Non- linear Differential equations and Analysis of a Disturb System Using Numerical Technique (Weighted Residual Method)

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Abstract: ***Nonlinear partial differential equations governing a disturb model of magneto- hydrodynamic and heat transfer in a channel flow have been treated by the weight residual method. The stability or other wise of the disturb system is our target to the effect of a variable absorption coefficient of radiation mode of heat transfer. The results from tables, in spite of the unreliability of calculation at law Reynolds numbers indicate quite clearly that, effect of a variable absorption coefficient is to increase any tendency towards stability. The present stability analysis relates to only a simple case of flow pattern. A Further and useful development would be to consider the influence of buoyancy forces and convective heat transfer upon the stability when the channel walls are differentially and non-uniformly heated. It would also be profitable to seek to reformulate the reduction of the differential eigenvalue problem to an algebraic eigenvalue problem by use of an alternative orthogonal function expansion.***

Keywords: Nonlinear, MHD, heat transfer, channel flow, weight residual method and stability.

1. Introduction

It is under stood for some time ago, that the radiative MHD is the interaction between the radiation field and the magneto hydrodynamic field which itself is concerned with the interaction of electrically conducting fluids and electromagnetic fields.

It is also most of the heat transfer modes are through conduction, convection and radiation. Usually the heat transfer by radiation is ignored in comparison with conduction and convections, especially when the temperature is not particularly high and the density of the fluid is not too low. However due to the international societies developments and the use of magneto hydrodynamic pumps and generator. Additionally they also serve to enhance our broad understanding of motions involving plasmas and conducting fluid generally. Consequently, the thermal radiation becomes an import mode of heat transfer; see for example Prof. Eric Fraga [12] and Sreedhara Rao [13]. For a wide range of researchers have a good article in this subject, such as, Qihua Zhang, Li Cao [14].

1.1 Statement of the Problem

This problem is about a mathematical model for the interaction between fluid flow , heat transfer and applied magnetic field through a horizontal channel.

1.2 Objective of the Project

The mean objective of this project are:

1) Define a horizontal channel of fluid with a parallel wall.
2) Apply a magnetic field uniformly across the channel.
3) The fluid is with heat transfer by conduction, convection and radiation.
4) Formulate as a nonlinear partial differential equations.
5) Apply a disturbance through the fluid and separate mathematically.
6) Apply the weight residual method to transform algebraic system, through which we have the results of the fluid stability or otherwise.

1.3 Significance of the Study

We expect that this project is useful for the engineering field work and implementation of the heat transfer by conduction, convection and radiation specially in mechanic applications. Also, we are looking to the systems which arise from the mechanical applications and in part, the disturbances of the systems and the stability of that.

2. Literature Review

The governing equations for the flow of an electrically conduction viscous and heat conducting fluid are well established, see for example Shercliff. The equations relating to effects of thermal radiation and energy conservation, whilst of more recent derivation are now also well understood, and may be found in the book by Vincentia and Kruger.

The basic steady state flow considered in this paper is that described by Helliwell & Mosa [4], in which models with both variable and constant absorption coefficient were studied. "The flow is parallel to the x-axis down a channel of great width in the z direction between walls distance 2h apart parallel to the x-plane of a Cartesian coordinate system". 'The bounds of the channel normal to the z-axis are taken to be electrodes of perfect electrical conductivity whilst the walls normal to the y-axis are supposed of a general electrical conductivity and perfect thermal conductivity". An external magnetic field is applied uniformly across the channel in a direction parallel to the y-axis the temperatures T of the walls are taken to be uniform but possibly different.

The velocity and magnetic field are known analytically, see [5], while the forms for temperature, radiative flux and radiative density are established numerically having been computed for several values of particular physical parameters. The same notation employed in [4] is retained for the analysis of the present part, unless stated otherwise.

The governing equations, relevant to the present problem for unsteady flow in three-dimensions, retaining the Boussinesq approximation, may be written as follows.
The equation of state takes the form

$$p = p_1\left\{1-\frac{\beta}{T_1}\left(T-T_1\right)\right\}. \quad (1)$$

The conservation equation for the mass is

$$div\underline{V} = 0. \quad (2)$$

The electromagnetic equations are Maxwell's equations, viz.

$$div\underline{B} = 0. \quad (3)$$

$$\nabla\times\underline{B} = \mu_e\underline{J}, \quad (4)$$

$$\nabla\times\underline{E} = -\frac{\partial\underline{B}}{\partial t}; \quad (5)$$

And Ohm's lam without hall effect

$$\underline{J} = \sigma\{\underline{E}+\underline{V}\times\underline{B}\}. \quad (6)$$

Elimination of the electric field vector $\underline{E}$ and the electric current density vector $\underline{J}$ from (4),(5) and (6), leads to the (so-called) magnetic vorticity equation

$$\frac{\partial\underline{B}}{\partial t} = curl\left(\underline{V}\times\underline{B}\right)+\frac{1}{\mu_e\sigma}\nabla^2\underline{B}. \quad (7)$$

Equation (7) may be written in the from

$$\frac{\partial\underline{B}}{\partial t} = \left(\underline{B}\cdot\nabla\right)\underline{V}-\underline{B}\left(\nabla\cdot\underline{V}\right)-\left(\underline{V}\cdot\nabla\right)\underline{B}+\underline{V}\left(\nabla\cdot\underline{B}\right)+\frac{1}{\mu_e\sigma}\nabla^2\underline{B}\cdot \quad (8)$$

Equation (8) together with equations (2) and (3) yield

$$\frac{\partial\underline{B}}{\partial t} = \left(\underline{B}\cdot\nabla\right)\underline{V}-\left(\underline{V}\cdot\nabla\right)\underline{B}+\frac{1}{\mu_e\sigma}\nabla^2\underline{B}\cdot \quad (9)$$

The momentum equation takes the form

$$\rho_1\left\{\frac{\partial\underline{V}}{\partial t}+\left(\underline{V}\cdot grad\right)\underline{V}\right\} = \underline{J}\times\underline{B}-gradP+\mu\nabla^2\underline{V}\cdot \quad (10)$$

Take the curl of the momentum equation (10) and make use of equation (4) for $\underline{J}$.
It follows that

$$\rho_1\left\{\frac{\partial}{\partial t}\left(curl\underline{V}\right)+curl\left[\left(\underline{V}\cdot grad\right)\underline{V}\right]\right\} = \frac{1}{\mu_e}curl\left\{\left(curl\underline{B}\right)\times\underline{B}\right\}+\mu curl\left[\nabla^2\underline{V}\right]. \quad (11)$$

The equation for conservation of energy can be written as

$$P_1C_v\left\{\frac{\partial T}{\partial t}+\underline{V}.gradT\right\} = K\nabla^2T-div\underline{q}+\frac{1}{\mu_e^2\sigma}\left(curl\underline{B}\right)^2+\Phi = 0, \quad (12)$$

Where Φ is the contribution from viscous dissipation.
The radiative transfer equation in terms of the radiative flux $\underline{q}$ and radiative density Σ are

$$\alpha C\Sigma+div\underline{q} = 4\tilde{\sigma}\alpha T^4, \quad (13)$$

$$\nabla C\Sigma+3\alpha\underline{q} = 0. \quad (14)$$

The equations governing the present analysis are therefore equations (2),(3),(9),(11),(12),(13), and (14), together with the from for variable absorption coefficient

$$\alpha = K_\alpha\rho_1^m\mathrm{T}^n, \quad (15)$$

3. Formulation of the Linearized Stability Equation

As a first study carried out in this paper only two-dimensional disturbances without z-variation are considered. The primary velocity, magnetic field, temperature, radiative flux and radiative density are now considered to have superimposed on them two-dimensional infinitesimal disturbances. Steady state quantities are indicated with a bar super-script and are known functions of positions, whilst perturbations are denoted by a tilde. Therefore we write

$$\begin{aligned}\underline{V} &= \underline{\bar{V}}+\underline{\tilde{V}}(x,y,t),\\ \underline{B} &= \underline{\bar{B}}+\underline{\tilde{B}}(x,y,t),\\ T &= \bar{T}+\tilde{T}(x,y,t),\\ \underline{q} &= \underline{\bar{q}}+\underline{\tilde{q}}(x,y,t),\\ \Sigma &= \bar{\Sigma}+\tilde{\Sigma}(x,y,t),\end{aligned} \quad (16)$$

Where x, y are the dimensional spatial coordinates and t is the time.

Direct substitution of (16) into the relevant equations, described earlier, generates the set of equations governing the present stability problem. After subtracting the terms corresponding to the steady flow and neglecting squares of small quantities, the linearized equations are

$$div\underline{\tilde{V}} = 0. \quad (17)$$

$$div\underline{\tilde{B}} = 0. \quad (18)$$

$$\frac{\partial\underline{\tilde{B}}}{\partial t} = \left(\underline{\tilde{B}}\cdot\nabla\right)\underline{\bar{V}}+\left(\underline{\bar{B}}\cdot\nabla\right)\underline{\tilde{V}}-\left(\underline{\tilde{V}}\cdot\nabla\right)\underline{\bar{B}}-\left(\underline{\bar{V}}\cdot\nabla\right)\underline{\tilde{B}}+\frac{1}{\mu_e\sigma}\nabla^2\underline{\tilde{B}}. \quad (19)$$

$$\rho_1\left\{\frac{\partial}{\partial t}\left(curl\underline{\tilde{V}}\right)+curl\left[\left(\underline{\tilde{V}}\cdot grad\right)\underline{\bar{V}}+\left(\underline{\bar{V}}\cdot grad\right)\underline{\tilde{V}}\right]\right\} = \frac{1}{\mu_e}curl\left\{\left(curl\underline{\tilde{B}}\right)\times\underline{\bar{B}}+\left(curl\underline{\bar{B}}\right)\times\underline{\tilde{B}}\right\}+\mu\nabla^2\left(curl\underline{\tilde{V}}\right) \quad (20)$$

$$\rho_1 c_v \left\{ \frac{\partial \mathrm{T}}{\partial t} + \underline{\overline{V}} \cdot grad\tilde{\mathrm{T}} + \underline{\tilde{V}} \cdot grad\overline{\mathrm{T}} \right\} = k\nabla^2 \tilde{\mathrm{T}} - div\underline{\tilde{q}} + \frac{2}{\mu_e^2 \sigma} \left(cur1\underline{\tilde{B}} \right)\left(cur1\underline{\overline{B}} \right) + \tilde{\tilde{\Phi}} \text{ , (21)}$$

Where the $\tilde{\tilde{\Phi}}$ denote contribution from viscous dissipation.

$$n\rho_1 k_\alpha \overline{\mathrm{T}}^{n-1} \tilde{\mathrm{T}} c\tilde{\Sigma} + \rho_1 k_\alpha \overline{\mathrm{T}}^n c\tilde{\Sigma} + div\underline{\tilde{q}} = 4(n+4)\tilde{\sigma}\rho_1 k_\alpha \overline{\mathrm{T}}^{n+3} \tilde{\mathrm{T}} \text{ . (22)}$$

$$\nabla^2 c\tilde{\Sigma} + 3nk_\alpha \rho_1 \overline{\mathrm{T}}^{n-1} \tilde{\mathrm{T}} \underline{\overline{q}} + 3k_\alpha \rho_1 \overline{\mathrm{T}}^n \underline{\tilde{q}} = 0 \text{ . (23)}$$

The system of equations (17) – (20) is the basic set previously investigated by Lock [1] in ordinary magentohydrodynamics. It can be noted from (17) – (20) that certain of these equations are implied by the others and that no difficulties result from the fact that there appear to be more equations than variables. Stuart [11], in the case when the magnetic field is parallel to velocity, commented explicitly upon the matter in relation to the analogous set of equations in his analysis. The particular relationships between the equations for the present analysis will be discussed in detail at a later stage of this development.

Non – dimensional variables are now introduced precisely the same as in [4] with the addition of

$$\xi = x|h, \quad \bar{t} = \frac{Ut}{h} \qquad (24)$$

Two-dimensional perturbation are considered of the from

$$u\underline{\tilde{u}} = \underline{\{U_x}(\eta), U_y(\eta)\} \exp ik(\xi - \overline{\overline{ct}}) \quad (25)$$

$$\underline{\tilde{b}} = \{b_x(\eta), b_y(\eta)\} \exp ik(\xi - \overline{\overline{ct}}) \quad (26)$$

$$\tilde{\theta} = \psi(\eta) \exp ik(\xi - \overline{\overline{ct}}) \text{ , (27)}$$

$$\tilde{\Sigma} = \phi(\eta) \exp ik\ (\xi - \overline{\overline{ct}}) \text{ , (28)}$$

The assumed forms of the disturbances imply a spatially periodic wave where k is a dimensionless wave number and is therefore a real quantity. the complex wave speed is $\bar{c} = c_r + ic_i$. A positive or negative c_i implies growth or decay of the disturbance , respectively.

This study is concentrated on the change of sign of c_i to separate the different regimes of fluid behavior

By substitution of the forms (25) - (28) into the relevant

Non-dimensional governing equations , the final set then be obtained.

Equations (17) – (20) yield

$$ikU_x + \frac{dU_y}{d\eta} = 0, \qquad (29)$$

$$ikb_x + \frac{db_y}{d\eta} = 0, \qquad (30)$$

$$\left[\frac{1}{R_m}\frac{d}{d\eta} + lkb\right]U_x - U_y\frac{db}{d\eta} + \left\{\frac{1}{R_m}\frac{d^2}{d\eta^2} - \frac{k^2}{R_m} + ik(\bar{c} - u)\right\}b_x + b_y\frac{du}{d\eta} = 0, \text{ (31)}$$

$$\left[\frac{1}{R_m} ikb\right]U_y + \left\{\frac{1}{R_m}\frac{d^2}{d\eta^2} - \frac{k^2}{R_m} + ik(\bar{c} - u)\right\}b_y = 0, \text{ (32)}$$

$$\left\{\frac{1}{R}\frac{d^3}{d\eta^3} - \left[\frac{k^2}{R} - ik(\bar{c} - u\right]\frac{d}{d\eta}\right\}U_x + \left\{\frac{-ik}{R}\frac{d^2}{d\eta^2} + \frac{ik^2}{R} + k^2(\bar{c} - u)\frac{d^2u}{d\eta^2}\right\}U_y +$$
$$\frac{M^2}{R}\left\{\frac{d^2}{d\eta^2} + ikbR_m\frac{d}{d\eta}\right\}b_x - \frac{M^2}{R}\left\{ik\frac{d}{d\eta} - kR_m b - R_m\frac{d^2b}{d\eta^2}\right\}b_y = 0, \text{ (33)}$$

Where u, b are denoted steady state values.

It can be noted , as remarked earlier , that the number of the equations is more than the number of the dependent variables which are U_x , U_y , b_x , b_y However elimination of U_x and b_x from the equations (29) – (31) yields an equation which is the derivative of equation (32).

Hence these equations are not independent and one of equations (29) – (31) may be omitted.
The remaining system of equations (29) – (33) can be reduced to a pair of differential equations in U_y and b_y alone. These are equation (32) together with the following equation resulting from the eliminations of U_x and b_x between (29),(30) and (33) :

$$\left\{\frac{i}{R}\frac{d^4}{d\eta^4} - \left[\frac{2ik^2}{R} + (\bar{c} - u)\right]\frac{d^2}{d\eta^2} + k^2\left[\frac{ik^2}{R} + k(\bar{c} - u)\right] - k\frac{d^2u}{d\eta^2}\right\}U_y +$$

$$\frac{M^2}{R}\left\{i\frac{d^3}{d\eta^3}-kR_m b\frac{d^2}{d\eta^2}-ik^2\frac{d}{d\eta}+kR_m\frac{d^2 b}{d\eta^2}+kR_m b\right\}b_y \quad =0. \quad (34)$$

Equations analogous to these have been studied in the analysis of the stability of the associated non-radiative problems with negligible R_m , by Lock [1 equations (21) and (22)]. In the case of $R_m \neq 0$ problem has been investigated by Sagalokov [2] and Potter and Kutchey [3]. The influence of radiation upon the stability is restricted to equations (21) , (22) and (23). for two-dimensional perturbations these reduce to four equations relating $U_x, U_y, b_x, b_y, \psi, Q_x, Q_y$ and ϕ , such that.

$$\left\{\frac{d^2}{d\eta^2}-U_y\frac{d}{d\eta}-k^2-\frac{FR}{{}_Y E_c}ik(U-\bar{c})\right\}\psi - NF\left\{ikQ_x\frac{dQ_y}{d\eta}\right\}-$$
$$2M^2F\left\{\frac{db_x}{d\eta}-ikb_y\right\}\frac{db}{d\eta}-2F\left\{\frac{dU_x}{d\eta}+ikU_y\right\}\frac{dU}{d\eta} \quad =0, \quad (35)$$

$$nw\theta^{n-1}\Sigma^{(1)}\psi+w\theta^n\phi+ikQ_x+\frac{dQ_y}{d\eta}=4(n+4)w\theta^{n+3}\psi, \quad (36)$$

$$ik\phi+3w\theta^n Q_x = \quad 0,$$
$$\frac{d\phi}{d\eta}+3w\theta^n Q_y + 3nw\theta^{n-1}Q^{(1)} \quad = \quad 0, \quad (37)$$

Where θ , $\sum^{(1)}$ and $Q^{(1)}$ are the steady state dimensionless temperature , radiative energy density and radiative flux , respectively.

From the equations (35) - (37) Q_x can be eliminated and the system reduced to the following set of differential equations

$$\left\{\frac{d^2}{d\eta^2}-U_y\frac{d}{d\eta}-k^2-\frac{FR}{{}_Y E_c}ik(u-\bar{c})\right\}\psi - NF\left\{\frac{dQ_y}{d\eta}+\frac{k^2\phi}{3w\theta^n}\right\}-$$
$$2F\left\{\left[\frac{dU_x}{d\eta}+ikU_y\right]\frac{dU}{d\eta}+M^2\left[\frac{db_x}{d\eta}-ikb_y\right]\frac{db}{d\eta}\right\} \quad = \quad 0, \quad (38)$$

$$nw\theta^{n-1}\Sigma^{(1)}\psi+\left[w\theta^n+\frac{k^2}{3n\theta^n}\right]\phi+\frac{dQ_y}{d\eta}=4(n+4)w\theta^{n+2}\psi, \quad (39)$$

$$\frac{d\theta}{d\eta}+3w\theta^n Q_y+3n\theta^{n-1}Q^{(1)}\psi \quad = \quad 0. \quad (40)$$

Therefore a system of seven differential equations governing the present stability problem for radiative flow may be taken as equations (24) , (30), (32), (34), (38), (39) and (40). Thus , with radiative effects present , additional equation arise. Moreover , since the system of equations (29), (30), (32), (34) can be solved separately from the system (38) - (40) , thermal radiation therefore cannot provide a stabilizing influence on the first system.

In the situation in which perturbations U_y and b_y arise the stability criteria arise from the homogeneous equation (32) and (34) alone which, as remarked, earlier, have been analysed by Lock and others. should however U_y and b_y be identically zero these two equations are automatically satisfied. Also equations (29), (30) then show that U_x and b_x must be identically zero as well.

When thermal radiation is a present , should disturbances in temperature , radiative energy density occur , equation (38) - (40) enter the analysis. But should velocity and magnetic filed not be disturbed , then the final set of equations (38) - (40) become homogeneous and supplementary stability criteria may arise from their solutions. the variable Q_y may be eliminated from them to yield a pair of equations for ϕ and ψ . These are;

$$\left\{\frac{d^2}{d\eta^2}-\frac{n}{\theta}\frac{d}{d\eta}-\left[k^2+3w^2\theta^{2n}\right]\right\}\phi+\left\{\begin{array}{l}3nw\theta^2Q^{(1)}\dfrac{d}{d\eta}+12(n+4)w^2\theta^{2n+3}\\ +2nw\theta^n-3nw^2\theta^{2n}\Sigma^{(1)}\end{array}\right\}\psi=0, \quad (41)$$

$$\left\{\frac{d^2}{d\eta^2}-k^2-\frac{FR}{{}_Y E_c}ik(u-\bar{c})-4(n+4)wNF\theta^{n+3}+nFN\theta^n\Sigma^{(1)}\right\}\psi - NwF\theta^n\phi=0 \quad (42)$$

Equations (41) and (42) are the equations be finally analysed for the purpose of the present stability problem. The steady state boundary conditions 'see [4]' remain unchanged for the present problem. Those relating to the radiative flux and energy density become.

$$\phi+\left(\frac{4}{e_1}-2\right)Q_y \quad = \quad 0, \ at \quad \eta=-1 \quad (43)$$
$$\phi+\left(\frac{4}{e_2}-2\right)Q_y \quad = \quad 0, \ at \quad \eta=1.$$

Using equation (40) , these two boundary conditions (43) may be written as :

$$\frac{d\phi}{d\eta}-\frac{3we_1}{2(2-e_1)}\phi=0 \;\; at \;\; \eta=-1 \qquad (44)$$

$$\frac{d\phi}{d\eta}-\frac{3e_2 w\theta^n}{2(2-e_2)}\phi=0 \;\; at \;\; \eta=1.$$

The other two boundary conditions on the temperature disturbance take the form

$$\psi \;=\; 0, \; at \;\; \eta=\pm 1. \qquad (45)$$

4. Stability Analysis of a Disturb System Using the Weighted Residual Method

Equations (41) and (42) define an eigen value problem. 'If the imaginary part of the eigenvalue $\bar{c}$ is positive, the flow will be unstable'.

An approximate solution to the stability problem may be produced using Weighted Residual Method. Finlayson [8] and Sukanek [9] give examples of Weighted Residual Method, applied to other stability problems. The variables ϕ and ψ are expanded in series in terms of complete sets of orthogonal trial functions which satisfy the boundary conditions (44) and (45) and an approximate formulation obtained by appropriate truncation. The coefficients of the various terms in the series expansions are chosen by adjusting the residual errors resulting from the substitution of these truncated expansions into the original differential equations to be orthogonal to the trial functions in the domain of interest.

The radiative energy density disturbance, ϕ, subject to the boundary conditions (44) may be expanded in a set of function, see for example Chandrasekhar (10), as follows:

It is well known that a simple Sturm-Liouville problem over the domain $|\eta|\le 1$ is associated with a differential equation of the form

$$\frac{d^2\phi}{d\eta^2}+\lambda^2\phi=O, \qquad (1)$$

with appropriate boundary conditions at $\eta=\pm 1$.

The solution in terms of a complete set of orthogonal functions may be written,

$$\phi=\sum_r\{a_r\cos\lambda_r\eta+b\sin\lambda_r\eta\}, \qquad (2)$$

Where a_r and b_r are constants, are λ_r and the relevant set of eigenvalues.

A typical term of equation (7.2) is

$$\phi_r=a_r\cos\lambda_r\eta+b_r\sin\lambda_r\eta.) \qquad (3)$$

Fit the boundary condition, (6.44) , to this typical term and one may obtain

$$(3a)\;\; b_r=\frac{a_r[\cos\lambda_r+z_1\lambda_r\sin\lambda_r]}{\sin\lambda_r-z_1\lambda_r\cos\lambda_r}, \text{ at } \eta=-1 \;\; (3a)$$

$$b_r=\frac{a_r[z_2\lambda_r\sin\lambda_r-\cos\lambda_r]}{\sin\lambda_r-z_2\lambda_r\cos\lambda_r}, \text{ at } \eta=1, \;\; (3b)$$

Where

$$z_1=\frac{-3we_1}{2(2-e_1)},\;\; z_2\frac{3we_2\Theta^n}{2(2-e_2)}.$$

Then, it follows from (3a) and (3b) that

$$(1+z_1z_2\lambda_r^2)\sin 2\lambda_r+(z_2-z_1)\lambda_r\cos 2\lambda_r=0. \qquad (4)$$

The equation thus yields the set of internal eigen values λ_r.

Using (3a), the truncated eigen function expansion for ϕ (expressed as a truncated Fourier series) may be written as

$$\phi_p=\sum_{r=1}^{p}A_r\{\sin\lambda_r(\eta+1)-z_1\lambda_r\cos\lambda_r(\eta+1)\} \qquad (5)$$

Where

$$A_r=\frac{a_r}{\sin\lambda_r-Z_1\lambda_r\cos\lambda_r}, \text{ for r = 1 , 2 , … , p}$$

and λ_r are the sequence of roots of equation (4).

The set of solution $\{\sin\lambda_r(\eta+1)-Z_1\lambda_r\cos\lambda_r(\eta+1)\}$ possess the orthogonal property, such that

$$\int_{-1}^{1}\{\sin\lambda_r(\eta+1)-Z_1\lambda_r\cos\lambda_r(\eta+1)\}\{\sin\lambda_s(\eta+1)-Z_1\lambda_s\cos\lambda_s(\eta+1)\}d\eta=0, \; for \; r\neq s, \; (6)$$

Where $S=1,\ldots,P,$ as may be easily verified.

In a similar way, the temperature disturbance ψ , subject to the boundary condition (45), may be expanded in a Fourier series. The truncated expansion is

$$\psi_p=\sum_{r=1}^{p}\left\{B_r\cos\frac{1}{2}(2r-1)\pi\eta+C_r\sin r\pi\eta\right\}, \;\; (7)$$

Where B_r and C_r are constants.

Both $\left\{\cos\frac{1}{2}(2r-1)\pi\eta\right\}$ and $\{\sin r\pi\eta\}$ form orthogonal sets on the interval $[-1,1]$, such that

$$\int_{-1}^{1}\cos\frac{1}{2}(2r-1)\pi\eta\cos\frac{1}{2}(2s-1)\pi\eta=\begin{cases}0, \; forr\#s\\ 1, \; forr=s.\end{cases} \qquad (8)$$

$$\int_{-1}^{1}\sin r\pi\eta\sin sd\eta=\begin{cases}0, \; forr\#s\\ 1, \; forr=s.\end{cases} \qquad (9)$$

$$\int_{-1}^{1} \cos\frac{1}{2}(2r-1)\pi\eta \sin s\pi\eta = 0.\ ) \quad (10)$$

If the series approximation to ϕ and ψ , equation (5)and (7), are substituted into the left hand sides of the differential eigenvalues equations (41) and (42), there remain residuals. Then denote by $R_p^{'}$ the residual arising from equation (41), and by $R_p^{''}$ that arising from equation (42).

It follows that

$$R_p^{'} = \sum_{r=1}^{p}\{-[\{\ \lambda_r^2+\kappa^2+3w^2\theta^{2n}\ \}\{\sin\lambda_r(\eta+1)-Z_1\lambda_r\cos\lambda_r(\eta+1)\ \}+$$

$$\frac{n\lambda_r}{\theta}\{\cos\lambda_r(\eta+1)+Z_1\lambda_r\sin\lambda_r(\eta+1)\}\ ]A_r +$$

$$\left[\left\langle 3nw\theta^n\frac{dQ^{(1)}}{d\eta}-3nw^2\theta^{2n}\Sigma^{(1)}+12(n+4)w^2\theta^{2n+3}\right\rangle coc\frac{1}{2}(2r-1)\pi\eta-\frac{3n(2r-1)\pi}{2}w\theta^nQ^{(1)}\sin\frac{1}{2}(2r-1)\pi\eta\right]$$

$$B_r+\left[\left\langle 3nw\theta^n\frac{dQ^{(1)}}{d\eta}-3nw^2\theta^{2n}\Sigma^{(1)}+12(n+4)w^2\theta^{2n+3}\right\rangle\sin r\pi\eta+3nr\pi w\theta^nQ^{(1)}\cos r\pi\eta\right]C_r\ \Big\},(11)$$

$$R_p^{''} = \sum_{r=1}^{p}\left\{\left[nNF\theta^n\Sigma^{(1)}-4(n+4)wNF\theta^{n+3}-\frac{1}{4}(2r-1)^2\pi^2-\kappa^2-\right]\right\}$$

$$\frac{FR}{\gamma E_c}i\kappa\left(u-\bar{c}\right)\Big]B_r\cos\frac{1}{2}\left(2r-1\right)\pi\eta+\left[nNF\theta^n\Sigma^{(1)}-r^2\pi^2-\kappa^2\frac{FR}{\gamma E_c}i\kappa\left(u-\bar{c}\right)-4\left(n+4\right)wNF\theta^{n+3}\right]$$

$$C_r\sin r\pi\eta+wNF\theta^n[\sin\lambda_r(\eta+1)-Z_1\lambda_r\cos\lambda_r(\eta+1)]A_r\Big\}.\ (12)$$

An application of Weighted Residual Method requires that the constants A_r , B_r and C_r be such as to satisfy the equations

$$\int_{-1}^{1} R_p^{'}\{\sin\lambda_s(\eta+1)-Z_1\lambda_s\cos\lambda_s(\eta+1)\}d\eta = 0.\ (13)$$

$$\int_{-1}^{1} R''\cos\frac{1}{2}(2s-1)\pi\eta = 0. \quad (14)$$

$$\int_{-1}^{1} R''\sin r\pi\eta = 0 \quad (15)$$

Equations (11) and (12) are substituted into equation (13) – (15) and the integrals evaluated. It is useful to define the following integrals.

$$X^{(1)}(s.r) = \int_{-1}^{1}\{-[\lambda_r^2+k^2+3w^2\theta^{2n}][\sin\lambda_r(\eta+1)-z_1\lambda_r\cos\lambda_1(\eta+1)]-$$

$$\frac{\eta\lambda_r}{\theta}[\cos\lambda_r(\eta+1)+Z_1\lambda_r(\eta+1)]\ \}\{\ \sin\lambda_r(\eta+1)-Z_1\lambda_s\cos\lambda_s(\eta+1)\ \}d\eta, \quad (16)$$

$$Y^{(1)}(s.r) = \int_{-1}^{1}\{\left[3nw\theta^n\frac{dQ^{(1)}}{d\eta}-3nw^2\theta^{2n}\Sigma^{(1)}+12(n+4)w^2\theta^{2n+4}\right]\cos\frac{1}{2}(2r-1)\pi\eta-$$

$$\frac{3n(2r-1)\pi}{2}w\theta^1\sin\frac{1}{2}(2n-1)\pi\eta\ \}\{\sin\lambda_s(n+1)-Z_1\lambda_s\cos\lambda_s(n+1)\}d\eta, \quad (17)$$

$$Z^{(1)}(s,r) = \int_{-1}^{1}\left\{\left[3nw\theta^n\frac{dQ^{(1)}}{d\eta}3nw^2\theta^{2n}\Sigma^{(1)}+12(n+4)w^2\theta^{2n+3}\right]\sin r\pi\eta+3nr\pi\theta^nQ^{(1)}\cos r\pi\eta\right\} \quad (18)$$

$$\{\sin\lambda_s(\eta+1)-Z_1\lambda_s\cos\lambda_s(\eta+1)\}d\eta,$$

$$X^{(2)}(s,r) = wNF\int_{-1}^{1}\left\{\sin\lambda_r(\eta+1)-Z_1\lambda_3\cos\lambda(\cos)\frac{1}{2}\right\}\cos\frac{1}{2}(2s-1)\pi\eta d\eta,\ (19)$$

$$Y^{(3)}(s,r)=\int_{-1}^{1}\left\{\frac{1}{4}(2r-1)^2\pi^2+k^2+4(n+4)wNF\theta^{n+3}-nNF\,\Sigma^{(1)}\right\}\cos\frac{1}{2}(2r-1)\pi\eta \quad (20)$$

$$]\cos\frac{1}{2}(2s-1)\pi\mu d\eta,$$

$$W^{(1)}(s,r)=\frac{FRk}{{}_yE_c}\int_{-1}^{1}u\cos\frac{1}{2}(2r-1)\pi\eta\cos\frac{1}{2}(2s-1)\pi\eta d\eta, \quad (21)$$

$$Z^{(2)}(s,r)=\int_{-1}^{1}\{4(n+4)wnF\theta^{n+3}-nF\theta^{n}\,\Sigma^{1}\}\sin r\pi\eta\cos\frac{1}{2}(2s-1)\pi\eta, \quad (22)$$

$$Y^{(3)}(s,r)=\int_{-1}^{1}\{4(n+4)wNF\theta^{n+3}-nNF\theta^{n}\,\Sigma^{(1)}\}\cos\frac{1}{2}(2r-1)\pi\eta\sin s\pi\eta d\eta, \quad (23)$$

$$X^{(3)}(s,r)=wNF\int_{-1}^{1}\{\sin\lambda_r(\eta+1)-z_1\lambda_r\cos\lambda_r(n+1)\}\sin s\pi\eta d\eta, \quad (24)$$

$$Z^{(3)}(s,r)=\int_{.1}^{1}\{r^2\pi^2+k^2+4(n+4)wNF\theta^{n+3}-nNF\theta^{n}\,\Sigma^{(1)}\}\sin r\pi\eta.\sin s\pi\eta d\eta, \quad (25)$$

$$w^{(2)}(s,r)=\frac{FRk}{yE_c}\int_{-1}^{1}u\sin r\pi\eta.\sin s\pi\eta d\eta, \quad (26)$$

In this way , the differential eigenvalue problem is reduced to an approximate algebraic problem expressed as equations (13)-(15) which can now be written in the form.

$$\sum_{r=1}^{p}\{x^{(1)}A_r+Y^{(1)}B_r+z^{(1)}c_r\}=0,\quad s=1,2,....,p \quad (27)$$

$$\sum_{r=1}^{p}\{[y^{(2)}+iw^{(1)}-v\delta_{sr}]B_r+z^{(2)}c_r-x^{(2)}A_r\}=0,$$
$$s=1,2,....,p \quad (28)$$

$$\sum_{r=1}^{p}\{Y^{(3)}B_r+[z^{(3)}+iw^{(2)}-v\delta_{sr}]c_r-X^{(3)}A_r\}=0,$$
$$s=1,2,....,p \quad (29)$$

Where $v=\dfrac{ic^{-}Rk}{yE_c}$ and δ_{sr} is the cronecker delta.

It should be noted that equations (27) do not contain the eignvalue v.

The coefficients A_r can thus be determined from the system (27) in term of the coefficients B_r and C_r and inserted in to equations (28) , (29). Then the latter two equations thus become the governing equations for the problem which is reduced to the classical form. for example in the first approximation with $s=r=1,$ Equation (27) yields

$$A_1=\frac{1}{x^{(1)}(1,1)}\left[y^{(1)}(1,1)B_1+Z(1)(1,1)C_1\right] \quad (30)$$

Equations (28) and (29) become

$$\left\{Y^{(2)}(1,1)+\frac{y^{(1)}(1,1)X^{(2)}(1,1)}{X^{(1)}(1,1)}+iw^{(1)}(1,1)-v\right\}B_1+$$
$$\left\{Z^{(2)}(1,1)+\frac{Z^{(1)}(1,1)X^{(2)}(1,1)}{X^{(1)}(1,1)}\right\}C_1=0, \quad (31)$$

$$\left\{Y^{(3)}(1,1)+\frac{Y^{(1)}(1,1)X^{(3)}(1,1)}{X^{(1)}(1,1)}\right\}B_1$$
$$+\left\{Z^{(3)}(1,1)+\frac{z^{(1)}(1,1)X^{(3)}(1,1)}{X^{(1)}(1,1)}+iw^{(2)}(1,1)-v\right\}c_1=0. \quad (32)$$

The matrix of coefficients of the homogenous equations (31) and (32) is

$$\begin{bmatrix} Y^{(2)}(1,1)+\dfrac{Y^{(1)}(1,1)X^{(2)}(1,1)}{X^{(1)}(1,1)}+iw^{(1)}(1,1)-vz^{(2)}(1,1)+\dfrac{Z^{(1)}(1,1)X^{(2)}(1,1)}{X^{(1)}(1,1)} \\ Y^{(3)}(1,1)+\dfrac{Y^{(1)}(1,1)X^{(3)}(1,1)}{X^{(1)}(1,1)}\,Z^{(3)}(1,1)+\dfrac{Z^{(1)}(1,1)X^{(3)}(1,1)}{X^{(1)}(1,1)}+iw^{(2)}(1,1)-v \end{bmatrix} \quad (33)$$

Since the stability , or otherwise , of the flow depends upon the sign of the imaginary part of $\bar{c}$ and $v=\dfrac{ik\bar{c}R}{yE_c}$,interest is centered on the sign of the real part of v. For the particular mode associated with an eigenvalue, if this has positive real part, the flow will be stable, but if is negative, the flow will be unstable. As already remarked, this simple example is, however, just a first approximation to the solution to the full problem.

As opposed to the differential eigenvalue problem, which possesses an infinite set of eigenvalue, the approximate algebraic problem, yields only a finite number of characteristic values. The number of eigenvalues depends on how many terms are used in the orthogonal functions expansions approximating the radiative energy density and temperature disturbances. If the real part of any one of these eignvalues is negative, the flow in large will be unstable. Therefore, in a search for a mode of the eigenvalue of the least positive real part.

5. Analysis, Results and Recommendations

Before describing the calculation solution of the algebraic eigenvalue problem, as a whole, it is necessary to determine the solutions of the steady state variables ,viz.velocity ,temperature , radiative flux and radiative energy density, as well as the internal eigen value λ_r.

The value of the physical parameters in this part of the paper are the same as those of paper[4],unless it is stated otherwise. For variable absorption coefficient n is taken to be five, whilst for constant absorption coefficient n is identically zero.

The numerical results of the solution of the algebraic equation (7.4), for λ, are dependent upon the wall emissivities ,Bouguer number and the temperature ratio of the walls. Although it can be solved for any value of these parameters in Table 1, the first positive half a dozen of the values of λ_r , for different values of Θ,are listed for fixed values of $e_1, e_2$and w. The results are arranged in a matrix from, which in its final derivation after elimination of the coefficients A_r has complex elements. The complex matrix is first reduced to upper Hessenberg form using stabilized elementary similarity transformation. The eigenvalues then found using the modified LR Algorithm for complex Hessenberg matrices. It has been found that the output of the calculation used to calculate the integrals to within a reasonable accuracy became unstable as r increases, due to the oscillatory nature. For the sixth approximation (r= 6), the convergence of the result fails. However if the modulus is examined the solution for the leading eigenvalues shows a good agreement between the third, fourth and fifth approximations and convergence is indicated. However one notes the rapid increase of the computing time required as r increases.

Thus it seems that a good compromise between computing time precision of results should be obtained by taking r as four. This means that a total of eight eigenvalues are calculated for each case.The present problem was solved numerically by Helliwell [1oc.cit]. In the absence of thermal conductivity ($f \longrightarrow \alpha$, in the present notation) and constant absorption coefficient. The results suggested that the thermal transfers under consideration are always stable. However, it was remarked earlier, [4], that it is very difficult to obtain the steady state distributions for extreme values of some of the physical parameters.

Thus a direct comparison with the earlier work of Helliwell cannot be made. Attention is paid to the sign of the real parts of the eigen values. If the least positive real part of an eigenvalue is approaching zero, the solution then indicates an approach to neutral stability. For given values of k the Reynolds number R is justed to determine the neutral stability appropriate point to appropriate accuracy. Although the computation may be carried out for a wide range of values of the physical parameters, a selection of them is taken as indicated in the Tables. The results of calculation are presented in Tables 1- 8, for both variable and constant absorption coefficient. It should be particularly noted that the results calculation are not reliable at low Reynolds number, because of the difficulties in the convergence calculation methods.The main object of the present analysis is to examine the effect of the introduction of a variable absorption coefficient upon the stability in comparison with that of a constant absorption coefficient. The results form Tables, in spite of the unreliability of computation at law Reynolds numbers indicate quite clearly that, effect of a variable absorption coefficient is to increase any tendency towards stability. The present stability analysis relates to only a simple case of flow pattern. A Further and useful development would be to consider the influence of buoyancy forces and convective heat transfer upon the stability when the channel walls are differentially and non-uniformly heated, viz. examine the stability of the complex flow configuration studied in the first part of this paper. It would also be profitable to seek to reformulate the reduction of the differential eigen value problem to an algebraic eigenvalue problem by use of an alternative orthogonal function expansion.

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Appendix

Table 1

The values of the internal eigen values λ_r ; equation (7,4),for different values of Θ.

Θ / r	0.5	1.0	1.5
1	1.41352	1.36835	1.27346
2	2.83573	2.75032	2.59616
3	4.27238	4.15504	3.97808
4	5.72558	5.58578	5.40766
5	7.19482	7.04126	6.87185
6	8.67840	8.51805	8.36084

e =1.0 w=0.1

Table 2

The real part of the eigen values, v.

F = 0.0025 B0 =1000 Θ=0.5

w = 0.1 M=1.0 K=0.0

e = 1.0 Y=$5/3\,E_c$=0.001

R / K	10	10^3	10^5
2.0	0.9021	28.4463	28.0784
1.6	0.5955	26.9106	26.6383
1.2	0.3445	25.5836	25.5183
0.8	0.1564	24.1925	24.7182
0.4	0.0396	20.7119	24.2379

Variable Absorption Coefficient Always Stable.

Table 3

The real part of the eigen values, v.

F=0.0025 B0 =10 Θ=0.5

w=0.1 M =1.0 K=0.0

e=1.0 Y=$5/3\,E_c$=0.001

R / K	0.1	10
2.0	0.0001	8.2462
1.6	0.0001	6.2511
1.2	0.00004	4.6297
0.8	0.00002	3.4359
0.4	0.000004	2.7066

Variable Absorption Coefficient

It seems that there is a tendency towards instability at low values of K and R. The computation becomes unstable however at these low values.

Table 4

The real part of the eigenvalues, v.

F=0.01 B0 =1000 Θ=0.5

w=0.1 M =1.0 K=0.0

e=1.0 Y=$5/3\,E_c$=0.001

R K	10	10^3	10^5
2.0	9.8099	28.5887	26.3475
1.6	6.9167	27.1427	24.9071
1.2	4.3788	26.0098	23.7871
0.8	2.2224	25.1730	22.9871
0.4	0.6151	24.4936	22.5071

Variable Absorption Coefficient Always stable.

Table 5

The real part of the eigen values, v.

F=0.0025 B0 =10 Θ=0.5

w=0.1 M =1.0 K=0.0

e=1.0 Y=$5/3\,E_c$=0.001

R / K	1	10
2.0	+ 0.0246	-23.5159
1.6	- 0.0597	- 46.5593
1.2	+ 0.1576	-81.4367
0.8	+ 0.3336	-114.8629
0.4	+ 0.5183	-139.1288

Constant Absorption Coefficient

There is an indication of stability at low values of Reynolds number. The computation becomes unstable however at these low values.

Table 6

The real part of the eigen values, v.

F=0.01 B0 =10 Θ=0.5

w=0.1 M =1.0 K=0.0

e=1.0 Y=$5/3\,E_c$=0.001

R / K	1	10
2.0	+ 0.6330	- 223.2128
1.6	- 0.5521	- 325.1491
1.2	-7.0433	- 461.3715
0.8	- 22.2400	- 593.5934
0.4	- 32.8725	- 690.4365

Constant Absorption Coefficient

There is an indication of stability at low values of Reynolds number. The computation becomes unstable however at these low values.

Table 7

The real part of the eigen values, v.

F=0.01 B0 =10 Θ=1.5

w=0.1 M =1.0 K=0.0

e=1.0 Y=$5/3\,E_c$=0.001

R / k	1	10
2.0	+ 0.3202	12.6306
1.6	- 27.3751	313.9828
1.2	-227.6479	2291.0284
0.8	- 408.6000	4094.6538
0.4	- 603.1960	6007.2109

Constant Absorption Coefficient

There is an indication of stability at low values of Reynolds number. The computation becomes unstable however at these low values.

Table 8

The real part of the eigen values, v.

F=0.25 B0 =10 $\ominus$=0.5

w=0.1 M =1.0 K=0.0

e=1.0 Y=$5/3$ E_c=0.001

k \ R	1	10
2.0	- 36.0376	-393.6033
1.6	- 890.7216	-9411.2733
1.2	-539.5492	-11863.8890
0.8	- 598.0973	-5904.9493
0.4	- 626.6697	-6147.6863

Constant Absorption Coefficient

It is always unstable, but there is an indication of decreasing stability at smaller values of Reynolds number. The computation becomes unstable however at these low values.