and } E = \{v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_5, v_5 v_1\}. \text{ Define } G: (\sigma, \mu) \text{ by } \sigma(v_1) = 0.3, \sigma(v_2) = 0.4, \sigma(v_3) = 0.5, \sigma(v_4) = 0.6, \sigma(v_5) = 0.7 \text{ and } \mu(v_1 v_2) = 0.3, \mu(v_2 v_3) = 0.4, \mu(v_3 v_4) = 0.3, \mu(v_4 v_5) = 0.6, \mu(v_5 v_1) = 0.3.
\]

\[
\text{Example 4.4: Consider } G^*:(V, E), \text{ where } V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\} \text{ and } E = \{v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_5, v_5 v_6, v_6 v_7, v_7 v_8, v_8 v_{10}, v_{10} v_2\}. \text{ Define } G: (\sigma, \mu) \text{ by } \sigma(v_1) = 0.2, \sigma(v_2) = 0.3, \sigma(v_3) = 0.4, \sigma(v_4) = 0.5, \sigma(v_5) = 0.6, \sigma(v_6) = 0.7, \sigma(v_7) = 0.8,
\]

\[
\mu(v_1 v_2) = 0.2, \mu(v_2 v_3) = 0.3, \mu(v_3 v_4) = 0.2, \mu(v_4 v_5) = 0.3, \mu(v_5 v_6) = 0.2, \mu(v_6 v_7) = 0.3, \mu(v_7 v_8) = 0.3, \mu(v_8 v_{10}) = 0.2, \mu(v_{10} v_2) = 0.2.
\]

\[
\text{Example 4.5: Consider } G^*:(V, E), \text{ where } V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\} \text{ and } E = \{v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_5, v_5 v_6, v_6 v_7, v_7 v_8, v_8 v_{10}, v_{10} v_2\}. \text{ Define } G: (\sigma, \mu) \text{ by } \sigma(v_1) = 0.2, \sigma(v_2) = 0.3, \sigma(v_3) = 0.4, \sigma(v_4) = 0.5, \sigma(v_5) = 0.6, \sigma(v_6) = 0.7, \sigma(v_7) = 0.8, \sigma(v_8) = 0.9 \text{ and } \mu(v_1 v_2) = 0.2, \mu(v_2 v_3) = 0.3, \mu(v_3 v_4) = 0.2, \mu(v_4 v_5) = 0.3, \mu(v_5 v_6) = 0.2, \mu(v_6 v_7) = 0.2, \mu(v_7 v_8) = 0.3, \mu(v_8 v_{10}) = 0.2, \mu(v_{10} v_2) = 0.2.
\]
Definition 4.6: If each vertex of G has the same total \(d_3\)-degree k, then G is said to be totally (3, k)-regular fuzzy graph.

**Example 4.7:** 1. A totally (3, k)-regular fuzzy graph need not be a (3, k)-regular fuzzy graph. Consider \(G : (V, E)\) where \(V = \{v_1, v_2, v_3, v_4, v_5\}\) and \(E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}\). Define \(G : (\sigma, \mu)\) by \(\sigma(v_1) = 0.5, \sigma(v_2) = 0.5, \sigma(v_3) = 0.5, \sigma(v_4) = 0.5, \sigma(v_5) = 0.5\) and \(\mu(v_1v_2) = 0.4, \mu(v_2v_3) = 0.4, \mu(v_3v_4) = 0.4, \mu(v_4v_5) = 0.4\). Each vertex has same total \(d_3\)-degree 1. Hence G is totally (3, 1)-regular fuzzy graph. But G is not (3, k)-regular fuzzy graph.

2. A (3, k)-regular fuzzy graph need not be a totally (3, k)-regular fuzzy graph. Consider \(G : (V, E)\) where \(V = \{v_1, v_2, v_3, v_4, v_5, v_6\}\) and \(E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}\).

3. A (3, k)-regular fuzzy graph which is totally (3, k)-regular fuzzy graph.

**Theorem 4.8:** Let \(G : (\sigma, \mu)\) be fuzzy graph on \(G : (V, E)\). Then \(\sigma(u) = c\), for all \(u \in V\) if and only if the following conditions are equivalent.
1. \(G : (\sigma, \mu)\) is (3, k)-regular fuzzy graph.
2. \(G : (\sigma, \mu)\) is totally (3, k+ c)-regular fuzzy graph.
Proof: Suppose that \( \sigma(u) = c \), for all \( u \in V \).

It is assumed that \( G : (\sigma, \mu) \) is a \((3, k)\)-regular fuzzy graph.

Then \( t_d(u) = k \), for all \( u \in V \).

So \( t_d(u) = d_d(u) + \sigma(u) \), for all \( u \in V \)

\[ \Rightarrow d_d(u) = k + c, \text{ for all } u \in V \]

Hence \( G : (\sigma, \mu) \) is a totally \((3, k + c)\)-regular fuzzy graph.

Thus, \((1) \Rightarrow (2)\) is proved.

Suppose \( G : (\sigma, \mu) \) is a totally \((3, k + c)\)-regular fuzzy graph.

Then, \( t_d(u) = k + c \), for all \( u \in V \).

\[ \Rightarrow d_d(u) = k + c, \text{ for all } u \in V \]

\[ \Rightarrow d_d(u) = k, \text{ for all } u \in V \]

Hence \( G : (\sigma, \mu) \) is a \((3, k)\)-regular fuzzy graph.

Thus, \((2) \Rightarrow (1)\) is proved.

Hence \((1)\) and \((2)\) are equivalent.

Conversely, it is assumed that \((1)\) and \((2)\) are equivalent.

Suppose that \( \sigma \) is not a constant function. Then \( \sigma(u) \neq \sigma(v) \), for atleast one pair \((u,v) \in V \) . Let \( G \) be a \((3, k)\)-regular fuzzy graph.

Then \( d_d(u) = d_d(v) = k \).

So \( t_d(u) = d_d(u) + \sigma(u) = k + \sigma(u) \) and \( t_d(v) = d_d(v) + \sigma(v) = k + \sigma(v) \).

Since \( \sigma(u) \neq \sigma(v) \), \( \Rightarrow k + \sigma(u) \neq k + \sigma(v) \).

\[ \Rightarrow d_d(u) \neq d_d(v) \]

So, \( G \) is not totally \((3, k)\)-regular fuzzy graph. Which is contradiction to our assumption.

Let \( G \) be a totally \((3, k)\)-regular fuzzy graph.

Then \( t_d(u) = t_d(v) \). \( \Rightarrow d_d(u) + \sigma(u) = d_d(v) + \sigma(v) \).

\[ \Rightarrow d_d(u) - d_d(v) = \sigma(v) - \sigma(u) \neq 0 \]

\[ \Rightarrow (u,v) \notin E \]

So, \( G \) is not \((3, k)\)-regular fuzzy graph. Which is contradiction to our assumption.

Hence \( \sigma \) is a constant function.

Theorem 5.4: If a fuzzy graph \( G \) is both \((3, k)\)-regular and totally \((3, k)\)-regular then \( \sigma \) is constant function.

Proof: Let \( G \) be \((3, k_1)\) -regular and totally \((3, k_2)\) -regular fuzzy graph.

Then \( d_d(u) = k_1 \) and \( t_d(u) = k_2 \), for all \( u \in V \).

Now \( t_d(u) = k_2 \), for all \( u \in V \).

\[ \Rightarrow d_d(u) + \sigma(u) = k_2 \], for all \( u \in V \).

\[ \Rightarrow k_1 + \sigma(u) = k_2 \], for all \( u \in V \).

\[ \Rightarrow \sigma(u) = k_2 - k_1 \], for all \( u \in V \).

Hence \( \sigma \) is a constant function.

Remark 4.10: The converse of theorem 4.9 is not true.

Consider \( G^*: (V, E) \) where \( V = \{ v_1, v_2, v_3, v_4, v_5, v_6 \} \) and \( E = \{ v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1 \} \). Define \( G : (\sigma, \mu) \) by \( \sigma(v_1) = 0.5, \sigma(v_2) = 0.5, \sigma(v_3) = 0.5, \sigma(v_4) = 0.5, \sigma(v_5) = 0.5, \sigma(v_6) = 0.5 \) and \( \mu(v_1v_2) = 0.2, \mu(v_2v_3) = 0.2, \mu(v_3v_4) = 0.3, \mu(v_4v_5) = 0.4, \mu(v_5v_6) = 0.4 \).

Here, \( d_d(v_1) = 0.5, d_d(v_2) = 0.4, d_d(v_3) = 0.5, d_d(v_4) = 0.4, d_d(v_5) = 0.4, d_d(v_6) = 0.4 \) and \( t_d(v_1) = 1.0, t_d(v_2) = 0.9, t_d(v_3) = 0.9, t_d(v_4) = 0.9, t_d(v_5) = 0.9, t_d(v_6) = 0.9 \).

Here, \( \sigma \) is a constant function.

But \( G \) is neither \((3, k)\)-regular fuzzy graph nor totally \((3, k)\)-regular fuzzy graph.

5. \((3, k)\)-regular fuzzy graphs on a path on 6 vertices with some specific membership functions

In this section \((3, k)\)-regularity and totally \((3, k)\)-regularity on fuzzy graph whose underlying crisp graph is a path on 6 vertices is studied with some specific membership functions.

Theorem 5.1: Let \( G : (\sigma, \mu) \) be a fuzzy graph such that \( G^*: (V, E) \) a path on 6 vertices. If \( \mu \) is constant function, then \( G \) is \((3, c)\)-regular fuzzy graph.

Proof: Suppose that \( \mu \) is a constant function, say \( \mu(uy) = c \), for all \( uy \in E \) . Then \( d_d(u) = c, \forall u \in V \) .

Hence \( G \) is a \((3, c)\)-regular fuzzy graph.

Remark 5.2: Converse the theorem 5.1 need not be true. For example, Consider \( G : (\sigma, \mu) \) be a fuzzy graph on \( G^*: (V, E) \) a path on 6 vertices.

![Figure 13](image-url)

Here, \( d_d(v_1) = 0.3, d_d(v_2) = 0.3, d_d(v_3) = 0.3, d_d(v_4) = 0.3, d_d(v_5) = 0.3, d_d(v_6) = 0.3 \) . Hence \( G \) is \((3, 0.3)\) - regular fuzzy graph. But \( \mu \) is not a constant function.

Theorem 5.3: Let \( G : (\sigma, \mu) \) be fuzzy graph such that \( G^*: (V, E) \) is a path on six vertices. If alternate edges have some membership values, then \( G \) is \((3, k)\)-regular fuzzy graph, where \( k = \min \{ c_1, c_2 \} \).

Proof: If alternate edges have same membership values, then \( \mu(ey) = c \) if \( i \) is odd \( \mu(ey) = c \) if \( i \) is even.

If \( c_1 = c_2 \), then \( \mu \) is constant function. So \( G \) is \((3, c_1)\)-regular fuzzy graph.

If \( c_1 < c_2 \), then \( d_d(v) = c_1 \), for all \( v \in V \) . So \( G \) is \((3, c_1)\)-regular fuzzy graph.

If \( c_1 > c_2 \), then \( d_d(v) = c_2 \), for all \( v \in V \) . So \( G \) is \((3, c_2)\)-regular fuzzy graph.

Theorem 5.4: Let \( G : (\sigma, \mu) \) be a fuzzy graph such that \( G^*: (V, E) \) a path on 6 vertices. If middle edge have membership value less than membership value of the remaining edges, then \( G \) is \((3, k)\)-regular fuzzy graph, where \( k = \text{ membership value of the middle edge} \).

For example, Consider \( G : (\sigma, \mu) \) be a fuzzy graph such that \( G^*: (V, E) \) a path on 6 vertices.

![Figure 14](image-url)
Here, \( d_3(v_1) = 0.2, \ d_3(v_2) = 0.2, \ d_3(v_3) = 0.2, \ d_3(v_4) = 0.2, \ d_3(v_5) = 0.2, \ d_3(v_6) = 0.2 \). Hence \( G \) is \((3, 0.2)\) - regular fuzzy graph.

**Remark 5.5:** If \( \sigma \) is not a constant function, then the \((3, k)\)-regular fuzzy graphs in the above theorems 5.1, 5.3 and 5.4 are not totally \((3, k)\)-regular fuzzy graphs.

For example, Consider \( G \): \((\sigma, \mu)\) be fuzzy graph such that \( G \): \((V,E)\) is a path on 6 vertices.

**Theorem 6.1:** Let \( G: (\sigma, \mu) \) be fuzzy graph such that \( G'': (V, E) \) is Wagner graph. If \( \mu \) is constant function, then \( G \) is \((3, k)\) - regular fuzzy graph.

**Proof:** Suppose that, \( \mu \) is constant function say \( \mu(uv) = c \) for all \( u,v \in E \). Then \( d_3(v) = 5c \), for all \( v \in V \). Hence \( G \) is \((3, 5c)\) - regular fuzzy graph.

**Remark 6.2:** Converse of the theorem 6.2 need not be true.

For example, Consider \( G: (\sigma, \mu) \) be fuzzy graph such that \( G': (V, E) \) is wagner graph.

**Theorem 7.1:** Let \( G: (\sigma, \mu) \) be fuzzy graph such that \( G'': (V, E) \) is Corona graph \( C_n \circ K_1 \) where \( n \geq 5 \). If \( \mu \) is a constant function, then \( G \) is \((3, 4c)\) - regular fuzzy graph.

**Proof:** Suppose that, \( \mu \) is a constant function say \( \mu(uv) = c \) for all \( u,v \in E \), then \( d_3(v) = 4c \), for all \( v \in V \). Hence \( G \) is \((3, 4c)\) - regular fuzzy graph.

**Remark 7.2:** Converse of the theorem 7.1 need not be true.

For example, Consider \( G: (\sigma, \mu) \) be fuzzy graph such that \( G': (V, E) \) is corona graph.
Remark 7.3: The theorem 7.1 does not hold for totally (3, k)-regular fuzzy graphs.

Here, \( d_3(v_1) = 0.8, \ d_3(v_2) = 0.8, \ d_3(v_3) = 0.8, \ d_3(v_4) = 0.8, \ d_3(v_5) = 0.8, \ d_3(v_6) = 0.8, \ d_3(v_7) = 0.8, \ d_3(v_8) = 0.8, \ d_3(v_9) = 0.8, \ d_3(v_{10}) = 0.8, \ d_3(v_{11}) = 0.8, \ d_3(v_{12}) = 0.8 \), and \( t_3(d(v_1)) = 1.0, \ t_3(d(v_2)) = 1.1, \ t_3(d(v_3)) = 1.2, \ t_3(d(v_4)) = 1.3, \ t_3(d(v_5)) = 1.4, \ t_3(d(v_6)) = 1.5, \ t_3(d(v_7)) = 1.6, \ t_3(d(v_8)) = 1.1, \ t_3(d(v_9)) = 1.2, \ t_3(d(v_{10})) = 1.3, \ t_3(d(v_{11})) = 1.4, \ t_3(d(v_{12})) = 1.5 \). Hence \( G \) is (3, 0.8) - regular fuzzy graph but not totally (3, k)-regular fuzzy graph. But \( \mu \) is a constant function.

8. (3, k)-regular fuzzy graphs on a Cycle of length \( \geq 5 \) with some specific membership function

In this section (3, k)-regularity and totally (3, k)-regularity on fuzzy graph whose underlying crisp graph is a Cycle of length \( \geq 5 \) is studied with some specific membership function.

Theorem 8.1: Let \( G: (\sigma, \mu) \) be fuzzy graph such that \( G^*: (V, E) \) is cycle of length \( \geq 5 \). If \( \mu \) is constant function, then \( G \) is (3, 2c) - regular fuzzy graph.

Proof: Suppose that, \( \mu \) is constant function say \( \mu(uv) = c \) for all \( uv \in E \), then \( d_3(v) = 2c \), for all \( v \in V \). Hence \( G \) is (3, 2c) – regular fuzzy graph.

Remark 8.2: Converse of the theorem 8.1 need not be true. For example, Consider \( G: (\sigma, \mu) \) be fuzzy graph such that \( G^*: (V, E) \) is odd cycle of length seven.

Here, \( d_3(v_1) = 0.6, \ d_3(v_2) = 0.6, \ d_3(v_3) = 0.6, \ d_3(v_4) = 0.6, \ d_3(v_5) = 0.6, \ d_3(v_6) = 0.6, \ d_3(v_7) = 0.6, \ d_3(v_8) = 0.6 \), and \( t_3(d(v_1)) = 0.9, \ t_3(d(v_2)) = 1.0, \ t_3(d(v_3)) = 1.1, \ t_3(d(v_4)) = 1.2, \ t_3(d(v_5)) = 1.3, \ t_3(d(v_6)) = 1.4 \). So \( G \) is (3, 0.6) – regular fuzzy graph. But \( \mu \) is not a constant function.

Remark 8.3: The theorem 8.1 does not hold for totally (3, k)-regular fuzzy graphs.

Here, \( d_3(v_1) = 0.6, \ d_3(v_2) = 0.6, \ d_3(v_3) = 0.6, \ d_3(v_4) = 0.6, \ d_3(v_5) = 0.6, \ d_3(v_6) = 0.6, \ d_3(v_7) = 0.6, \ d_3(v_8) = 0.6, \ d_3(v_9) = 0.6, \ d_3(v_{10}) = 0.6 \), and \( t_3(d(v_1)) = 1.0, \ t_3(d(v_2)) = 1.1, \ t_3(d(v_3)) = 1.2, \ t_3(d(v_4)) = 1.3, \ t_3(d(v_5)) = 1.4 \). So \( G \) is (3, 0.6)-regular fuzzy graph, but not totally (3, k)-regular fuzzy graph. But \( \mu \) is a constant function.

Theorem 8.4: Let \( G: (\sigma, \mu) \) be fuzzy graph such that \( G^*: (V, E) \) is even cycle of length \( \geq 5 \). If alternate edges have same membership values, then \( G \) is (3, k)-regular fuzzy graph.

Proof: If alternate edges have same membership values, then \( \mu(e_i) = \begin{cases} c_1 & \text{if } i \text{ is odd} \\ c_2 & \text{if } i \text{ is even} \end{cases} \).

If \( c_1 < c_2 \), then \( \mu \) is constant function. So \( G \) is (3, 2c)-regular fuzzy graph.

If \( c_1 > c_2 \), then \( d_3(v) = 2c_1 \), for all \( v \in V \). So \( G \) is (3, 2c)-regular fuzzy graph.

Remark 8.5: The theorem 8.4 does not hold for totally (3, k)-regular fuzzy graphs.

For example, Consider \( G: (\sigma, \mu) \) be fuzzy graph such that \( G^*: (V, E) \) is even cycle of length \( \geq 5 \).
not totally (3, k)-regular fuzzy graph.

**Remark 8.6:** Let \( G:(\sigma,\mu) \) be fuzzy graph such that \( G^*:(V,E) \) is odd cycle of length \( \geq 5 \). If alternate edges have same membership values, then \( G \) is (3, k)-regular fuzzy graph. For example, Consider \( G:(\sigma,\mu) \) be fuzzy graph such that \( G^*:(V,E) \) is odd cycle of length 5.

**Proof:** Let \( \mu(e) = \begin{cases} c_i, & \text{if } i \text{ is odd} \\ c_2, & \text{if } i \text{ is even} \end{cases} \), then \( G \) is (3, \( 2c_i \))-regular fuzzy graph.

**Case 1:** Let \( G:(\sigma,\mu) \) be fuzzy graph such that \( G^*:(V,E) \) is even cycle of length \( \geq 5 \).

\[
\begin{align*}
\delta_1(v_3) &= \{c_1 \land c_1 \land c_2 \} + \{c_2 \land c_1 \land c_2 \} = c_1 + c_2 = 2c_1 \\
\delta_2(v_4) &= \{\mu(e_1) \land \mu(e_2) \land \mu(e_{2n-1})\} + \{\mu(e_2) \land \mu(e_1) \land \mu(e_{2n-1})\} \\
&= \{c_1 \land c_1 \land c_2 \} + \{c_1 \land c_2 \land c_1 \} = c_1 + c_2 = 2c_1.
\end{align*}
\]

For \( i = 4, 5, 6, \ldots, 2n \),

\[
\delta_i(v_1) = \{\mu(e_1) \land \mu(e_{2i-2}) \land \mu(e_{2i})\} + \{\mu(e_{2i}) \land \mu(e_1) \land \mu(e_{2i})\} \\
= \{c_1 \land c_2 \land c_1 \} + \{c_1 \land c_2 \land c_1 \} = c_1 + c_2 = 2c_1.
\]

Hence, \( \delta_i(v_1) = 2c_1 \) for all \( v \in V \).

So \( G \) is (3, \( 2c_i \))-regular fuzzy graph.

**Remark 8.7:** The remark 8.6 does not hold for totally (3, k)-regular fuzzy graphs. For example, Consider \( G:(\sigma,\mu) \) be fuzzy graph such that \( G^*:(V,E) \) is odd cycle of length 5.

**Figure 23**

Here \( \delta_1(v_3) = 0.4, \delta_1(v_4) = 0.4, \delta_3(v_1) = 0.4, \delta_3(v_4) = 0.4, \delta_5(v_3) = 0.4. \) \( G \) is (3, k)-regular fuzzy graph.

**Figure 24**

Here \( \delta_1(v_1) = 0.4, \delta_1(v_2) = 0.4, \delta_3(v_1) = 0.4, \delta_3(v_2) = 0.4, \delta_5(v_1) = 0.4, \delta_5(v_2) = 0.4, \delta_3(v_3) = 0.9, \delta_3(v_4) = 1.0, \delta_5(v_3) = 1.1, \delta_5(v_4) = 1.2 \). So \( G \) is (3, \( 2c_i \))-regular fuzzy graph.

**Theorem 8.8:** Let \( G:(\sigma,\mu) \) be fuzzy graph such that \( G^*:(V,E) \) is any odd cycle of length \( \geq 5 \).

Let \( \mu(e_i) = \begin{cases} c_1, & \text{if } i \text{ is odd} \\ c_2, & \text{if } i \text{ is even} \end{cases} \), then \( G \) is (3, \( 2c_i \))-regular fuzzy graph.

**Proof:** Let \( \mu(e_i) = \begin{cases} c_1, & \text{if } i \text{ is odd} \\ c_2, & \text{if } i \text{ is even} \end{cases} \), then \( G \) is (3, \( 2c_i \))-regular fuzzy graph.

**Case 2:** Let \( G:(\sigma,\mu) \) be fuzzy graph such that \( G^*:(V,E) \) is an odd cycle of length \( \leq 5 \). Let \( e_1, e_2, e_3, \ldots, e_{2n-1} \) be the edges of the odd cycle of \( G^* \) in that order.

\[
\begin{align*}
\delta_1(v_1) &= \{\mu(e_1) \land \mu(e_2) \land \mu(e_{2n-1})\} + \{\mu(e_2) \land \mu(e_1) \land \mu(e_{2n-1})\} \\
&= \{c_1 \land c_1 \land c_2 \} + \{c_1 \land c_2 \land c_1 \} = c_1 + c_2 = 2c_1 \\
\delta_3(v_2) &= \{\mu(e_1) \land \mu(e_2) \land \mu(e_{2n-1})\} + \{\mu(e_2) \land \mu(e_1) \land \mu(e_{2n-1})\} \\
&= \{c_1 \land c_1 \land c_2 \} + \{c_1 \land c_2 \land c_1 \} = c_1 + c_2 = 2c_1.
\end{align*}
\]

For \( i = 4, 5, 6, \ldots, 2n \),

\[
\delta_i(v_1) = \{\mu(e_1) \land \mu(e_{2i}) \land \mu(e_{2i-2})\} + \{\mu(e_{2i}) \land \mu(e_1) \land \mu(e_{2i-2})\} \\
= \{c_1 \land c_1 \land c_2 \} + \{c_1 \land c_2 \land c_1 \} = c_1 + c_2 = 2c_1.
\]

2. Consider \( G:(\sigma,\mu) \) be fuzzy graph such that \( G^*:(V,E) \) is an odd cycle of length five.

**Figure 25**

Here \( \delta_1(v_1) = 0.4, \delta_1(v_2) = 0.4, \delta_1(v_3) = 0.4, \delta_1(v_4) = 0.4, \delta_2(v_3) = 0.4, \delta_2(v_4) = 0.4, \delta_3(v_1) = 0.7, \delta_3(v_2) = 0.8, \delta_3(v_3) = 0.9, \delta_3(v_4) = 1.0, \delta_4(v_1) = 1.1, \delta_4(v_2) = 1.2 \). So \( G \) is not totally (3, \( 2c_i \))-regular fuzzy graph.

**Figure 26**

Here \( \delta_1(v_1) = 0.4, \delta_1(v_2) = 0.4, \delta_1(v_3) = 0.4, \delta_1(v_4) = 0.4, \delta_2(v_3) = 0.4, \delta_2(v_4) = 0.4, \delta_3(v_1) = 0.7, \delta_3(v_2) = 0.8, \delta_3(v_4) = 0.9, \delta_4(v_1) = 0.7 \). So \( G \) is not totally (3, \( 2c_i \))-regular fuzzy graph.

**9. Conclusion and Future Studies**

In this paper, (3, k)-regular fuzzy graphs and totally (3, k)-regular fuzzy graphs are compared through various examples. A necessary and sufficient condition under which they are equivalent is provided. Also we provide (3, k)-regular fuzzy graphs and totally (3, k)-regular fuzzy graphs
in which underlying crisp graphs are a path on six vertices, a corona graph, a wagner graph and a cycle of length \( \geq 5 \) is studied with some specific membership function. Some properties of \((3, k)\)-regular fuzzy graphs studied and they are examined for totally \((3, k)\)-regular fuzzy graphs. The results discussed may be used to study about various fuzzy graphs invariants. For further investigation, the following open problem is suggested.

“\((r, m)\)-regular fuzzy graph and totally \((r, m)\)-regular fuzzy graph, for \( m > 3 \) may be investigated”.

References


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