Some New Results about the Convergence of Fuzzy Measurable Functions Sequence on Fuzzy Measure on Fuzzy Sets

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Abstract: The goal of this paper is to study the Convergence of fuzzy measurable functions sequence on fuzzy sets and get on some new results.

Keywords: Fuzzy measure, exhaustive, order continuous, null additive, autocontinuous from below, accountably weakly null-additive fuzzy measure, null-continuous.

1. Introduction

In measure theory, several types of convergence were introduced for sequence of measurable functions on a measure space and some basic relations among these types were established [3].

In the proof of the theorem (9) we need that $\mu$ is countably weakly null-additive because as is well known, sugeno's fuzzy measure loses additivity in general therefore if $\mu(A_n) = 0$ for all

$$n \geq 1 \implies \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = 0, \{A_n\} \subset F$$

Fuzzy measure generalization of measure theory. This generalization is obtained by replacing the additivity axiom of measure theory with weak axiom of monotonicity and continuity [1].

The fuzzy measure, defined on $\sigma$-field, was introduced by Sugeno [20]. Ralescu and Adams [21] generalized the concepts of fuzzy measure and fuzzy integral to the case that the value of a fuzzy measure can be infinite, and to realize an approach from Subjective.

Jun Li [4] study order continuous and strongly order continuous of monotone set function and convergence of measurable functions sequence.

Jun Li, Masami Yasuda, Qingshan Jiang, Hisakichi Suzuki and Zhenyuan Wang [2], Deli Zhang and Caimei Guo [5], studied some Convergence of sequence of measurable functions on Fuzzy measure spaces and generalized convergence theorems obtained a series of new results.

After that, many authors studied Convergence of sequence of measurable functions on Fuzzy measure spaces and proved some results about it as G. J. Klor [6, 7], Jun Li, Radko Mesiar, Endre Pap and Erich Peter Klement [8], L.Y. Kui[9], L.Y. Kui and L. Baoding [10].

In this paper, we mention the definition of Fuzzy Measure on Fuzzy Set and study three types of convergence of sequence of fuzzy measurable functions defined on fuzzy sets; the concepts of "almost" and "Pseudo" are introduced also too the Convergence almost everywhere.

Convergence in fuzzy measure and almost uniformly convergence and get on some new results about them.

Definition (1): [17, 18]
Let $\Omega$ be anon empty set, a fuzzy set $A$ in $\Omega$(ora fuzzy subset of $\Omega$) is a function from $\Omega$ into $I$, i.e. $A \in I^\Omega$. $A(\chi)$ is interpreted as the degree of membership of element $x$ in a fuzzy set $A$ for each $x \in \Omega$, a fuzzy set $A$ in $\Omega$ is can be represented by the set of pairs:

$$A = \{(x, A(x)): x \in \Omega \}$$

Note that every ordinary set is fuzzy set, i.e. $P(\Omega) \subseteq I^\Omega$.

Definition (2): [11, 12]
A family $F$ of fuzzy sets in a set $\Omega$ is called a fuzzy $\sigma$–field on a set $\Omega$ if:
1) $\phi, \Omega \in F$.
2) If $A \in F$, then $A^c \in F$.
3) If $\{A_n\} \subset F$, $n = 1, 2, 3, ..., \text{then } \bigcup_{n=1}^{\infty} A_n \in F$.

Evidently, an arbitrary $\sigma$ –field must be fuzzy $\sigma$ –field. A fuzzy measurable Space is a pair$(\Omega, F)$, where $\Omega$ is a set and $F$ is a fuzzy $\sigma$–field on $\Omega$, a fuzzy set $A$ in $\Omega$ is called fuzzy measurable (fuzzy measurable with respect to the fuzzy $\sigma$–field) if $A \in F$, i.e. any member of $F$ is called a fuzzy measurable set.

Definition (3): [13]
Let $(\Omega, F)$ be a fuzzy measurable space. A set function $\mu: F \rightarrow [0, \infty]$ is said to be a fuzzy measure on $(\Omega, F)$ if it satisfies the following properties:
1) $\mu(\emptyset) = 0$
2) If $A, B \in F$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$

Definition (4): [14]
Let $(\Omega, F)$ be a fuzzy measurable space. A set function $\mu: F \rightarrow [0, \infty]$ is said to be
1. Exhaustive if $\mu(A_n) \rightarrow 0$ whenever $\{A_n\}$ is infinite sequence of disjoint sets in $F$
2. Order-continuous if \( \mu(A_n) \to 0 \) whenever \( A_n \in \mathcal{F} \), \( n = 1, 2, \ldots \) and \( A_n \downarrow \emptyset \).

**Definition (5):** [14, 15]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu: \mathcal{F} \to [0, \infty)\) is said to be Null-additive, if \(\mu(A \cup B) = \mu(A)\) whenever \( A, B \in \mathcal{F} \) such that \( A \cap B = \emptyset \) and \(\mu(B) = 0\).

**Definition (6):** [1]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu: \mathcal{F} \to [0, \infty)\) is said to be weakly null-additive, if for any \( A, B \in \mathcal{F} \),

\[
\mu(A) = \mu(B) = 0 \implies \mu(A \cup B) = 0
\]

**Remark (70):**
The concept of null-null additive stems from a wings textbook which the book[1] derived from, in which it is said to be weak null additive. But we consider that it is more precise and vivid to call it "null-null additive"

**Definition (8):** [16]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space, A set function \(\mu: \mathcal{F} \to [0, \infty)\) is said to be Countably weakly null-additive, if for any \(\{A_n\} \subset \mathcal{F}, \mu(A_n) = 0\),

\[
\forall n \geq 1 \implies \mu\left( \bigcup_{n=1}^{\infty} A_n \right) = 0
\]

**Definition (9):** [16]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu: \mathcal{F} \to [0, \infty)\) is said to be Null-continuous, if \(\mu(U_{n=1}^{\infty} A_n) = 0\) for every increasing sequence \(\{A_n\}\) in \(\mathcal{F}\) such that \(\mu(A_n) = 0\), for all \(n \geq 1\).

**Definition (10):** [19]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu: \mathcal{F} \to [0, \infty)\) is said to be Autocontinuous from above (resp. autocontinuous from below), if \(\mu(B_n) \to 0\) implies \(\mu(A \cup B_n) = \mu(A)\) (resp. \(\mu(A \cap B_n) = \mu(A)\)), whenever \( A \in \mathcal{F}, \{B_n\} \subset \mathcal{F}\), \(\mu\) is called autocontinuous if it is both autocontinuous from above and autocontinuous from below.

**Definition (11):** [14]
Let \(C(\Omega)\) be the collection of all real valued functions defined on a set \(\Omega\). Let \(f, f_n \in C(\Omega)\), \(n \in \mathbb{N}\) and \(A \subset \Omega\), we say that

1. \(\{f_n\}\) converges point wise to \(f\) on \(A\), if for every \(x \in A\) and for every \(\varepsilon > 0\) there is \(k \in \mathbb{Z}^+\) such that \(|f_n(x) - f(x)| < \varepsilon\) for all \(n > k\).

We write \(\lim_{n \to \infty} f_n(x) = f(x)\) or \(f_n \to f\) on \(A\).

2. \(\{f_n\}\) Uniformly convergent to \(f\) on \(A\), for every \(\varepsilon > 0\) there is \(k \in \mathbb{Z}^+\) such that \(|f_n(x) - f(x)| < \varepsilon\) for all \(n > k\) and all \(x \in A\).

We write \(f_n(x) \xrightarrow{\text{u}} f(x)\) on \(A\).

3. \(\{f_n\}\) is point wise Cauchy sequence on \(A\), if for every \(x \in A\) and for every \(\varepsilon > 0\) there is \(k \in \mathbb{Z}^+\) such that \(|f_n(x) - f_m(x)| < \varepsilon\) for all \(m, n > k\), we write \(f_n \xrightarrow{\text{w}} f\) on \(A\).

This has meaning only if \(f_n: \Omega \to \mathbb{R}\) is finite valued, because \(\mathbb{R}\) is complete it is clear if \(\{f_n\}\) is a Cauchy sequence point wise on \(\Omega\), there must be on \(f: \Omega \to \mathbb{R}\) such that \(f_n \to f\) on \(\Omega\).

4. \(\{f_n\}\) is a uniformly Cauchy sequence on \(A\), if for every \(\varepsilon > 0\) there is \(k \in \mathbb{Z}^+\) such that \(|f_n(x) - f_m(x)| < \varepsilon\) for all \(n, m > k\) and all \(x \in A\). We write \(f_n \xrightarrow{\text{u}} f\) on \(A\).

**Definition (12):** [1, 19]
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space, let \(f, f_n \in C(\Omega)\), \(n \in \mathbb{N}\) and let \(A \in \mathcal{F}\), we say that

1. \(\{f_n\}\) Converges almost everywhere to \(f\) on \(A\), denoted by \(f_n \xrightarrow{a.e.} f\) on \(A\), if there is a subset \(B \subset A\) such that \(\mu(B) = 0\) and \(f_n \to f\) on \(A/B\).

2. \(\{f_n\}\) Converges pseudo almost everywhere to \(f\) on \(A\), denoted by \(f_n \xrightarrow{p.a.e.} f\) on \(A\), if there is a subset \(B \subset A\) such that \(\mu(A/B) = \mu(A)\) and \(f_n \to f\) on \(A/B\).

3. \(\{f_n\}\) Converges almost uniformly to \(f\) on \(A\), denoted by \(f_n \xrightarrow{a.u.} f\) on \(A\), if there is a sequence \(\{A_n\}\) in \(\mathcal{F}\) with \(\lim_{n \to \infty} \mu(A_n) = 0\) and \(f_n \to f\) on \(A/A_n\) for any fixed \(n = 1, 2, \ldots\).

4. \(\{f_n\}\) converges pseudo almost uniformly to \(f\) on \(A\), denoted by \(f_n \xrightarrow{p.a.u.} f\) on \(A\), if there is a sequence \(\{A_n\}\) in \(\mathcal{F}\) with \(\lim_{n \to \infty} \mu(A_n) = 0\) and \(f_n \to f\) on \(A_n\) for any fixed \(n = 1, 2, \ldots\).

5. \(\{f_n\}\) convergence in measure to \(f\) on \(A\), denoted by \(f_n \xrightarrow{m} f\) on \(A\), if \(\lim_{n \to \infty} \mu(\{x \in \Omega: |f_n(x) - f(x)| \geq \varepsilon\}) = 0\) for each \(\varepsilon > 0\).

6. \(\{f_n\}\) convergence pseudo in measure to \(f\) on \(A\), denoted by \(f_n \xrightarrow{p.m} f\) on \(A\), if \(\lim_{n \to \infty} \mu(\{x \in \Omega: |f_n(x) - f(x)| < \varepsilon\}) = \mu(A)\) for each \(\varepsilon > 0\).

**note that,** in the above definitions, when \(A = \Omega\) we can omit “on A” from the statements.

**2. Main Results**

**Lemma (1):** [1]
Let \((\Omega, \mathcal{F})\) be a measurable space, if \(\mu: \mathcal{F} \to [0, \infty)\) is a non-decreasing set function, then the following statements are equivalent:

1. \(\mu\) is null additive
2. \(\mu(A \cup B) = \mu(A)\)Whenever \(A, B \in \mathcal{F}\) and \(\mu(B) = 0\).
3. \(\mu(A/B) = \mu(A)\)Whenever \(A \subset \mathcal{F}\) such that \(B \subset A\) and \(\mu(B) = 0\).
4. \(\mu(A/B) = \mu(A)\)Whenever \(A \in \mathcal{F}\) and \(\mu(B) = 0\).
5. \(\mu(A\Delta B) = \mu(A)\)Whenever \(A \in \mathcal{F}\) and \(\mu(B) = 0\).

**Theorem (2):**
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space such that \(\mu\) is null additive, let \(f, f_n \in C(\Omega), n \in \mathbb{N}\) and let \(A \in \mathcal{F}\), if \(f_n \xrightarrow{a.e.} f\) on \(A\) then \(f_n \xrightarrow{p.a.e.} f\) on \(A\).

**Proof:**
Since \(f_n \xrightarrow{a.e.} f\) on \(A\), there exists a subset \(B \subset A\) such that \(\mu(B) = 0\) and \(f_n \to f\) on \(A/B\).
Since \( \mu \) is null additive, hence
\[ \mu(A \cup B) = \mu(A), \quad \text{whenever } A, B \in \mathcal{F} \text{ such that } A \cap B = \emptyset \text{ and } \mu(B) = 0. \]

By using lemma (1), we get on
\[ \mu(A/B) = \mu(A \cup B) = \mu(A). \]

Consequently
\[ f_n \longrightarrow f \text{ on } A. \]

**Theorem (3):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space such that \( \mu \) is autocontinuous from below, let \( f, f_n \in \mathcal{C}(\Omega), n \in \mathbb{N} \) and let \( A \in \mathcal{F} \), if \( f_n \longrightarrow f \) on \( A \) then \( f_n \longrightarrow f \) on \( A \).

**Proof:**

Since \( f_n \longrightarrow f \) on \( A \), then there is a subset \( B \subseteq A \) such that \( \mu(B) = 0 \) and \( f_n \rightarrow f \) on \( A/B \).

Since \( \mu \) is autocontinuous from below, hence
\[ \lim_{n \to \infty} \mu(A/A_n) = \mu(A), \quad \text{whenever } A \in \mathcal{F}, A_n \in \mathcal{F}, A_n \subseteq A, n = 1, 2, \ldots \text{ and } \lim_{n \to \infty} \mu(A_n) = 0. \]

Take \( B = A_n \), \( n = 1, 2, \ldots \), we have
\[ \mu(B) = \lim_{n \to \infty} \mu(A_n) = 0. \]

Therefore
\[ A/B \subseteq \emptyset. \]

Consequently
\[ f_n \longrightarrow f \text{ on } A. \]

**Theorem (4):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space, let \( f, f_n \in \mathcal{C}(\Omega), n \in \mathbb{N} \) and let \( A \in \mathcal{F} \) if \( f_n \longrightarrow f \) on \( A \) and \( f_n \longrightarrow f \) on \( A \). Then \( \mu \) is order continuous and autocontinuous from below.

**Proof:**

Since \( f_n \longrightarrow f \) on \( A \), then there is a sequence \( \{A_n\} \) be a sequence of sets in \( \mathcal{F} \)

With \( \lim_{n \to \infty} \mu(A_n) = 0 \).

i.e. \( \mu(A_n) \to 0 \), as \( n \to \infty \)

Therefore \( A_n \downarrow \emptyset \)

Consequently \( \mu \) is order continuous from below.

Let \( A \in \mathcal{F}, \{A_n\} \) be a sequence of sets in \( \mathcal{F} \) with \( A_n \subseteq A \)

Through \( f_n \longrightarrow f \) on \( A \), we have
\[ \lim_{n \to \infty} \mu(A/A_n) = \mu(A). \]

Which is \( \mu \) is autocontinuous from below.

**Theorem (5):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space, let \( f, f_n \in \mathcal{C}(\Omega), n \in \mathbb{N} \) and let \( A \in \mathcal{F} \) such that
\[ \lim_{n \to \infty} \mu(A/A_n) = \mu(A), \quad \text{whenever } \{A_n\} \text{ is a sequence of sets in } \mathcal{F} \text{ with } \lim_{n \to \infty} \mu(A_n) = 0. \]

Then \( f_n \longrightarrow f \) on \( A \) then \( f_n \longrightarrow f \) on \( A \).

**Proof:**

Since \( f_n \longrightarrow f \) on \( A \), then there is a sequence \( \{A_n\} \) in \( \mathcal{F} \) with
\[ \lim_{n \to \infty} \mu(A_n) = 0 \text{ such that } f_n \rightarrow f \text{ on } A \text{ for any fixed } n = 1, 2, \ldots \]

Since \( \lim_{n \to \infty} \mu(A/A_n) = \mu(A) \), we have
\[ A \cap A_n \in \mathcal{F} \text{ and } \mu(A \cap A_n) \leq \mu(A_n). \]

So we have
\[ \lim_{n \to \infty} \mu(A \cap A_n) = 0. \]

and therefore, by the condition given in this theorem, we have
\[ \lim_{n \to \infty} \mu(A/A_n) = \lim_{n \to \infty} \mu(A \Delta (A_n \cap A)) = \mu(A). \]

Consequently
\[ f_n \longrightarrow f \text{ on } A. \]

**Theorem (6):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space, let \( f, f_n \in \mathcal{C}(\Omega), n \in \mathbb{N} \) and let \( A \in \mathcal{F} \) such that \( \mu \) is exhaustive, if \( f_n \longrightarrow f \) on \( A \) then \( f_n \longrightarrow f \) on \( A \).

**Proof:**

Since \( f_n \longrightarrow f \) on \( A \), then there is a sequence \( \{A_n\} \) of sets in \( \mathcal{F} \) with \( \lim_{n \to \infty} \mu(A_n) = \mu(A) \) such that \( f_n \rightarrow f \) on \( A \) for any fixed \( n = 1, 2, \ldots. \)

Since \( \mu \) is exhaustive, \( \{A_n\} \) is a pairwise disjoint sequence in \( \mathcal{F} \), with \( \lim_{n \to \infty} \mu(A_n) = 0 \)

Consequently
\[ f_n \longrightarrow f \text{ on } A. \]

**Theorem (7):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space, let \( f, f_n \in \mathcal{C}(\Omega), n \in \mathbb{N} \) and let \( A \in \mathcal{F} \) such that \( \mu \) is exhaustive, for any decreasing sequence \( \{A_n\} \) of sets in \( \mathcal{F} \) for which \( \lim_{n \to \infty} \mu(A_n) = 0 \), if \( f_n \rightarrow f \) on \( A \) then \( f_n \longrightarrow f \) on \( A \).

**Proof:**

Since \( f_n \rightarrow f \) on \( A \), then there is a sequence \( \{A_n\} \) of sets in \( \mathcal{F} \) with \( \lim_{n \to \infty} \mu(A_n) = 0 \) such that \( f_n \rightarrow f \) on \( A \) for any fixed \( n = 1, 2, \ldots. \)

Since
\[ A/A_n \uparrow A/(\bigcap_{n=1}^{\infty} A_n) \]

and
\[ \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0. \]

By using lemma (1) continuity of \( \mu \), it follows that
\[ \lim_{n \to \infty} \mu(A/A_n) = \mu\left( A/(\bigcap_{n=1}^{\infty} A_n) \right) = \mu(A). \]

Consequently
\[ f_n \longrightarrow f \text{ on } A. \]

**Theorem (8):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space such that \( \mu \) is weakly null additive, let \( f, f_n \in \mathcal{C}(\Omega), n \in \mathbb{N} \) and let \( A \in \mathcal{F} \), if \( f_n \rightarrow f \) on \( A \) then \( f_n \rightarrow f \) on \( A \).

**Proof:**

Since \( f_n \rightarrow f \) on \( A \), then there is a subset \( B \in \mathcal{F} \) such that \( \mu(B) = 0 \) and for any \( x \in B \lim_{n \to \infty} f_n(x) = f(x) \)

For any \( \varepsilon > 0 \)

Since
\[ \{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\} \cap A \subseteq B \cup \{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\} \cap A \]

By monotonicity and weakly null additive, we have
\[\mu((x \in \Omega: |f_n(x) - f(x)| \geq \varepsilon) \cap A) \to 0 \text{ as } n \to \infty\]

Consequently \(f_n \xrightarrow{a.e.} f\) on \(A\).

**Theorem (9):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space such that \(\mu\) is countably weakly null-additive, let \(f, f_n \in C(\Omega), n \in \mathbb{N}\) and let \(A \in \mathcal{F}\), if \(f_n \xrightarrow{a.e.} f\) on \(A\) then

1. If \(g\) is real \(-\)-valued measurable function and \(f_n \xrightarrow{a.e.} g\) then \(f = g\) a.e.
2. If \(g\) is real \(-\)-valued measurable function such that \(f = g\) a.e. then \(f_n \xrightarrow{a.e.} g\).
3. If \(\{g_n\}\) is a sequence of real \(-\)-valued measurable functions such that \(g_n = f_n\) a.e for each \(n\) then \(g_n \xrightarrow{a.e.} f\).
4. If \(g, \{g_n\}\) is a sequence of real \(-\)-valued measurable function such that \(f_n = g_n\) a.e for each \(n\) and \(f = g\) a.e. then \(g_n \xrightarrow{a.e.} g\).

**Proof:**

1. Since \(f_n \xrightarrow{a.e.} f\) on \(A\), then there is a subset \(B \subseteq A\) such that \(\mu(B) = 0\) and \(f_n(x) \xrightarrow{} f(x)\) for all \(x \in A/B\) implies \(f_n(x)\) is Cauchy sequence for all \(x \in A/B\) \(\Rightarrow f_n\) is Cauchy a.e.
2. Since \(f_n \xrightarrow{a.e.} f\) on \(A\), then there is a subset \(B \subseteq A\) such that \(\mu(B) = 0\) and \(f_n(x) \xrightarrow{} f(x)\) for all \(x \in A/B\) implies \(f_n\) is Cauchy a.e.
3. Since \(f_n \xrightarrow{a.e.} f\) on \(A\), then there is a subset \(B \subseteq A\) such that \(\mu(B) = 0\) and \(f_n(x) \xrightarrow{} f(x)\) for all \(x \in A/B\) implies \(f_n\) is Cauchy a.e.
4. Since \(f = g\) a.e. then \(f_n \xrightarrow{a.e.} g\).
5. Since \(g, \{g_n\}\) is a sequence of real \(-\)-valued measurable function such that \(f_n = g_n\) a.e for each \(n\) and \(f = g\) a.e. then \(g_n \xrightarrow{a.e.} g\).

**Theorem (10):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space such that \(\mu\) is countably weakly null-additive, let \(f_n, g_n, f, g \in C(\Omega), n \in \mathbb{N}\) and let \(A \in \mathcal{F}, C \in \mathbb{R}\) if \(f_n \xrightarrow{a.e.} f\) and \(g_n \xrightarrow{a.e.} g\) on \(A\) then

1. \(c. f_n \xrightarrow{a.e.} c. f\).
2. \(f_n + g_n \xrightarrow{a.e.} f + g\).
3. \(|f_n| \xrightarrow{a.e.} |f|\).
4. If \(f_n = g_n\) a.e for all \(n\), then \(f = g\) a.e.

**Proof:**

1. Since \(f_n \xrightarrow{a.e.} f\), then there is a subset \(B \subseteq A\) such that \(\mu(B) = 0\) and \(f_n(x) \xrightarrow{} f(x)\) for all \(x \in A/B\) then \(c. f_n(x) \xrightarrow{} c. f(x)\) for all \(x \in A/B\) \(\Rightarrow c. f_n \xrightarrow{a.e.} c. f\).
2. Since \(f_n \xrightarrow{a.e.} f\) on \(A\), then there is a subset \(B \subseteq A\) such that \(\mu(B) = 0\) and \(f_n(x) \xrightarrow{} f(x)\) for all \(x \in A/B\) implies \(f_n \xrightarrow{a.e.} f\).
3. Since \(g_n \xrightarrow{a.e.} g\) on \(A\), then there is a subset \(C \subseteq A\) such that \(\mu(C) = 0\) and \(g_n(x) \xrightarrow{} g(x)\) for all \(x \in A/C\) implies \(g_n \xrightarrow{a.e.} g\).
4. Since \(f_n = g_n\) a.e for all \(n\), then \(f = g\) a.e.

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such that \( \mu(B) = 0 \) and \( f_n(x) \to f(x) \) for all \( x \in A/B \)

\[ \therefore \left| f_n(x) \right| \to |f(x)| \text{ for all } x \in A/B \]

\[ \therefore \left| f_n \right| \overset{a.e.}{\to} |f|. \]

(4) Since \( f_n \overset{a.e.}{\to} f \) on \( A \), then there is a subset \( B \subseteq A \) such that \( \mu(B) = 0 \) and \( f_n(x) \to f(x) \) for all \( x \in A/B \).

Since \( g_n \overset{a.e.}{\to} g \) on \( A \), then there is a subset \( C \subseteq A \) such that \( \mu(C) = 0 \) and

\[ g_n(x) \to g(x) \text{ for all } x \in A/C \]

Since \( f_n = g_n \) a.e. for each \( n \) then there is a sequence \( \{B_n\} \) such that

\[ \mu(B_n) = 0 \text{ and } f_n(x) = g_n(x) \text{ for all } x \in B_n \]

Let

\[ D = B \cup C \bigcup \left( \bigcup_{n=1}^{\infty} B_n \right) \]

Since \( \mu \) is countably weakly null-additive, we have

\[ \mu(D) = \mu(B \cup C \bigcup \left( \bigcup_{n=1}^{\infty} B_n \right)) \]

By (1) \( g - f_n \overset{a.e.}{\to} g - f \)

(3) Since \( f_n \overset{a.e.}{\to} f \), from theorem (10), we have \( |f_n| \overset{a.e.}{\to} |f| \)

Since \( |f_n| \leq |g| \) a.e. by (2), we get on

\[ |f| \leq |g| \] a.e.

(4) Since \( f_n \leq f_{n+1} \) a.e. for each \( n \), then there is \( A_n \in \Omega \) such that \( \mu(A_n) = 0 \text{ and } f_n(x) \leq f_{n+1}(x) \) for all \( x \in A_n \).

Let

\[ D = B \cup \left( \bigcup_{n=1}^{\infty} A_n \right) \]

Since \( \mu \) is countably weakly null-additive, we have

\[ \mu(D) = 0 \text{ for all } x \in D \]

Implies \( f_n(x) \overset{\text{a.e.}}{\to} f(x) \) and \( f_n(x) \to f(x) \)

Hence \( f_n \overset{\text{a.e.}}{\to} f \)

(5) Since \( f_n \overset{a.e.}{\to} f \), \( g_n \overset{a.e.}{\to} g \) and

\[ f_n = g_n \text{ a.e.} \] then by (4) from theorem (10), we get on

\[ f = g \text{ a.e.} \]

Since \( f_n \geq 0 \) a.e then by (1)

\[ f \geq 0 \text{ a.e. and } g \geq 0 \text{ a.e.} \]

\[ \text{Proof:} \]

Since \( f_n \overset{a.e.}{\to} f \), then there is \( B \subseteq \Omega \) such that \( \mu(B) = 0 \) and \( f_n(x) \to f(x) \) for all \( x \in B \)

(1) Since \( f_n \geq 0 \) a.e for each \( n \), then there is \( B_n \subseteq \Omega \)

such that \( \mu(B_n) = 0 \) and \( f_n(x) \geq 0 \) for all \( x \in B_n \)

Let

\[ D = B \bigcup \left( \bigcup_{n=1}^{\infty} B_n \right) \]

Since \( \mu \) is countably weakly null-additive, we have

\[ \mu(D) = 0 \text{ for all } x \in D \]

\[ f(x) = \lim_{n \to \infty} f_n(x) \geq 0 \text{ for all } x \in D \]

Therefor

\[ f \geq 0 \text{ a.e.} \]

(2) Since \( f_n \leq g \) a.e then \( g - f_n \geq 0 \) a.e.

Since \( f_n \overset{a.e.}{\to} f \)

\[ \therefore \left| f_n \right| \overset{a.e.}{\to} |f| \]

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Since $\mu$ is countably weakly null-additive, we have $\mu(D) = 0$ for all $x \notin D$, and $f_n(x) \to f(x), f_n(x) \to g(x)$ uniformly for any $x \notin D$

\[ \mu(D) = 0 \quad \text{for all } x \notin D \]

$\Rightarrow f(x) = g(x)$ for any $x \notin D$

$\therefore f = g$ a.e.

(2) Since $f_n \to f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$ with $\lim_{n \to \infty} \mu(A_n) = 0$ such that $f_n \to f$ on $A/A_n$ for any fixed $n = 1, 2, \ldots$. I.e. $f_n(x) \to f(x)$ uniformly for any $x \in A/A_n$. Since $f = g$ a.e, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f(x) = g(x)$ for all $x \in A \cap B$.

Let

\[ B_n = \emptyset \quad \text{for all } n \geq 2 \]

\[ B_1 = B, \quad B_\infty = \emptyset, B_3 = \emptyset, \ldots \]

\[ \bigcup_{n=1}^{\infty} B_n = B \]

\[ \Rightarrow \mu(B) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \]

\[ \therefore \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = 0 \]

Let

\[ D_n = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n \]

Where $\{D_n\}$ be a sequence of sets in $\mathcal{F}$

\[ \lim_{n \to \infty} \mu(D_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n\right) \]

Since $\mu$ is continuous from below at $A$, we have

\[ A = \bigcup_{n=1}^{\infty} A_n, \mu(A) = \lim_{n \to \infty} \mu(A_n) \]

\[ \Rightarrow \mu(A) = 0 \]

\[ \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0 \]

\[ \Rightarrow \lim \mu(D_n) = \mu(A \cup B) \]

Since $\mu$ is countably weakly null-additive, we have $\lim_{n \to \infty} \mu(D_n) = 0$ for any $x \notin D_n$ $f_n(x) \to f(x) = g(x)$ Uniformly for any $x \notin D_n$

Therefore

\[ f_n \to f \quad \text{a.e.} \]

(3) Since $f_n \to f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$ with $\lim_{n \to \infty} \mu(A_n) = 0$ such that $f_n \to f$ on $A/A_n$ for any fixed $n = 1, 2, \ldots$ I.e. $f_n(x) \to f(x)$ uniformly for any $x \in A/A_n$. Since $f_n = g_n$ a.e for each $n$, then there is a sequence $\{B_n\}$ in $\mathcal{F}$ such that $\mu(B_n) = 0$ and $f_n(x) = g_n(x)$ for all $x \in A/B_n$.

Let

\[ D_n = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n \]

Where $\{D_n\}$ be a sequence of sets in $\mathcal{F}$

\[ \lim_{n \to \infty} \mu(D_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n\right) \]

Since $\mu$ is continuous from below at $A$, we have

\[ A = \bigcup_{n=1}^{\infty} A_n, \mu(A) = \lim_{n \to \infty} \mu(A_n) \]

\[ \Rightarrow \mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0 \]

Since $\mu$ is countably weakly null-additive, we have $\mu(B_n) = 0$ for all $n \geq 1$

\[ \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = 0 \]

\[ \lim_{n \to \infty} \mu(D_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n\right) = 0 \]

Let $C_n = \emptyset$, for all $n \geq 2$

\[ C_1 = C, C_2 = \emptyset, C_3 = \emptyset, \ldots \]

\[ \bigcup_{n=1}^{\infty} C_n = C \]

\[ \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} C_n\right) = \mu(C) \]

\[ \therefore \mu\left(\bigcup_{n=1}^{\infty} C_n\right) = 0 \]

Let

\[ D_n = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} C_n \cup \bigcup_{n=1}^{\infty} B_n \]

Where $\{D_n\}$ be a sequence of sets in $\mathcal{F}$

\[ \lim_{n \to \infty} \mu(D_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} C_n \cup \bigcup_{n=1}^{\infty} B_n\right) \]

Since $\mu$ is continuous from below at $A$, we have

\[ \bigcup_{n=1}^{\infty} A_n = A, \mu(A) = \lim_{n \to \infty} \mu(A_n) \]

\[ \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu(A) \]
Also countably weakly null-additive, this mean
\[ \mu(B_n) = 0, \text{ for all } n \geq 1 \]
\[ \implies \mu \left( \bigcup_{n=1}^{\infty} B_n \right) = 0 \]

\[ \therefore \lim_{n \to \infty} \mu(D_n) = \mu \left( A \cup C \cup \left( \bigcup_{n=1}^{\infty} B_n \right) \right) \]

Since is countably weakly null-additive
\[ \implies \lim_{n \to \infty} \mu(D_n) = 0 \text{ for any } x \notin D_n, \]
and \( g_n(x) = f_n(x) \implies f(x) = g(x) \) Uniformly for any \( x \in D_n \)

Therefore \( g_n(x) \to g(x) \) Uniformly for any \( x \notin D_n \)
\[ \implies g_n \to g. \]

**Theorem (13):**
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space such that \(\mu\) is countably weakly null-additive and continuous from below at \(A\), let \(f_n, g_n, f, g \in \mathcal{B}(\Omega), n \in \mathbb{N}\) and let \(A \in \mathcal{F}, C \in \mathcal{C}\) if \(f_n \to f\) and \(g_n \to g\) then
1. \(c.f_n \to c.f\).
2. \(f_n + g_n \to f + g\).
3. \(f_n \to f\) for all \(\varepsilon > 0\), define

\[ B = \{ x \in \Omega : |f(x) - g(x)| \geq \varepsilon \} \cap A \]
\[ B_n = \{ x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon \} \cap A \]
\[ C_n = \{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \cap A \]

Since \(|f(x) - g(x)| \leq |f_n(x) - f(x)| + |f_n(x) - g(x)|\)
This implies that
\[ B \subseteq B_n \cup C_n \mu(B) \leq \mu(B_n) \cup C_n \]
\[ \implies \mu(B) \to 0, \mu(C_n) \to 0 \text{ as } n \to \infty \]
Since \(\mu\) is countably weakly null-additive, we have \(\mu(B_n) \cup C_n \to 0 \text{ as } n \to \infty\)
Therefore \(\mu(B) \to 0 \text{ as } n \to \infty\)
\[ N(f-g) = \{ x \in \Omega : (f-g)(x) \neq 0 \} \]
\[ = \bigcup_{n=1}^{\infty} \{ x \in \Omega : |f_n(x) - g(x)| \geq \frac{1}{n} \} \cap A \]
\[ \implies \mu(N(f-g)) = 0 \implies f = g \text{ a.e.} \]

2. Since \(f = g \text{ a.e.} \implies \text{there exists } B \in \mathcal{F} \text{ with} \mu(B) = 0 \text{ and } (f(x) \neq g(x)) \text{ for all } x \in B \text{ for any } \varepsilon > 0, \text{ we have} \]
\[ \{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \cap A \]
\[ \subseteq B \cup \{ x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon \} \cap A \]
\[ \subseteq B \cup \{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \cap A \]
\[ \mu \left( \bigcup_{n=1}^{\infty} \{ x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon \} \right) \leq \mu(B) \]
\[ \mu \left( \bigcup_{n=1}^{\infty} \{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \right) \leq \mu(B) \]
\[ \implies \mu(B) \to 0, \mu(C_n) \to 0 \text{ as } n \to \infty \]
Therefore \(\mu(B) \to 0 \text{ as } n \to \infty\)
\[ \mu(B) = 0 \implies \mu(A) = 0 \text{ a.e.} \]
\[ f_n \to f \text{ then} \]
\[ \mu \left( \bigcup_{n=1}^{\infty} \{ x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon \} \right) = 0, \mu(B) = 0 \]

Since \(\mu\) is countably weakly null-additive, we have
\[ \implies \mu \left( \bigcup_{n=1}^{\infty} \{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \right) \to 0 \text{ as } n \to \infty \]
\[ \implies f_n \to g. \]

3. Since \(f_n = g_n\) a.e for all \(n\), then there exists \(A_n \in \mathcal{F}\)
with \(\mu(A_n) = 0 \text{ and } f_n(x) \neq g_n(x) \text{ for all } x \in A_n\)

**Theorem (14):**
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space, let \(f, f_n \in \mathcal{C}(\Omega), n \in \mathbb{N}\) and let \(A \in \mathcal{F}\) such that \(\mu\) is countably weakly null-additive and continuous from below at \(A\), if \(f_n \to f\) then
1. \(f_n \to f\) and \(g_n \to g\)
2. \(f = g \text{ a.e.} \)
3. \(f_n = g_n\) a.e for all \(\varepsilon > 0\)

\[ g_n(x) \to g(x) \text{ Uniformly for any } x \notin D_n \]
\[ \implies g_n \to g. \]
Since $\mu$ is discontinuous from below at $A$, we have
\[ A = \bigcup_{n=1}^{\infty} A_n, \mu(A) = \lim_{n \to \infty} \mu(A_n) \]
\[ \Rightarrow \mu(A) = \mu\left( \bigcup_{n=1}^{\infty} A_n \right) \]
Since $\mu$ is countably weakly null-additive
\[ \Rightarrow \mu(x) = 0 \text{ for any } x > 0, \]
we have
\[ C = \left\{ (x \in \Omega : |g_n(x) - f(x)| \geq \varepsilon \cap B) \right\} \]
\[ C_n = \left\{ (x \in \Omega : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2} \cap B) \right\} \]
\[ D_n = \left\{ (x \in \Omega : |g_n(x) - f_n(x)| \geq \frac{\varepsilon}{2} \cap B) \right\} \]
Since
\[ |g_n(x) - f(x)| \leq |f_n(x) - f(x)| + |g_n(x) - f_n(x)| \]
\[ \Rightarrow C \subseteq C_n \cup D_n \Rightarrow \mu(C) \leq \mu(C_n \cup D_n) \]
Since $f_n \mu \to f, g_n \to g$
\[ \Rightarrow \mu(C_n \cup D_n) \to 0 \text{ as } n \to \infty \]
Therefore $\mu(x) = 0$.

**Theorem (15):**
Let $(\Omega, F, \mu)$ be a fuzzy measure space such that $\mu$ is weakly null-additive, let $f, g, f_n, g_n \in \mathcal{C}(\Omega), n \in \mathbb{N}$ and let $A \in F$, if $f_n \mu \to f, g_n \mu \to g \in \mathbb{R}$ then
\[ (1) \ c.f_n \mu \to c.f \]
\[ (2) \ |f_n - f| \to 0 \]

**Proof:**
(1) This is clear if $c = 0$, if $c \neq 0$, let $x > 0$.
Since $f_n \mu \to f$ and
\[ \left\{ (x \in \Omega : c.f_n(x) - f(x) \geq \varepsilon \cap A) \right\} \]
\[ = \left\{ (x \in \Omega : |f_n(x) - f(x)| \geq \frac{\varepsilon}{|c|} \cap A) \right\} \]
This implies that
\[ \mu\left( \left\{ x \in \Omega : c.f_n(x) - f(x) \geq \varepsilon \cap A \right\} \right) \]
\[ = \mu\left( \left\{ x \in \Omega : f_n(x) - f(x) \geq \frac{\varepsilon}{|c|} \cap A \right\} \right) \to 0 \text{ as } n \to \infty \]
So that
\[ c.f_n \mu \to c.f \]
(2) Since
\[ |f_n(x) - f(x)| \leq |f_n(x) - f(x)| \]
This implies that
\[ \left\{ x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon \cap A \right\} \]
\[ \subseteq \left\{ x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon \cap A \right\} \]
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