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Roles of Reachable Sets in Linear Control Systems.

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Abstract: ***The control function plays a great role in the study reachable sets. In this paper, it was shown that reachable set $\mathbb{R}(t)$ in a linear control system is closed, convex, bounded, symmetric and compact. This is true provided the associated control function has these properties.***

Keywords: Reachable sets, Banach Space, Attainable Set, Weakly Compact, integrable function

1. Introduction

Let us consider the linear control system

$$\dot{x} = A(t)\, x(t) + B(t)\, u(t),\; x(0) = x_0,\; t \geq 0 \quad (1.1)$$

where $\dot{x}(t) = \frac{d}{dt}x(t),\; x(t) \epsilon E^n$, the n-dimensional vector space. $A(t)$ *is* $n \times n$ matrix function, B(t) is $n \times m$ matrix function whose elements are integrable on every finite interval. The function u is an m-vector-valued measurable function constrained to lie in a compact set $U \epsilon E^m$. We note that under these conditions, $\dot{x}(t)$ in (1.1) has a unique solution. [1]. Let X(t) be the fundamental matrix of the homogeneous equation

$$\dot{x} = A(t)\, x(t),\; X(0) = I \quad (1.2)$$

where I is identity matrix. We note that if A is a constant matrix $X(t) = e^{At}$. For any measurable control function $u(t) \epsilon U$, the solution of (1.1) is given uniquely, using variation of parameter method, by

$$x(t,u) = X(t)x_0 + X(t)\int_0^t X^{-1}(s)\, B(s)\, u(s)\, ds \quad (1.3)$$

where, as usual, $x(0,u) = x_0$. [1]

We start with the following notations and definitions.

1.1 Notation and Definitions.

1.1(a) Notations:

$\mathbb{P} = \{P \subset E^n : P \text{ is closed}\}$.

$g : [T_0, T_1] \to P$, is called a target set function.

$\mathcal{A}(t) = \{x(t,u) : u \epsilon U$, u is measurable$\}$.

$\mathbb{R}(t) = \int_0^t Y(s)\, u(s)\, ds,\quad u \epsilon U.$

$\mathbb{R} = \{\mathbb{R}(t) : t \geq 0\} = \bigcup_{t \geq 0} \mathbb{R}(t).$

$U = \{u : [0,\infty) \to E^m : u \text{ is measurable}, |u_j| \leq 1, j = 1,2,3,\cdots,n.\}$

$\mathbb{R}^0(t) = \left\{\int_0^t X^{-1}(s)\, B(s)\, u^0(s)\, ds,\; u^0 \epsilon U^0\right\}.$

1.1(b) Definitions. [1]

Definition 1.
The set $\mathcal{A}(t) = \{x(t,u) : u \epsilon U$, u is measurable$\}$ is called an attainable set.

Definition 2.
The solution $x(t,u)$ for which the initial condition $x_0 : x(t, t_0, x_0, u) \epsilon g(t)\; \forall t \geq 0$ is called the core of the target.

Definition 3
A set U is symmetric about O if $u \epsilon U \Rightarrow -u \epsilon U$.

Definition 4. [1]
$u \epsilon U^n$ is convex if whenever $x, y \epsilon U$, the line segment $tx + (1-t)\, y \epsilon U$, where $0 \leq t \leq 1$.

Definition 5.
If P is a function defined on $0 \leq v \leq t$ and

$$P(v) = \begin{cases} u(v) : & v \epsilon [0,s) \\ 0 : & v \epsilon [s,t] \end{cases}$$

then P is said to be admissible.

Definition 6. [3]
A function f is said to be square integrable function if $f \epsilon L_2[0,t]$. *where*

$$L_2[0,t] = \left\{f : [0,t] \to E^n : \|f\| = \left(\int_0^t |f(s)|^2\right)^{1/2} < \infty\right\}.$$

Definition 7. [5]
$\{U_n\} = M \subset L_2$ is said to be weakly convergent to u if

$$\lim_{k \to \infty} \int_0^t f(s)\, U_k(s) ds = \int_0^t f(s)\, u(s) ds.$$

for each $f \epsilon L_2$.

Definition 8. [5]
M is said to be weakly compact if every sequence of M has a weakly convergence subsequence with limit in M.
Note that M is a Banach Space.

Definition 9: (Addition Formula of Reachable Sets)
Let $\dot{x} = Ax + Bu$ be autonomous system; A, B constant. A is an $n \times n$ matrix function and B(t) is $n \times m$ matrix. If $0 \epsilon U,\ t \geq 0,\ s \geq 0, then$

$\mathbb{R}(t) + e^{-At}\,\mathbb{R}(s) = \mathbb{R}(t+s)$

2. Propositions and Theorems

We have the following propositions and theorems concerning reachable sets for consideration. We are referring to the following linear control systems.

$\dot{x} = A(t)\,x(t) + B(t)\,u(t),\ x(0) = x_0$ (2.1)

Now, the controllability problem can be stated as follows: Find the conditions on A and B such that ∃ an admissible control such that the solution x(t, u) in (1.3) satisfies $x(t,u)\ \epsilon\ g(t)$. We note that the attainable set at a time t, consists of all solutions of (1.1) which we obtain by using all values of admissible controls u. We remark that hitting the target at time t i.e. $x(t,u)\ \epsilon\ g(t)$ is equivalent to $(t) \cap g(t) \neq \varphi$?. So, the controllable problem is now stated as follows: Is there a t ≥ 0 for which $\mathcal{A}(t) \cap g(t) \neq \varphi$?.

We also have the optimality problem, which can be stated as follows:

If $t^* = \inf\{t = g(t) \cap \mathcal{A}(t) \neq \varphi\}$ is there an admissible control u* such that
$x(t^*, u^*)\epsilon\, g(t^*)$?

We consider the solution x(t, u*) which is called the optimal trajectory. Finding optimal controls and their corresponding trajectories is called the time-optimal problem.

Now, let

$$\left.\begin{aligned} Y(t) &= X^{-1}(t)\,B(t) \\ W(t) &= X^{-1}(t)\,g(t) - x_0 \end{aligned}\right\} \qquad (2.2)$$

We note that $\mathbb{R}(t)$ is called the reachable set of (1.1) at time t. The optimal problem is equivalent to:
$\int_0^t Y(s)\,u(s)\,ds,\ \epsilon\, W(t)$ in minimum time.

We also note that
$\mathcal{A}(t) = X(t)\ (x_0 + \mathbb{R}(t))$
$= X(t)\ \{x_0 + y : y\ \epsilon\ \mathbb{R}(t)\}.$

We also note that

$g(t) \cap \mathcal{A}(t) \neq \varphi$ is equivalent to

$W(t) \cap \mathbb{R}(t) \neq \varphi,$

where W(t) is a defined in (2.2)

We recall that $\mathbb{R} = \{\mathbb{R}(t) : t \geq 0\} = \bigcup_{t \geq 0} \mathbb{R}(t)$.

Lemma 1.
$0\ \epsilon\ \mathcal{A}(t)\ \ iff\ -x_0\ \epsilon\ \mathbb{R}(t).$

Proof:
Let $0\ \epsilon\ \mathcal{A}(t)$

$0 = X(t)\ \{x_0 + y\}\ for\ some\ y\ \epsilon\ \mathbb{R}(t).$

Since the matrix X(t) is non singular, then

$$0 = X(t)\ (x_0 + y)$$
$$\Rightarrow X^{-1}(t).0 = X^{-1}(t)X(t)\ \ (x_0 + y)$$
$$\Rightarrow 0 = x_0 + y$$
$$\Rightarrow y = -x_0\ ;\ y\ \epsilon\ \mathbb{R}(t)$$
$$\Rightarrow -x_0\ \epsilon\ \mathbb{R}(t).$$

Conversely, suppose $-x_0\ \epsilon\ \mathbb{R}(t)$. Then for some $x(t,u)\ \epsilon\ \mathcal{A}(t)$,

$$x(t,u) = X(t)\ [\,x_0 + (-x_0).]$$
$$x(t,u) = X(t).0 = 0.$$

Since $x(t,u)\ \epsilon\ \mathcal{A}(t)\ and\ x(t,u) = 0$. Then, clearly $0\ \epsilon\ \mathcal{A}(t)$, completing the proof.

Lemma 2. [4]
(a) If the control set U is convex, so is $\mathbb{R}(t)$.

Proof :
Let $x, y\ \epsilon\ \mathbb{R}(t)$, then

$x = \int_0^t Y(s)\ U_1\,(s)\ ds,\ U_1\ \epsilon\ U.$

$y = \int_0^t Y(s)\ U_2\,(s)\ ds,\ U_2\ \epsilon\ U.$

We are to show that
$\lambda\,x + (1-\lambda)y\ \epsilon\ \mathbb{R}(t), 0 < \lambda < 1.$

Now,

$$\lambda \int_0^t Y(s)\ U_1(s)\ ds + (1-\lambda) \int_0^t Y(s)\ U_2(s)\ ds.$$

$$= \int_0^t Y(s)\ [\,\lambda U_1(s) + (1-\lambda)U_2(s)\,]\ ds.$$

u is convex $\Rightarrow \lambda\ U_1(s) + (1-\lambda)\ U_2(s)\ \ \epsilon\ U.$

Set $v = \lambda\,U_1(s) + (1-\lambda)\ U_2(s)\ \ \epsilon\ U.$

$\lambda\,x + (1-\lambda)y = \int_0^t Y(s)\ \ v(s)\ ds,\ \ v\ \epsilon\ U$

$\epsilon\ \mathbb{R}(t)$ QED.

(b) If $0\ \epsilon\ U$, then for all $0 \leq s \leq t$

$0\ \epsilon\ \mathbb{R}(s) \leq \mathbb{R}(t).$

Proof:
Since $0 \epsilon U$

Recall $\mathbb{R}(t) = \{\int_0^t Y(s)\ u(s)\ ds; u \epsilon U\}$.

So, $0 \epsilon \mathbb{R}(s)$ $[choose\ u(s) \neq since\ 0 \epsilon U]$

Let $x \epsilon \mathbb{R}(s); then$

$x = \int_0^s e^{-A\varsigma} B(\varsigma) u(\varsigma) d\varsigma; u : [0,t] \to U$

Since $x = \int_0^s X^{-1}(\varsigma)\ B(\varsigma) u(\varsigma) d\varsigma$

and $X(\varsigma) = e^{A\varsigma}\ for\ A\ constant$

$Y(\varsigma) = X^{-1}(\varsigma)\ B(\varsigma)$

$\int_0^s X^{-1}(\varsigma)\ B(\varsigma) d\varsigma\ +\ \int_s^t X^{-1}(\varsigma)\ B(\varsigma) d\varsigma, 0 \leq \varsigma \leq t$

$x = \int_s^t X^{-1}(\varsigma) B(\varsigma)\ P(\varsigma) d\varsigma\ \ \epsilon \mathbb{R}(t)$.

where $P(\varsigma)$ is an defined in Definition 5.

So
$x \epsilon \mathbb{R}(s) \Longrightarrow x \epsilon \mathbb{R}(t)$. Thus

$\mathbb{R}(s) \subseteq \mathbb{R}(t), 0 \leq s \leq t$.

c) If u is bonded, so is $\mathbb{R}(t)$. [4]

d) If u is symmetric, so is $\mathbb{R}(t)$. [4]

The proof of (c) and (d) are hereby omitted.

Proposition 1
The unit ball in L_2 is weakly compact.
i.e.
$M = \{ f \epsilon L_2 : \|f\| \leq 1\}$ is weakly compact.

$N = \{ f \epsilon L_2 : \|f\| \leq b < \infty\}$ is weakly compact.

Theorem 1.
If U is convex and compact, then the reachable set $\mathbb{R}(t)$ is convex and compact.

Proof of Theorem 1
We have seen that if u is convex and bounded, then so is $\mathbb{R}(t)$. We need only prove that $\mathbb{R}(t)$ is closed. Let x_k be a sequence of points in $\mathbb{R}(t)$ $such\ that\ x_k \to x\ as\ k \to \infty$.

We wish to show that $x \epsilon \mathbb{R}(t)$.

$x_k \epsilon \mathbb{R}(t) \Longrightarrow\ x_k = \int_0^t Y(s) u_k(s)\ ds\ \ for$

some $u_k \epsilon U$.

$U = \{u : [0,t) \to E^m;\ \|u_j\| < 1,\ i = 1,2,\cdots,m$

$U \subset\ L_2 \Longrightarrow \|u\| \leq K\ for\ some\ K.$

$\Longrightarrow U\ is\ weakly\ compact.$

Hence, the sequence $\{u_k\} \subset U$ has a subsequence $\{u_{k_j}\}$ which converges weakly to a point u in U.
Hence $\int_0^t Y(s)\ u_k(s)\ ds \to \int_0^t Y(s)\ u(s)\ ds\ as\ k \to \infty$.

Therefore
$x = \int_0^t Y(s)\ u(s)\ ds, some\ u \epsilon U$

$\Longrightarrow x \epsilon \mathbb{R}(t)$

Hence $\mathbb{R}(t)$ is closed and so is compact.

Theorem 2.
The reachable set $\mathbb{R} : [0,\infty) \to E^n$ is continuous.

Proof:
We show that $\forall\ \varepsilon > 0\ \exists\ \ \delta(\varepsilon) > 0\ such\ that$
$\rho(\mathbb{R}(t_1), \mathbb{R}(t_2)) < \varepsilon\ if\ \ |t_1 - t| < \delta(\varepsilon).$

For each $t_1 \geq 0, t_2 \geq 0\ \ if\ x(t,u) \epsilon \mathbb{R}(t)$, then

$x(t,u) = \int_0^t Y(s)\ u(s)\ ds$

$\|\ x(t_1,u)\ -\ x(t_2,u) \| = \left\|\int_{t_2}^{t_1} Y(s)\ u(s)\ ds\right\|.$

Since $\exists\ \lambda$ such that $\|\ u\|\ < \lambda, u \epsilon U\ compact.$

$\|\ x(t_1,u)\ -\ x(t_2,u) \| \leq \int_{t_2}^{t_1} \lambda\ \|Y(s)\| ds.$

Since an integral is an absolutely continuous function, it follows that
$\forall\ \varepsilon > 0\ \exists\ \ \delta(\varepsilon) > 0\ such\ that\ if\ |t_1 - t_2| <$
$\delta, we\ have\ \ \left\|\int_{t_2}^{t_1} \lambda\right\|\ \|Y(s)\| ds < \varepsilon.$

Hence
$\forall\ \varepsilon > 0\ \exists\ \delta = \ \delta(\varepsilon) > 0\ such\ that\ if\ |t_1 - t_2| < \delta,$

$\|\ x(t_1,u)\ -\ x(t_2,u) \| < \varepsilon$

Hence $\rho(\mathbb{R}(t_1), \mathbb{R}(t_2)) < \varepsilon$. Thus $\mathbb{R}(t)$ is continuous.

Theorem 3

Let A and B be constant, ie

$$\dot{x} = A\,x(t) + B\ u(t),\ \ x = x_0 \qquad (2.3)$$

Let $0 \epsilon U$. If 0ϵ Int. $\mathbb{R}(\varsigma)$ for small ς, then

1) $\mathbb{R}(t) \leq Int. \mathbb{R}(s)\ if\ 0 \leq t \leq s,$

2) $\mathbb{R} = \bigcup_{t\ \geq 0} \mathbb{R}(t)$ is open.

Proof
(i) Let $s > 0, \theta < s$. The addition formula for reachable sets states that $\mathbb{R}(\theta) + e^{-A\theta}\ \ \mathbb{R}(s - \theta) = \mathbb{R}(s)$.

Let $\varsigma > 0$ be small and is such that 0ϵ Int. $\mathbb{R}(\varsigma)$ $where\ \varsigma = s - \theta.$

Therefore $\exists$ a ball Q of small radius and such that

$$Q \leq \mathbb{R}(\varsigma).$$

If $t < \theta, then\ \mathbb{R}(t)\ <\ \mathbb{R}(\theta).$

Hence $\mathbb{R}(\theta) + e^{-A\theta}Q \leq \mathbb{R}(s)$.

Since $e^{-A\theta}$ is a homeomorphism, $e^{-A\theta}Q$ is a ball P about 0.

So
$\mathbb{R}(t) + P \subseteq \mathbb{R}(s)$.

Hence $\mathbb{R}(t) + P$ is a ball around $\mathbb{R}(t)$. Therefore, there is a ball N around $\mathbb{R}(t)$ which is contained in $\mathbb{R}(s)$ so that $\mathbb{R}(t) \subseteq Int\ \mathbb{R}(s)$.

(ii) To prove that $\mathbb{R}$ is open.

$$\mathbb{R}(t) = \bigcup_{t \geq 0} \mathbb{R}(t) \leq \bigcup_{s \geq 0} Int\ \mathbb{R}(s) \leq \bigcup_{s \geq 0} \mathbb{R}(s) = \mathbb{R}.$$

$\therefore \mathbb{R} = \bigcup_{s \geq 0} Int\ \mathbb{R}(s)$. The arbitrary union of open sets is open since interior $\mathbb{R}(s)$ is open. $\mathbb{R}$ is open.

3. Conclusion

We have given many roles played by reachable set $\mathbb{R}(t)$ on Linear control systems. So, the study of Linear control system cannot be said to be complete without the knowledge of reachable sets and their contributions to the systems. This set has so many roles on linear system which this contribution did not catch. We shall be happy if such roles will be investigated in near future.

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