Bayesian Analysis of Exponential Distribution Using Informative Prior

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Abstract: ***The exponential distribution is a well known distribution as a life time model in life testing experiments. In this paper. Bayesian and Classical analysis is discussed. For comparison Bayesian credible interval, coefficient of skewness, posterior predictive distribution is considered using informative prior as Gamma for exponential model.***

Keywords: Posterior, prior, Bayesian loss functions

1. Introduction

The objective of this study is to obtain the posterior distributions for the parameters of the exponential model using informative prior (Gamma). Bayes estimators are obtained under squared error loss function and quadratic loss function. Lee (1989) and Berger (1985) have discussed Bayes estimator in detail. Aslam et al.(2010)considered Bayes estimator of the parameter of exponential distribution using non-informative prior and they have compared with classical estimation. In this paper Bayesian credible intervals for the parameter of a exponential distribution and the posterior predictive distribution is also discussed. Bayes estimator under loss function is also considered. Simulation study is also done for this model.

2. Bayes Estimation

2.1. Model

The distribution of the time-to-failure system usually follows exponential distribution.Let $x_1, x_2, \ldots x_n$ be a random sample of size n drawn from exponential distribution with scale parameter θ denoted by $\exp(\theta)$, θ is independent. The probability density function of x is given by

$$f(x/\theta) = \theta e^{-\theta x};\ \theta > 0, x \geq 0 \qquad (1)$$

The likelihood function of $x_1, x_2, \ldots x_n$ given θ is given by

$$L(x_1, x_2, \ldots x_n / \theta) = \theta^n e^{-\theta \sum_{i=1}^{\infty} x} \qquad (2)$$

2.2. Prior distribution

The prior distribution of the parameter is the probability distribution that represents uncertainty about the parameter before the current data are examined. Here the parameter θ is assumed to follow gamma distribution with p.d.f. given by

$$g(\theta) = \frac{\beta^\alpha}{\Gamma\alpha} e^{-\theta\beta} \theta^{\alpha-1}; 0 < x < \infty \qquad (3)$$

2.3 Posterior distribution of θ:

The posterior distribution of θ given $x_1, x_2, \ldots x_n$ is given by

$$f(\theta / x) = \frac{L(x_1, x_2, \ldots x_n / \theta) g(\theta)}{\int_0^\infty L(x_1, x_2, \ldots x_n / \theta) g(\theta) d\theta}$$

$$= \frac{\theta^n e^{-\theta \sum_{i=1}^{n} x_i} e^{-\theta\beta} \beta^\alpha \theta^{\alpha-1} \Gamma\alpha}{\Gamma\alpha \beta^\alpha \int_0^\infty \theta^n e^{-\theta \sum_{i=1}^{n} x_i} e^{-\theta\beta} \theta^{\alpha-1} d\theta} \quad \text{using (3.2) and (3.3)}$$

$$f(\theta / x) = \frac{\theta^{n+\alpha-1} e^{-\theta(\beta + \sum_{i=1}^{n} x_i)} (\beta + \sum_{i=1}^{n} x_i)^{n+\alpha}}{\Gamma n + \alpha} \qquad (4)$$

Which is density of the gamma distribution with parameters $(n+\alpha)$ and $(\beta + \sum_{i=1}^{n} x_i)$ .Here we find the posterior distribution θ for the given data is Gamma[$(n+\alpha)$, $(\beta + \sum_{i=1}^{n} x_i)$] (5)

3. The Posterior Predictive Distribution Using the Gamma Prior

The posterior predictive distribution for $Y = x_{n+1}$ is given by

$$f(y/x) = \int_0^\infty f(y/\theta) f(\theta/x) d\theta \qquad (6)$$

$$= \frac{(\beta + \sum_{i=1}^{n} x_i)^{n+\alpha}}{\Gamma n + \alpha} \int_0^\infty \theta e^{-\theta y} \theta^{(n+\alpha)-1} e^{-\theta(\beta + \sum_{i=1}^{n} x_i)} d\theta$$

$$= \frac{(\beta + \sum_{i=1}^{n} x_i)^{n+\alpha}}{\Gamma n + \alpha} \int_0^\infty \theta^{(n+\alpha+1)-1} e^{-\theta(\beta + y + \sum_{i=1}^{n} x_i)} d\theta$$

$$= \frac{(\beta+\sum_{i=1}^{n} x_i)^{n+\alpha}\,\Gamma n+\alpha+1}{(\beta+y+\sum_{i=1}^{n} x_i)^{n+\alpha+1}\,\Gamma n+\alpha}$$

$$f(y/x) = \frac{(n+\alpha)(\beta+\sum_{i=1}^{n} x_i)^{n+\alpha}}{(\beta+y+\sum_{i=1}^{n} x_i)^{n+\alpha+1}} \qquad (7)$$

Which is exponential gamma distribution i.e. EG(α,β) with parameters $\alpha = n+\alpha$ and $\beta = \beta + y + \sum_{i=1}^{n} x_i$ (8)

4. Bayesian Credible Interval For θ

Kapur and Lamberson (1977) show that if $x_1, x_2, \ldots x_n$ are independent and identically distributed exponential random variables, $2\theta\sum x_i$ has a chi square distribution with 2n degrees of freedom. $100(1-\gamma)$ Percentage highest density (HDR) for posterior distribution of the parameter θ is

$$\frac{\chi^2_{2(\alpha+n)}(1-\frac{c}{2})}{2(\beta+\sum x_i)} < \theta < \frac{\chi^2_{2(\alpha+n)}(\frac{c}{2})}{2(\beta+\sum x_i)}, 0<\lambda, x<\infty \qquad (9)$$

5. Bayesian Estimation Under Squared Error Loss Function And Quadratic Loss Function:

Let $L(\theta,k)$ be a loss function and E [$L(\theta,k)$] is a risk function, then the Bayes decision is a decision k^*, which minimizes risk function and k^* is the best decision. If the decision is choice of an estimator then the Bayes decision is a Bayes estimator.

5.1. Bayes estimate of θ under squared error loss function:

Bayes estimate of θ under squared error loss function is given by

$$\hat{\theta}_{BSE} = E[\, f(\theta / \underline{x})\,] \qquad (10)$$

$$= \frac{1}{\Gamma n+\alpha}\int_0^\infty \theta\theta^{(n+\alpha)-1} e^{-\theta(\beta+\sum_{i=1}^{n} x_i)} (\beta+\sum_{i=1}^{n} x_i)^{n+\alpha}\, d\theta$$

using (4)

$$= \frac{(\beta+\sum_{i=1}^{n} x_i)^{n+\alpha}}{\Gamma n+\alpha}\int_0^\infty \theta^{(n+\alpha+1)-1} e^{-\theta(\beta+\sum_{i=1}^{n} x_i)}\, d\theta$$

$$= \frac{(\beta+\sum_{i=1}^{n} x_i)^{n+\alpha}\,\Gamma n+\alpha+1}{\Gamma n+\alpha(\beta+\sum_{i=1}^{n} x_i)^{n+\alpha+1}}$$

$$\hat{\theta}_{BSE} = \frac{n+\alpha}{\beta+y+\sum_{i=1}^{n} x_i} \qquad (11)$$

5.2. Bayes estimate of θ under quadratic loss function:

Bayes estimate of θ under quadratic loss function is given by

$$\hat{\theta}_{BQL} = \frac{n+\alpha-2}{\beta+\sum x_i} \qquad (12)$$

6. Numerical Example

Random sample of the male mice data of days (in hundred) until death due to Thymic Lymohoma exposed to 300 rads of radiation is taken from Kalbfleisch and Prentice (2002):

1.59,1.89,1.91,1.98,2.00,2.07,2.20,2.35,2.45,2.50,2.56,2.61, 2.65,2.66,2.80,3.43,3.56,3.83, 4.03,4.14,4.28,4.32.

For n=22 observations, the sum is 61.81.The posterior distribution of parameter θ for given data ($x = x_1, x_2, \ldots x_{22}$) using equation (5) is the Gamma distribution with parameters $\alpha = 11$ and $\beta = 4$ (i.e.) Gamma [$(n+\alpha),(\beta+\Sigma x_i)$] is gamma [33, 65.81]

6.1 The posterior predictive distribution using gamma prior:

Hence posterior predictive distribution of Y= x_{23} is EG (33, 65.81)

6.2 Bayesian interval estimate using gamma prior with α =11 and β =4

The Bayesian interval estimates of the parameter theta is presented in the table 1

Table 1: Bayesian interval estimate using gamma prior with α =11 and β =4

95% Credibility interval	99% Credibility interval
(0.3451707,0.6864375)	(0.3670064,0.7546758)

It is found that the length of 95% and 99% high density region for θ are approximately equal.

Table 2: Bayes estimator under loss function

Loss function $L(\theta,k)$	Bayes estimator $\hat{\theta}$	Bayes estimator $\hat{\theta}$
$L_1 = (\theta-k)^2$	$\frac{n+\alpha}{\beta+\sum x_i}$	0.50144
$L_2 = (1-\frac{\theta}{k})^2$	$\frac{n+\alpha-2}{\beta+\sum x_i}$	0.45585

Table 2, shows Bayes estimator of loss functions for squared error loss function (BSE-L_1) and quadratic loss function (BQL-L_2).

6.3 Coefficient of skewness

The coefficients of skewness are calculated from the posterior and posterior predictive distributions. The coefficient of skewness is calculated using the formula is presented in table 3

Coefficient of skewness = $\gamma = 2\sqrt{\frac{1}{\alpha}}$ and is given in table 3

Table 3: Coefficient of skewness for posterior and predictive distribution using gamma prior

Posterior parameters $(\alpha = 33, \beta = 65.81)$	Coefficient of skewness γ
Posterior	0.3482
Predictive posterior	2.1969

From table 3 as $\gamma > 0$, we find that the posterior and posterior predictive distributions are positively skewed.

7. Simulation Study

In this study random samples were generated from exponential distribution and the performance of squared error loss function (BSE) and quadratic loss function (BQL) were compared based on Bayes estimators. Sample size n=5,10,15,20,50,100 to represent both small and large sample. The values of the parameter are discussed for the following cases :

(i) when n=20, α =2, β =3 and θ =1,1.5,2,3.

(ii) when n=10,15,20,30,50,75,100, α =1.5, β =2 and θ =1.5.

The number of replications used was N=1000. The results are presented below for the estimated parameters and their corresponding mean square error is calculated by

$$MSE = E[(\hat{\theta} - \theta)]^2$$

The simulation was run using the package R(3.41) (free available from http:/www.r-project.org). The results are presented below

7.1 Case (i) The performance of the estimators for different values of parameter 'θ' is shown in the table:4.

Table 4: Estimated value and MSE of BSE and BQL of parameter 'θ' when θ =1,1.5,2,3, n=20, α =2 and β =3

θ	Criteria	$\hat{\theta}_{BSE}$	$\hat{\theta}_{BQL}$
1	Estimated value	0.9223	0.8384
	MSE	0.0060	0.0261
1.5	Estimated value	1.3563	1.2330
	MSE	0.0206	0.0713
2	Estimated value	1.7014	1.5468
	MSE	0.0891	0.2054
3	Estimated value	2.2503	2.0457
	MSE	0.5620	0.9106

It is seen from the table: 3.4., that as the parameter increases Bayes estimator for BSE and BQL increases.

7.2 Case (ii) The performance of the estimators for different values of 'n' is shown in the table:5

Table 5: Estimated value an MSE of BSE and BQL of sample 'n' when θ =1.5, α =1.5, β =2, n=10,15,20,30,50,75,100

n	Criteria	$\hat{\theta}_{BSE}$	$\hat{\theta}_{BQL}$
10	Estimated value	1.3924	1.1502
	MSE	0.01158	0.1223
15	Estimated value	1.3754	1.2087
	MSE	0.0155	0.0849
20	Estimated value	1.4126	1.2812
	MSE	0.0076	0.0479
30	Estimated value	1.4407	1.3492
	MSE	0.0035	0.0227
50	Estimated value	1.4515	1.3951
	MSE	0.0024	0.0110
75	Estimated value	1.4786	1.4400
	MSE	0.0005	0.0036
100	Estimated value	1.4921	1.4627
	MSE	0.00006	0.0014

From the above table it seen that as the sample size increases parameters of BSE and BQL decreases. On comparing the values of MSE, $\hat{\theta}_{BSE}$ gives smaller value than $\hat{\theta}_{BQL}$

From the above table according to MSE, the relation among the estimators is

$$MSE(\hat{\theta}_{BSE}) \leq MSE(\hat{\theta}_{BQL})$$

8. Conclusions

From the above analysis through numerical example that Posterior and posterior predictive distribution is positively skewed. Bayesian interval estimate, the lengths of 95% and 99% credibility intervals for θ is also approximately equal Bayes estimators based on loss functions BSE and BQL are calculated. Through simulation study, it is found that as the sample size increases for fixed parameter theta, alpha and beta, MES of BSE is less than BQL. It is shown from simulation study that MSE decreases with increases in sample size. Also it is found that as censor time increases MSE decreases.

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