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Some New Results about the Convergence of Fuzzy Measurable Functions Sequence on Fuzzy Measure on Fuzzy Sets

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Abstract: ***The goal of this paper is to study the Convergence of fuzzy measurable functions sequence on fuzzy sets and get on some new results.***

Keywords: Fuzzy measure, exhaustive, order continuous, null additive, autocontinuous from below, accountably weakly null-additive fuzzy measure, null-continuous.

1. Introduction

In measure theory, several types of convergence were introduced for sequence of measurable functions on a measure space and some basic relations among these types were established [3].

In the proof of the theorem (9) we need thatμ is countably weakly null-additive because as is well known, sugeno's fuzzy measure loses additivity in general therefore if $\mu(A_n) = 0$ for all

$$n \geq 1 \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0, \{A_n\} \subset \mathcal{F}$$

Fuzzy measure generalization of measure theory. This generalizationis obtained by replacing the additivity axiom of measure theory with weak axiom of monotonicity and continuity [1]

The fuzzy measure, defined on σ-field, was introduced by Sugeno [20]. Ralescu and Adams [21] generalized the concepts of fuzzy measure and fuzzy integral to the case that the value of a fuzzy measure can be infinite, and to realize an approach from Subjective.

Jun Li [4] study order continuous and strongly order continuous of monotone set function and convergence of measurable functions sequence

Jun Li, Masami Yasuda, Qingshan Jiang, Hisakichi Suzuki and Zhenyuan Wang [2], Deli Zhang and Caimei Guo [5], studied some Convergence of sequence of measureable functions on Fuzzy measure spaces and generalized convergence theorems" obtained a series of new results.

After that, many authors studied Convergence of sequence of measureable functions on Fuzzy measure spaces and proved some results about it asG. J. Klir [6, 7], Jun Li, Radko Mesiar, Endre Pap and Erich Peter Klement [8], L.Y. Kui[9], L.Y. Kui and L. Baoding [10],

In this paper, we mention the definition of Fuzzy Measure on Fuzzy Set and study three types of convergence of sequence of fuzzy measurable functions defined on fuzzy sets; the concepts of "almost" and
"Pseudo" are introduced also too the
Convergence almost everywhere,
Convergence in fuzzy measure and almost uniformly convergence and get on some new results about them.

Definition (1): [17, 18]
Let Ω be anon empty set, a fuzzy set A in Ω(ora fuzzy subset inΩ) is a function from Ω into I, i.e. $A \in I^{\Omega}$. $A(x)$ is interpreted as the degree of membership of element x in a fuzzy set A for each $x \in \Omega$. a fuzzy set A in Ω is can be represented by the set of pairs:

$$A = \{(x, A(x)) : x \in \Omega\}$$

Note that every ordinary set is fuzzy set, i.e. $P(\Omega) \subseteq I^{\Omega}$.

Definition (2): [11, 12]
A family $\mathcal{F}$ of fuzzy sets in a set Ω is called a fuzzy σ −field on a set Ω If,
1) $\phi, \Omega \in \mathcal{F}$.
2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3) If $\{A_n\} \subset \mathcal{F}, n = 1, 2, 3, \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Evidently, an arbitrary σ −field must be fuzzy σ −field. A fuzzy measurable Space is a pair$(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a fuzzy σ −field on Ω. a fuzzy set A in Ω is called fuzzy measurable (fuzzy measurable with respect to the fuzzy σ −field) if $A \in \mathcal{F}$, i.e. any member of $\mathcal{F}$ is called a fuzzy measurable set.

Definition (3): [13]
Let$(\Omega, \mathcal{F})$be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty]$ is said to be a fuzzy measure on $(\Omega, \mathcal{F})$ if it satisfies the following properties:
(1) $\mu(\emptyset)$=0
(2) If $A, B \in \mathcal{F}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$

Definition (4): [14]
Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be
1. Exhaustive if
$\mu(A_n) \longrightarrow 0$ whenever $\{A_n\}$is infinite sequence of disjoint sets in $\mathcal{F}$

2. Order-continuous if $\mu(A_n) \longrightarrow 0$, whenever $A_n \in \mathcal{F}$, $n = 1, 2, \ldots$ and $A_n \downarrow \emptyset$.

Definition (5): [14, 15]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be Null-additive, if $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, and $\mu(B) = 0$.

Definition (6): [1]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be weakly null-additive, if for any $A, B \in \mathcal{F}$,

$$\boldsymbol{\mu(A) = \mu(B) = 0 \Longrightarrow \mu(A \cup B) = 0}$$

Remark (70):

The concept of null-null additive stems from a wings textbook which the book[1] derived from, in which it is said to be weak null additive. But we consider that it is more precise and vivid to call it "null-null additive"

Definition (8): [16]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be Countably weakly null-additive, if for any $\{A_n\} \subset \mathcal{F}, \mu(A_n) = 0$

$$, \text{for all } n \geq 1 \Longrightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$$

Definition (9): [16]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be Null-continuous, if $\mu(\bigcup_{n=1}^{\infty} A_n) = 0$ for every increasing sequence $\{A_n\}$ in $\mathcal{F}$ such that $\mu(A_n) = 0$, for all $n \geq 1$.

Definition (10): [19]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be
Autocontinuous from above (resp.
autocontinuous from below), if $\mu(B_n) \to 0$ implies $\mu(A \cup Bn \to \mu(A)$ (resp.
$\mu(A \cap B^c{}_n) \to \mu(A))$, whenever $A \in \mathcal{F}, \{B_n\} \subset \mathcal{F}$, μ is called autocontinuous if it is both autocontinuous from above and autocontinuous from below.

Definition (11): [14]

Let $\mathbb{C}(\Omega)$ be the collection of all real valued functions defined on a set Ω. Let $f, f_n \in \mathbb{C}(\Omega)$, $n \in \mathbb{N}$ and $A \subset \Omega$, we say that

1- $\{f_n\}$ converges point wise to f on A, if for every $x \in A$ and for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f(x) < \varepsilon$ for all $n > k$.

We write $\lim_{n\to\infty} f_n(x) = f(x)$ or $f_n \to f$ on A.

2- $\{f_n\}$ Uniformly convergent to f on A, if for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$ and all $x \in A$.

We write $f_n(x) \overset{u}{\to} f(x)$ on A.

It is clear that every uniformly convergent sequence is point wise convergent, but the convers is not true.

3- $\{f_n\}$ is point wise Cauchy sequence on A, if for every $x \in A$ and for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m > k$, we write $f_n\ p.c$ on A.

● This has meaning only If $f_n: \Omega \to \mathbb{R}$ is finite valued, because $\mathbb{R}$ is complete it is clear that if $\{f_n\}$ is a Cauchy sequence point wise on Ω, there must be on $f: \Omega \to \mathbb{R}$ such that $f_n \to f$ on Ω.

4- $\{f_n\}$ is a uniformly Cauchy sequence on A, if for every $\varepsilon > 0$ there is $k \in Z^+$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m > k$ and all $x \in A$. We write $f_n\ u.c.$ on A.

Definition (12): [1, 19]

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space, let $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, we say that

(1) $\{f_n\}$ Converges almost everywhere to f on A, denoted by $f_n \overset{a.e}{\to} f$ on A, if there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n \to f$ on A/B.
(2) $\{f_n\}$ Converges pseudo almost everywhere to f on A, denoted by $f_n \xrightarrow{p.a.e} f$ on A, if there is a subset $B \subseteq A$ such that $\mu(A/B) = \mu(A)$ and $f_n \to f$ on A/B.
(3) $\{f_n\}$ converges almost uniformly to f on A, denoted by $f_n \overset{a.u}{\longrightarrow} f$ on A, if there is a sequence $\{A_n\}$ in $\mathcal{F}$ with $\lim_{n\to\infty} \mu(A_n) = 0$ Such that $f_n \overset{u}{\to} f$ on A/A_n for any fixed $n = 1, 2, \ldots$.
(4) $\{f_n\}$ converges pseudo almost uniformly to f on A, denoted by $f_n \xrightarrow{p.a.u} f$ on A, if there is a sequence $\{A_n\}$ in $\mathcal{F}$ with
$\mathrm{Lim}_{n\to\infty} \mu(A/A_n) = \mu(A)$ Such that
$f_n \overset{u}{\to} f$ On A/A_n for any fixed
$n = 1, 2, \ldots$.
(5) $\{f_n\}$ convergence in measure to f on A, denoted by $f_n \overset{\mu}{\to} f$ on A, if $\lim_{n\to\infty} \mu(\{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\} \cap A = 0$ for each $\varepsilon > 0$.
(6) $\{f_n\}$ convergence pseudo in measure to f on A, denoted by $f_n \overset{p.\mu}{\to} f$ on A, if $\lim_{n\to\infty} \mu(\{x \in \Omega : |f_n(x) - f(x)| < \varepsilon\} \cap A = \mu A$ for each $\varepsilon > 0$

note that, in the above definitions, when $A = \Omega$ we can omit "on A " from the statements.

2. Main Results

Lemma (1): [1]

Let $(\Omega, \mathcal{F})$ be a measurable space. if $\mu: \mathcal{F} \to [0, \infty]$ is a non-decreasing set function, then the following statements are equivalent :

(1) μ is null additive
(2) $\mu(A \cup B) = \mu(A)$ Whenever $A, B \in \mathcal{F}$ and $\mu(B) = 0$.
(3) $\mu(A/B) = \mu(A)$ Whenever $A, B \in \mathcal{F}$ such that $B \subseteq A$ and $\mu(B) = 0$.
(4) $\mu(A/B) = \mu(A)$ Whenever $A, B \in \mathcal{F}$ and $\mu(B) = 0$.
(5) $\mu(A \Delta B) = \mu(A)$ Whenever $A, B \in \mathcal{F}$ and $\mu(B) = 0$.

Theorem (2):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is null additive, let
$f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, if $f_n \overset{a.e}{\to} f$ on A then $f_n \xrightarrow{p.a.e} f$ on A.

Proof:

Since $f_n \overset{a.e}{\to} f$ on A, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n \to f$ on A/B.

Since μ is null additive, hence
$\mu(A \cup B) = \mu(A)$, whenever $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$ and $\mu(B) = 0$
By using lemma (1), we get on

$$\mu(A/B) = \mu(A \cup B) = \mu(A)$$

Consequently
$f_n \xrightarrow{p.a.e} f$ on A.

Theorem (3):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μis autocontinuous from below, let $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, if $f_n \xrightarrow{a.e} f$ on A then $f_n \xrightarrow{p.a.e} f$ on A.

Proof:
Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n \to f$ on A/B.
Since μis autocontinuous from below, hence
$\lim_{n\to\infty} \mu(A/A_n) = \mu(A)$, whenever
$A \in \mathcal{F}, A_n \in \mathcal{F}, A_n \subseteq A, n = 1, 2, ..$and $\lim_{n\to\infty} \mu(A_n) = 0$
Take $B = A_n$, $n = 1, 2, \dots$, we have

$$\mu(B) = \lim_{n\to\infty} \mu(A_n) = 0$$

$$\Rightarrow \mu(A/B) = \lim_{n\to\infty} \mu(A/A_n) = \mu(A)$$

Consequently
$f_n \xrightarrow{p.a.e} f$ on A.

Theorem (4):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space, let $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$if $f_n \xrightarrow{a.e} f$ on $A$and $f_n \xrightarrow{p.a.u} f$ on A.Thenμ is order continuous and autocontinuous from below

Proof:
Since $f_n \xrightarrow{a.e} f$ on $A$and $f_n \xrightarrow{p.a.u} f$ on A.
Since $f_n \xrightarrow{p.a.u} f$ then there is a sequence $\{A_n\}$ be a sequence of sets in $\mathcal{F}$
With $\lim_{n\to\infty} \mu(A_n) = 0$,
i.e. $\mu(A_n) \longrightarrow 0, as\ n \to \infty$
Therefore $A_n \downarrow \emptyset$
Consequently μis order continuous
To prove μ is autocontinuous from below
Let$A \in \mathcal{F}, \{A_n\}$ be a sequence of sets in $\mathcal{F}$ with $A_n \subseteq A$
Through $f_n \xrightarrow{p.a.u} f$ on A, we have

$$\lim_{n\to\infty} \mu(A/A_n) = \mu(A)$$

Which is μ is autocontinuous from below.

Theorem (5):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space,
$f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$ such thatlim$_{n\to\infty}$ $\mu(A\Delta A_n) = \mu(A)$, whenever $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ with $\lim_{n\to\infty} \mu(A_n) = 0$, if $f_n \xrightarrow{a.u} f$ on A then $f_n \xrightarrow{p.a.u} f$ on A.

Proof:
Since $f_n \xrightarrow{a.u} f$ on A, then there is a sequence $\{A_n\}$ in $\mathcal{F}$ with $\lim_{n\to\infty} \mu(A_n) = 0$ such that $f_n \xrightarrow{u} f$ on A/A_n for any fixed $n = 1, 2, \dots$
Since $\lim_{n\to\infty} \mu(A\Delta A_n) = \mu(A)$, we have
$A \cap A_n \in \mathcal{F}$ and $\mu(A \cap A_n) \le \mu(A_n)$
So we have

$$\lim_{n\to\infty} \mu(A \cap A_n) = 0$$

and therefore, by the condition given in this theorem, we have

$$\lim_{n\to\infty} \mu(A/A_n) = \lim_{n\to\infty} \mu(A\Delta(A \cap A_n)) = \mu(A)$$

Consequently
$f_n \xrightarrow{p.a.u} f$ on A.

Theorem (6):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space, $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$such that μ is exhaustive, if $f_n \xrightarrow{p.a.u} f$ on A then $f_n \xrightarrow{a.u} f$ on A.

Proof:
Since $f_n \xrightarrow{p.a.u} f$ on A, then there is a sequence $\{A_n\}$of sets in $\mathcal{F}$ with $\lim_{n\to\infty} \mu(A/A_n) = \mu(A)$ such that $f_n \xrightarrow{u} f$ on A/A_n for any fixed $n = 1, 2, \dots$
Since μ is exhaustive, let $\{A_n\}$ is
A pairwise of disjoint sequence in$\mathcal{F}$, with $\lim_{n\to\infty} \mu(A_n) = 0$
Consequently
$f_n \xrightarrow{a.u} f$ on A.

Theorem (7):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space, $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$such that μ is null additive, for any decreasing sequence $\{A_n\}$of sets in$\mathcal{F}$ for whichlim$_{n\to\infty}$ $\mu(A_n) = 0$, if $f_n \xrightarrow{a.u} f$ on A then $f_n \xrightarrow{p.a.u} f$ on A.

Proof:
Since $f_n \xrightarrow{a.u} f$ on A, then there is a sequence $\{A_n\}$of sets in $\mathcal{F}$ with $\lim_{n\to\infty} \mu(A_n) = 0$ such that $f_n \xrightarrow{u} f$ on A/A_n for any fixed $n = 1, 2, \dots$
Since

$$A/A_n \uparrow A/(\bigcap_{n=1}^{\infty} A_n)$$

and

$$\mu(\bigcap_{n=1}^{\infty} A_n) = 0$$

By using lemma (1)continuity of μ, it follows that

$$\lim_{n\to\infty} \mu(A/A_n) = \lim_{n\to\infty} \mu\left(A/(\bigcap_{n=1}^{\infty} A_n)\right) = \mu(A)$$

Consequently

$$f_n \xrightarrow{p.a.u} f \text{ on } A.$$

Theorem (8):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is weakly null additive, let $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, if $f_n \xrightarrow{a.e} f$ on A then $f_n \xrightarrow{\mu} f$ on A.

Proof:
Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset B in $\mathcal{F}$ such that $\mu(B) = 0$ and for any $x \in B$ $\lim_{n\to\infty} f_n(x) = f(x)$
For any $\varepsilon > 0$
Since

$$\{x \in \Omega : |f_n(x) - f(x)| \ge \varepsilon\} \cap A \subseteq B \cup \{x \in \Omega : |f_n(x) - f(x)| \ge \varepsilon\} \cap A$$

By monotonicity and weakly null additive, we have

$\mu(\{x \in \Omega: |f_n(x) - f(x)| \geq \varepsilon\} \cap A) \to 0 \ \ as \ \ n \to \infty$
Consequently $f_n \xrightarrow{\mu} f$ on A.

Theorem (9):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is countably weakly null-additive, let $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, if $f_n \xrightarrow{a.e} f$ on A then
(1) f_n is a Cauchy $a.e$.
(2) If g is real –valued measurable function and $f_n \xrightarrow{a.e} g$ then $f = g \ a.e$.
(3) If g is real –valued measurable function such that $f = g \ a.e$ then $f_n \xrightarrow{a.e} g$.
(4) If $\{g_n\}$ is a sequence of real –valued measurable functions such that $f_n = g_n \ a.e$ for each n then $g_n \xrightarrow{a.e} f$.
(5) If $g, \{g_n\}$ is a sequence of real –valued measurable function such that
$f_n = g_n \ a.e$ for each n and $f = g \ a.e$ then $g_n \xrightarrow{a.e} g$.

Proof:
(1) Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset $B \subseteq A$ such that
$\mu(B) = 0$ and $f_n(x) \dashrightarrow f(x)$ for all $x \in A/B \Rightarrow f_n(x)$ is Cauchy sequence for all $x \in A/B \Rightarrow f_n$ is Cauchy $a.e$.
(2) Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n(x) \dashrightarrow f(x)$ for all $x \in A/B$
Since $f_n \xrightarrow{a.e} g$ on A, then there is a subset $C \subseteq A$ such that $\mu(C) = 0$ and $f_n(x) \dashrightarrow g(x)$ for all $x \in A/C$
Let $D = B \cup C$
$\to \mu(D) = \mu(B \cup C)$, Whenever
$\mu(B) = 0, \mu(C) = 0$
Since μ is countably weakly null-additive, we have
$\mu(D) = 0$ for any $x \in A/D$

$$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x), \lim_{n\to\infty} f_n(x) = g(x)$$

So $f(x) = g(x)$, for all $x \notin D$

$$\Rightarrow f = g \ a.e.$$

(3) Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n(x) \dashrightarrow f(x)$ for all $x \in A/B$
Since $f = g \ a.e$ then there is a subset $C \subseteq A$ such that $\mu(C) = 0$ and
$f(x) = g(x)$ for all $x \in A/C$
Let $D = B \cup C$

$$\Rightarrow \mu(D) = \mu(B \cup C)$$

Since μ is countably weakly null-additive, we have
$\mu(D) = 0$ for any $x \in A/D$

$$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x) = g(x)$$

So $\lim_{n\to\infty} f_n(x) = g(x)$ for all $x \notin D$
Consequently
$f_n \xrightarrow{a.e} g$.
(4) Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n \dashrightarrow f$ on A/B
Since $f_n = g_n \ a.e$ then there is a sequence $\{B_n\}$ such that $\mu(B_n) = 0$ and
$f_n(x) = g_n(x)$ for all $x \notin B_n$
Let

$$D = B \cup C \cup \left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$\Rightarrow \mu(D) = \mu\left((B \cup C) \cup \left(\bigcup_{n=1}^{\infty} B_n\right)\right)$$

Since μ is countably weakly null-additive, we have
$\mu(D) = 0$ for all $x \notin D$
$\Rightarrow \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x) = f(x)$ for all $x \notin D$
$\Rightarrow g_n(x) \to f(x)$ for all $x \notin D$
Consequently
$g_n \xrightarrow{a.e} f$.
(5) Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n(x) \dashrightarrow f(x)$ for all $x \notin B$
Since $f_n = g_n \ a.e$ for each n then there is $B_n \subseteq A$ such that $\mu(B_n) = 0$ for all n and $f_n(x) = g_n(x)$ for all $x \notin B_n$
Since $f = g \ a.e$, then there is a subset $C \subseteq A$ such that $\mu(C) = 0$ and
$f(x) = g(x)$ for all $x \notin C$
Let

$$D = B \cup C \cup \left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$\Rightarrow \mu(D) = \mu\left(B \cup C \cup \left(\bigcup_{n=1}^{\infty} B_n\right)\right)$$

Since μ is countably weakly null-additive, we have
$\mu(D) = 0$ for all $x \notin D$
$\therefore \mathbf{\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x) = f(x) = g(x)}$ for all $x \notin D$
$\Rightarrow g_n(x) \to g(x)$ for all $x \notin D$
Consequently
$g_n \xrightarrow{a.e} g$.

Theorem (10):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is countably weakly null-additive, let $f_n, g_n, f, g \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, $C \in \mathbb{R}$ if $f_n \xrightarrow{a.e} f$ and $g_n \xrightarrow{a.e} g$ on A then
(1) $c.f_n \xrightarrow{a.e} c.f$.
(2) $f_n + g_n \xrightarrow{a.e} f + g$.
(3) $|f_n| \xrightarrow{a.e} |f|$.
(4) If $f_n = g_n \ a.e$ for all n, then
$f = g \ \ a.e$.

Proof:
(1) Since $f_n \xrightarrow{a.e} f$, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n(x) \dashrightarrow f(x)$ for all $x \in A/B$ then $c.f_n(x) \dashrightarrow c.f(x)$ all $x \in A/B$
$\Rightarrow c.f_n \xrightarrow{a.e} c.f$.
(2) Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n(x) \dashrightarrow f(x)$ for all $x \in A/B$
Since $g_n \xrightarrow{a.e} g$ on A, then there is a subset $C \subseteq A$ such that $\mu(C) = 0$ and $g_n(x) \dashrightarrow g(x)$ for all $x \in A/C$
Let $D = B \cup C$

$$\Rightarrow \mu(D) = \mu(B \cup C)$$

Since μ is countably weakly null-additive, we have
$\mu(D) = 0$ for any $x \in A/D$
$\Rightarrow f_n(x) \dashrightarrow f(x)$ and $g_n(x) \dashrightarrow g(x)$ for all $x \notin D$
So $f_n(x) + g_n(x) \dashrightarrow f(x) + g(x)$, for all $x \notin D$

$$\Rightarrow f_n + g_n \xrightarrow{a.e} f + g.$$

(3) Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset $B \subseteq A$

such that$\mu(B) = 0$ and $f_n(x) \dashrightarrow f(x)$for all $x \in A/B$
$\Longrightarrow |f_n(x)| \dashrightarrow |f(x)|$ for all $x \in A/B$
$\therefore |f_n| \xrightarrow{a.e} |f|$.
(4) Since $f_n \xrightarrow{a.e} f$ on A, then there is a subset $B \subseteq A$ such that$\mu(B) = 0$ and $f_n(x) \dashrightarrow f(x)$ for all $x \in A/B$
Since$g_n \xrightarrow{a.e} g$ on A, then there is a subset $C \subseteq A$ such that$\mu(C) = 0$ and
$g_n(x) \dashrightarrow g(x)$ for all $x \in A/C$
Since $f_n = g_n$ $a.e$ for each n then there is a sequence $\{B_n\}$ such that
$\mu(B_n) = 0$And$f_n(x) = g_n(x)$for all $x \notin B_n$
Let

$$D = B \cup C \cup \left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$\Longrightarrow \mu(D) = \mu\left(B \cup C \cup \left(\bigcup_{n=1}^{\infty} B_n\right)\right)$$

Since μ is countably weakly null-additive, we have
$\mu(D) = 0$For all $x \notin D$
$\therefore f(x) = lim_{n\to\infty} f_n(x)$and
$lim_{n\to\infty} f_n(x) = lim_{n\to\infty} g_n(x) = g(x) For\ all\ x \notin D$
$\Longrightarrow f(x) = g(x)$For all $x \notin D$
Consequently
$f = g\ a.e$.

Theorem (11):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is countably weakly null-additive, let $f_n, g_n, f, g \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}, C \in \mathbb{R}$ if $f_n \xrightarrow{a.e} f$ and
(1) If$f_n \geq 0\ a.e$ then $f \geq 0\ a.e$.
(2) If$f_n \leq g\ a.e$ for each n then
$f \leq g\ a.e$.
(3) If$|f_n| \leq |g|\ a.e$ then
$|f| \leq |g|\ a.e.$
(4) If $f_n \leq f_{n+1}\ a.e$ for each n, then $f_n \uparrow f\ \ a.e.$
(5) If$f_n \xrightarrow{a.e} f, g_n \xrightarrow{a.e} g$ and
$f_n = g_n\ a.e, f_n \geq 0\ a.e$ Then$f \geq 0\ a.e$ and $g \geq 0\ a.e.$

Proof:
Since$f_n \xrightarrow{a.e} f$, then there is $B \subseteq \Omega$ such that $\mu(B) = 0$ and$f_n(x) \to f(x)$ for all $x \notin B$
(1) Since $f_n \geq 0\ a.e$ for each n, then there is $B_n \subseteq \Omega$ such that $\mu(B_n) = 0$And$f_n(x) \geq 0$ for all $x \notin B_n$
Let

$$D = B \cup \left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$\Longrightarrow \mu(D) = \mu\left(B \cup \left(\bigcup_{n=1}^{\infty} B_n\right)\right)$$

Since μ is countably weakly null-additive, we have
$\mu(D) = 0$ for all $x \notin D$
$\Longrightarrow f(x) = \lim_{n\to\infty} f_n(x) \geq 0$For all $x \notin D$
Therefor

$$f \geq 0\ \ a.e.$$

(2) Since $f_n \leq g\ \ a.e$

$$\Longrightarrow g - f_n \geq 0\ \ a.e,$$

Since $f_n \xrightarrow{a.e} f$

$$\Longrightarrow g - f_n \xrightarrow{a.e} g - f$$

By (1) $g - f \geq 0\ \ a.e$

$$\Longrightarrow f \leq g\ \ a.e$$

(3) Since $f_n \xrightarrow{a.e} f$, from theorem (10), we have$|f_n| \xrightarrow{a.e} |f|$
Since $|f_n| \leq |g|\ a.e$ by (2), we get on

$$|f| \leq |g|\ a.e$$

(4) Since $f_n \leq f_{n+1}\ a.e$ for each n, then there is $A_n \subset \Omega$ such that $\mu(A_n) = 0$and $f_n(x) \leq f_{n+1}(x)$ for all $x \notin A_n$
Let

$$D = B \cup \left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$\Longrightarrow \mu(D) = \mu\left(B \cup \left(\bigcup_{n=1}^{\infty} A_n\right)\right)$$

Since μ is countably weakly null-additive, we have
$\mu(D) = 0$For all $x \notin D$
Implies $f_n(x) \uparrow f(x)$ and $f_n(x) \to f(x)$
Hence

$$f_n \uparrow f\ \ a.e.$$

(5) Since $f_n \xrightarrow{a.e} f, g_n \xrightarrow{a.e} g$ and $f_n = g_n\ a.e$, then by (4) from theorem (10), we get on $f = g\ a.e$
Since $f_n \geq 0\ a.e$ then by (1)

$$\Longrightarrow f \geq 0\ a.e \text{ and } g \geq 0\ a.e$$

Theorem (12):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is countably weakly null-additive and continuous from below at A, let $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, if $f_n \xrightarrow{a.u} f$ then
(1) If $f_n \xrightarrow{a.u} g$ then $f = g\ a.e$.
(2) If $f = g\ a.e$ then$f_n \xrightarrow{a.u} g$.
(3) If $f_n = g_n\ a.e$ for each n then$g_n \xrightarrow{a.u} f$.
(4) If $f_n = g_n\ a.e$ for each n and
$f = g\ a.e$Then $g_n \xrightarrow{a.u} g$

Proof:
(1) Since $f_n \xrightarrow{a.u} f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$with $\lim_{n\to\infty} \mu(A_n) = 0$ such that $f_n \xrightarrow{u} f$ on A/A_n for any fixed $n = 1, 2, \ldots$
Since $f_n \xrightarrow{a.u} g$, then there is sequence of sets $\{B_n\}$ in $\mathcal{F}$with$\boldsymbol{\lim_{n\to\infty} \mu(B_n) = 0}$ such that $f_n \xrightarrow{u} f$ on A/B_n for any fixed $n = 1, 2, \ldots$
Since μ is continuous from below at A, we have

$$A = \bigcup_{n=1}^{\infty} A_n, \mu(A) = \lim_{n\to\infty} \mu(A_n)$$

Take

$$B = \bigcup_{n=1}^{\infty} B_n$$

$$\Longrightarrow \mu(B) = \lim_{n\to\infty} \mu(B_n)$$

$$\therefore \mu(A) = \mu(B) = 0$$

Let

$$D = (\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} B_{n,})$$

$$\Longrightarrow D = (A \cup B)$$

$$\Longrightarrow \mu(D) = \mu(A \cup B)$$

Since μ is countably weakly null-additive, we have
$\mu(D) = 0$Forall $x \notin D$, and
$f_n(x) \to f(x), f_n(x) \to g(x)$ uniformly for any $x \notin D$
Since $\mu(D) = 0$ for all $x \notin D$
$\Longrightarrow f(x) = g(x)$ for any $x \notin D$

$$\therefore f = g\ a.e.$$

(2) Since $f_n \overset{a.u}{\longrightarrow} f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$with $\lim_{n\to\infty} \mu(A_n) = 0$ such that $f_n \overset{u}{\to} f$ on A/A_n for any fixed $n = 1, 2, \ldots$
I.e.$f_n(x) \to f(x)$ uniformly for any $x \in A/A_n$.
Since $f = g\ a.e$, then there is subset $B \subseteq A$such that$\mu(B) = 0$ and
$f(x) = g(x)$for all $x \in A/B$
Let

$$B_n = \emptyset, \quad \text{For all } n \geq 2$$
$$B_1 = B, B_2 = \emptyset, B_3 = \emptyset, \ldots$$
$$\bigcup_{n=1}^{\infty} B_n = B$$
$$\Longrightarrow \mu(B) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right)$$
$$\therefore \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = 0$$

Let

$$D_n = (\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} B_n)$$

Where $\{D_n\}$ be a sequence of sets in $\mathcal{F}$

$$\lim_{n\to\infty} \mu(D_n) = \mu\left((\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} B_n)\right)$$

Since μ is continuous from below at A, we have

$$A = \bigcup_{n=1}^{\infty} A_n\,, \mu(A) = \lim_{n\to\infty} \mu(A_n)$$
$$\mu(A) = 0$$
$$\Longrightarrow \mu(A) = \mu(\bigcup_{n=1}^{\infty} A_n)$$
$$\Longrightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$$
$$\Longrightarrow \lim_{n\to\infty} \mu(D_n) = \mu(A \cup B)$$

Since μ is countably weakly null-additive, we have
$\lim_{n\to\infty} \mu(D_n) = 0$ for any $x \notin D_n$
$f_n(x) \dashrightarrow f(x) = g(x)$ Uniformly for any $x \notin D_n$
Therefore

$$f_n \overset{a.u}{\longrightarrow} g.$$

(3) Since $f_n \overset{a.u}{\longrightarrow} f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$with $\lim_{n\to\infty} \mu(A_n) = 0$ such that $f_n \overset{u}{\to} f$ on A/A_n for any fixed $n = 1, 2, \ldots$
I.e.$f_n(x) \to f(x)$ uniformly for any $x \in A/A_n$
Since $f_n = g_n\ a.e$ for each n, then there is a sequence $\{B_n\}$ in $\mathcal{F}$such that
$\mu(B_n) = 0$And$f_n(x) = g_n(x)$for all $x \in A/B_n$
Let

$$D_n = (\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} B_n)$$

Where $\{D_n\}$ be a sequence of sets in $\mathcal{F}$

$$\lim_{n\to\infty} \mu(D_n) = \mu\left((\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} B_n)\right)$$

Since μ is continuous from below at A, we have

$$A = \bigcup_{n=1}^{\infty} A_n \text{ and } \mu(A) = \lim_{n\to\infty} \mu(A_n)$$
$$\Longrightarrow \mu(A) = \mu(\bigcup_{n=1}^{\infty} A_n) = 0$$

Since μ is countably weakly null-additive, we have

$$\mu(B_n) = 0, \text{for all n} \geq 1$$
$$\Longrightarrow \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = 0$$
$$\lim_{n\to\infty} \mu(\mathrm{D_n}) = \mu\left((\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} \mathrm{B_n})\right)$$

Since μ is countably weakly null-additive, we have
$\lim_{n\to\infty} \mu(D_n) = 0$For any $x \notin D_n$
$g_n(x) = f_n(x) \dashrightarrow f(x)$ Uniformly for any $x \notin D_n$
Therefore

$$\Longrightarrow g_n \overset{a.u}{\longrightarrow} f$$

(4) Since $f_n \overset{a.u}{\longrightarrow} f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$with $\lim_{n\to\infty} \mu(A_n) = 0$ such that $f_n \overset{u}{\to} f$ on A/A_n for any fixed $n = 1, 2, \ldots$
I.e.$f_n(x) \to f(x)$ uniformly for any $x \in A/A_n$
Since$f_n = g_n\ a.e$ for each n, then there is a sequence $\{B_n\}$ in $\mathcal{F}$such that
$\mu(B_n) = 0$For each n and
$f_n(x) = g_n(x)$for all $x \in A/B_n$
Since$f = g\ a.e$, then there is $C \subseteq A$such that $\mu(C) = 0$ and
$f(x) = g(x)$ For all $x \in A/C$

Let$C_n = \emptyset$, for all $n \geq 2$

$$C_1 = C, C_2 = \emptyset, C_3 = \emptyset, \ldots$$
$$\bigcup_{n=1}^{\infty} C_n = C$$
$$\Longrightarrow \mu(\bigcup_{n=1}^{\infty} C_n) = \mu(C)$$
$$\therefore \mu(\bigcup_{n=1}^{\infty} C_n) = 0$$

Let

$$D_n = (\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} C_n) \cup (\bigcup_{n=1}^{\infty} B_n)$$

Where $\{D_n\}$ be a sequence of sets in $\mathcal{F}$

$$\boldsymbol{\lim_{n\to\infty} \mu(D_n) = \mu\left((\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} C_n) \cup (\bigcup_{n=1}^{\infty} B_n)\right)}$$

Since μ is continuous from below at A, we have

$$\bigcup_{n=1}^{\infty} A_n = A, \mu(A) = \lim_{n\to\infty} \mu(A_n)$$
$$\Longrightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \mu(A)$$

$$\therefore \mu(\bigcup_{n=1}^{\infty} A_n) = 0$$

Also μcountably weakly null-additive, this mean

$$\mu(B_n) = 0, \text{for all n} \geq 1$$

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = 0$$

$$\therefore \lim_{n\to\infty} \mu(D_n) = \mu\left(A \cup C \cup (\bigcup_{n=1}^{\infty} B_n)\right)$$

Since is μcountably weakly null-additive

$\Longrightarrow \lim_{n\to\infty} \mu(D_n) = 0$For any $x \notin D_n$,

$and\ g_n(x) = f_n(x) \dashrightarrow f(x) = g(x)$ Uniformly for any $x \notin D_n$

Therefore

$g_n(x) \dashrightarrow g(x)$ Uniformly for any $x \notin D_n$

$$\Longrightarrow g_n \overset{a.u}{\longrightarrow} g.$$

Theorem (13):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is countably weakly null-additive and continuous from below at A, let $f_n, g_n, f, g \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, $C \in \mathbb{R}$ if $f_n \overset{a.u}{\longrightarrow} f$ and $g_n \overset{a.u}{\longrightarrow} g$ then

(1) $c.f_n \overset{a.u}{\longrightarrow} c.f.$

(2) $f_n + g_n \overset{a.u}{\longrightarrow} f + g.$

(3) $|f_n| \overset{a.u}{\longrightarrow} |f|.$

Proof:

(1) Since $f_n \overset{a.u}{\longrightarrow} f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$with $\lim_{n\to\infty} \mu(A_n) = 0$ such that $f_n \overset{u}{\rightarrow} f$ on A/A_n for any fixed $n = 1, 2, \ldots$

I.e.$f_n(x) \rightarrow f(x)$ uniformly for any $x \in A/A_n$

$\Longrightarrow c.f_n \overset{a.e}{\longrightarrow} c.f.$

(2) Since $f_n \overset{a.u}{\longrightarrow} f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$with $\lim_{n\to\infty} \mu(A_n) = 0$ Such that $f_n \overset{u}{\rightarrow} f$ on A/A_n for any fixed $n = 1, 2, \ldots$

I.e.$f_n(x) \rightarrow f(x)$ uniformly for any $x \in A/A_n$

Since $g_n \overset{a.u}{\longrightarrow} g$, then there is a of sets $\{B_n\}$ in $\mathcal{F}$with$\lim_{n\to\infty} \mu(B_n) = 0$ such that $g_n \overset{u}{\rightarrow} g$ on A/B_n for any fixed $n = 1, 2, \ldots$

I.e.$g_n(x) \rightarrow g(x)$ uniformly for any $x \in A/B_n$

Let

$$D_n = (\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} B_n)$$

Where $\{D_n\}$ be a sequence of sets in $\mathcal{F}$

$$\Longrightarrow \lim_{n\to\infty} \mu(D_n) = \mu\left((\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} B_n)\right)$$

Since μ is continuous from below at A, we have

$$\bigcup_{n=1}^{\infty} A_n = A, \mu(A) = \lim_{n\to\infty} \mu(A_n)$$

$$\therefore \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$$

$$\mu(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu(B_n) = 0$$

$$\Longrightarrow \lim_{n\to\infty} \mu(D_n) = \mu\left(A \cup (\bigcup_{n=1}^{\infty} B_n)\right)$$

Since is μcountably weakly null-additive

$\Longrightarrow \lim_{n\to\infty} \mu(D_n) = 0$For any $x \notin D_n$

And $f_n(x) \dashrightarrow f(x), g_n(x) \dashrightarrow g(x)$ uniformly for all $x \notin D_n$

So $f_n(x) + g_n(x) \dashrightarrow f(x) + g(x)$uniformly for all $x \notin D_n$

$$\Longrightarrow f_n + g_n \overset{a.u}{\longrightarrow} f + g.$$

(3) Since $f_n \overset{a.u}{\longrightarrow} f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$with $\lim_{n\to\infty} \mu(A_n) = 0$ such that $f_n \overset{u}{\rightarrow} f$ on A/A_n for any fixed $n = 1, 2, \ldots$

I.e.$f_n(x) \rightarrow f(x)$ uniformly for any $x \in A/A_n$

So $|f_n(x)| \dashrightarrow |f(x)|$ uniformly for any $x \in A/A_n$

$$\therefore |f_n| \overset{a.u}{\longrightarrow} |f|.$$

Theorem (14):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space, let $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$such that μ is countably weakly null-additive and continuous from below at A, if $f_n \overset{\mu}{\rightarrow} f$ then

(1) If $f_n \overset{\mu}{\rightarrow} g$ then $f = g\ a.e.$

(2) If $f = g\ a.e$ then$f_n \overset{\mu}{\rightarrow} g.$

(3) If $f_n = g_n\ a.e$ for all $n$then $g_n \overset{\mu}{\rightarrow} f.$

Proof:

(1) Given any $\varepsilon > 0$, define

$$B = \{x \in \Omega: |f(x) - g(x)| \geq \varepsilon\} \cap A$$

$$B_n = \left\{x \in \Omega: |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \cap A$$

$$C_n = \left\{x \in \Omega: |f_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\} \cap A$$

Since

$|f(x) - g(x)| \leq |f_n(x) - f(x)| + |f_n(x) - g(x)|$

This implies that

$$B \subseteq B_n \cup C_n \Rightarrow \mu(B) \leq \mu(B_n \cup C_n)$$

Since $f_n \overset{\mu}{\rightarrow} f, f_n \overset{\mu}{\rightarrow} g$

$$\Longrightarrow \mu(B_n) \longrightarrow 0, \mu(C_n) \longrightarrow 0 \text{ as n} \rightarrow \infty$$

Since μ is countably weakly null-additive, we have

$$\mu(B_n \cup C_n) \longrightarrow 0 \text{ as n} \rightarrow \infty$$

Therefore $\mu(B) \longrightarrow 0$ as n $\rightarrow \infty$

$$N(f - g) = \{x \in \Omega: (f - g)(x) \neq 0\}$$

$$= \bigcup_{n=1}^{\infty} \left\{x \in \Omega: |f(x) - g(x)| \geq \frac{1}{n}\right\} \cap \mathbf{A}$$

$$\Rightarrow \mu(N(f - g)) = 0 \Longrightarrow f = g\ a.e.$$

(2) Since $f = g\ a.e \Longrightarrow$ there exists $B \in F$ with $\mu(B) = 0$ and $f(x) \neq g(x)$ for all $x \in B$ for any $\varepsilon > 0$, we have

$\{x \in \Omega: |f_n(x) - g(x)| \geq \varepsilon\} \cap A$

$\subset B \cup (\{x \in \Omega: |f_n(x) - f(x)| \geq \varepsilon\} \cap A)$

$$\therefore \mu(\{x \in \Omega: |f_n(x) - g(x)| \geq \varepsilon\} \cap A)$$

$$\leq \mu(B \cup (\{x \in \Omega: |f_n(x) - f(x)| \geq \varepsilon\} \cap \mathbf{A})$$

Since $f_n \overset{\mu}{\rightarrow} f$ then

$\mu(\{x \in \Omega: |f_n(x) - f(x)| \geq \varepsilon\} \cap A) = 0, \mu(B) = 0$

Since $\mu$is countably weakly null-additive, we have

$$\Longrightarrow \mu(\{x \in \Omega: |f_n(x) - g(x)| \geq \varepsilon\} \cap A) \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\Longrightarrow f_n \overset{\mu}{\rightarrow} g.$$

(3) Since $f_n = g_n$ a.e for all n, then there exists $A_n \in F$ with $\mu(A_n) = 0$ and $f_n(x) \neq g_n(x)$ for all $x \in A_n$

Since μ iscontinuous from below at A, we have

$$A = \bigcup_{n=1}^{\infty} A_n \;, \mu(A) = \lim_{n\to\infty} \mu(A_n)$$

$$\Longrightarrow \mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

Since μ is countably weakly null-additive

$$\Longrightarrow \mu(A) = 0, \text{for any } \varepsilon > 0,$$

we have

$$C = (\{x \in \Omega: |g_n(x) - f(x)| \geq \varepsilon\} \cap B)$$

$$C_n = (\{x \in \Omega: |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\} \cap B)$$

$$D_n = (\{x \in \Omega: |g_n(x) - f_n(x)| \geq \frac{\varepsilon}{2}\} \cap B)$$

Since

$$|g_n(x) - f(x)| \leq |f_n(x) - f(x)| + |g_n(x) - f_n(x)|$$

$$\Longrightarrow C \subseteq C_n \cup D_n \Rightarrow \mu(C) \leq \mu(C_n \cup D_n)$$

Since $f_n \xrightarrow{\mu} f, f_n = g_n$

$$\Longrightarrow \mu(C_n) \longrightarrow 0, \mu(D_n) \longrightarrow 0 \text{ as } n \to \infty$$

Since μ is countably weakly null-additive, we have

$$\mu(C_n \cup D_n) \longrightarrow 0 \text{ as } n \to \infty$$

Therefore $\mu(C) \longrightarrow 0$ as $n \to \infty$

$$\Longrightarrow g_n \xrightarrow{\mu} f.$$

Theorem (15):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is weakly null-additive, let $f, f_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, if $f_n \xrightarrow{\mu} f, g_n \xrightarrow{\mu} g C \in \mathbb{R}$ then

(1) $c.f_n \xrightarrow{\mu} c.f$

(2) $|f_n| \xrightarrow{\mu} |f|$

Proof:

(1) This is clear if $c = 0$, if $c \neq 0$, let $\varepsilon > 0$.

Since $f_n \xrightarrow{\mu} f$ and

$(\{x \in \Omega: |c.f_n(x) - c.f(x)| \geq \varepsilon\} \cap A)$

$= \left(\left\{x \in \Omega: |f_n(x) - f(x)| \geq \frac{\varepsilon}{|c|}\right\} \cap A\right)$

This implies that

$$\mu(\{x \in \Omega: |c.f_n(x) - c.f(x)| \geq \varepsilon\} \cap A)$$

$$= \mu\left(\left\{x \in \Omega: |f_n(x) - f(x)| \geq \frac{\varepsilon}{|c|}\right\} \cap A\right) \to 0 \; as \; n \to \infty$$

So that

$$c.f_n \xrightarrow{\mu} c.f$$

(2) Since

$$\big||f_n(x)| - |f(x)|\big| \leq |f_n(x) - f(x)|$$

This implies that

$$\{x \in \Omega: \big||f_n(x)| - |f(x)|\big| \geq \varepsilon\} \cap A$$

$$\subseteq \{x \in \Omega: |f_n(x) - f(x)| \geq \varepsilon\} \cap A$$

Since $f_n \xrightarrow{\mu} f$ so

$$\mu(\{x \in \Omega: \big||f_n(x)| - |f(x)|\big| \geq \varepsilon\} \cap A) \to 0 \; as \; n \to \infty$$

Therefore

$$|f_n| \xrightarrow{\mu} |f|.$$

Theorem (16):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is weakly null-additive, let $f, g, f_n, g_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, suppose that $f_n + g_n \xrightarrow{\mu} 0$ whenever $f_n \xrightarrow{\mu} 0$ and $g_n \xrightarrow{\mu} 0$, then μ is autocontinuous from below

Proof:

Let $\{A_n\}$ be a sequence with $\lim_{n\to\infty} \mu(A_n) = 0$, given any $\varepsilon > 0$

Suppose μ is not autcontinuous from below

Take $\lim_{n\to\infty} \mu(A \cup A_n) > 0$

There is no loss of generality $A, A_n \in \mathcal{F}$ and $A \cap A_n = \emptyset$

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

and

$$g_n(x) = \begin{cases} 0 & \text{if } x \notin A_n \\ 1 & \text{if } x \in A_n \end{cases}$$

Then $f_n \xrightarrow{\mu} 0$ and $g_n \xrightarrow{\mu} 0$, thus

$$f_n + g_n = \begin{cases} 0 & \text{if } x \notin A \cup A_n \\ 1 & \text{if } x \in A \cup A_n \end{cases}$$

So $f_n + g_n \xrightarrow{\mu} 0$

$\lim_{n\to\infty} \mu(A \cup A_n)$

$= \boldsymbol{\lim_{n\to\infty} \mu(\{x \in \Omega: |f_n(x) + g_n(x)| \geq \varepsilon\} \cap A) = 0}$

$$\Longrightarrow \lim_{n\to\infty} \mu(A \cup A_n) = 0$$

Which is contradiction with assumption that $\lim_{n\to\infty} \mu(A \cup An)>0$

Consequently

μ is autocontinuous from below

Theorem (17):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that μ is countably weakly null-additive, let $f, g, f_n, g_n \in \mathbb{C}(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, suppose that $f_n + g_n \xrightarrow{\mu} 0$ whenever $f_n \xrightarrow{\mu} 0$ and $g_n \xrightarrow{\mu} 0$, then μ is null continuous.

Proof:

Let $A_1 = \left\{x \in \Omega: |f_n(x)| \geq \frac{\varepsilon}{2}\right\} \cap A$

$$A_2 = \left\{x \in \Omega: |g_n(x)| \geq \frac{\varepsilon}{2}\right\} \cap A$$

$$\boldsymbol{A_3 = \{x \in \Omega: |f_n(x) + g_n(x)| \geq \varepsilon\} \cap A}$$

Since $f_n \xrightarrow{\mu} 0, g_n \xrightarrow{\mu} 0$ and $f_n + g_n \xrightarrow{\mu} 0$, we have

$$\therefore \lim_{n\to\infty} \mu(\left\{x \in \Omega: |f_n(x)| \geq \frac{\varepsilon}{2}\right\} \cap A) = 0$$

$$\therefore \lim_{n\to\infty} \mu(\left\{x \in \Omega: |g_n(x)| \geq \frac{\varepsilon}{2}\right\} \cap A) = 0$$

$$\therefore \boldsymbol{\lim_{n\to\infty} \mu(\{x \in \Omega: |f_n(x) + g_n(x)| \geq \varepsilon\} \cap A) = 0}$$

$\therefore \; \mu(A_1) = 0, \mu(A_2) = 0, \mu(A_3) = 0$

$\because \mu$ is countably weakly null-additive, for all $n \geq 1$

$$\Longrightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$$

$\therefore \mu$ is null-continuous.

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