

Generalized Radical_g-Supplemented Modules

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Abstract: In this paper we introduce generalized Radicals-supplemented and \oplus generalized Radical_g – supplemented module as generalizations of generalized Radical-supplemented module, \oplus generalized Radical supplemented module respectively some properties of these types of modules are investigated.

Keywords: generalized Radicals Radical_g supplemented module, supplemented module, \oplus generalized

1. Introduction

Let R be an associative ring with identity and let M be a unital left R -module. A submodule N of M is said to be small in M (briefly $N \ll M$), if whenever $M = N + L$ for $L \leq M$ implies $L = M$ [1]. By $\text{Rad}(M)$ we denote the Jacobson radical of M , $\text{Rad}(M)$ is the intersection of all maximal submodules equivalently $\text{Rad}(M)$ is the sum of all small submodules. A submodule N of M is called essential in M denoted by $N \leq_e M$, if $N \cap K \neq 0$ for every submodule $K \neq 0$ [2]. A submodule L of M is called generalized small denoted by $L \ll_g M$, if for essential submodule T of M with the property $M = L + T$ implies that $T = M$ [6]. It is clear that every small submodule is generalized small, but the converse is not true generally. The intersection of all maximal essential submodules of M is called generalized radical of M denoted by $\text{Rad}_g(M)$, If M has no maximal, then $\text{Rad}_g(M) = M$, In fact $\text{Rad}_g(M)$ is the sum of all g -small submodule of M [1]. The concept of generalized supplemented module appeared in [3]. Let N be any submodule of M , if there exists a submodule L of M such that $M = N + L$ with $N \cap L \leq \text{Rad}(L)$, then L is called generalized supplement of L in M . A module is called generalized supplemented, if every submodule of M has a generalized supplement in M [3]. Also in [3] a module M is called generalized \oplus supplemented, if every submodule of M has a generalized supplemented in M that is direct summand of M [5]. The concept of G -Radical supplemented module was introduced in [4] as a generalization of generalized supplemented modules, M is said to be generalized Radical supplemented (briefly) Rad_g supplemented, if $\forall N \leq M, \exists L \leq M$ such that $M = N + L$ and $N \cap L \leq \text{Rad}(L)$. In this paper we will give another notation of modules as generalization of Rad_g – supplemented module. Also $\oplus G\text{-Rad}_g$ –supplemented will be introduced some properties of these types of modules will be proved.

2. $G\text{-Rad}_g$ -supplemented Modules

In this section we will introduce $G\text{-Rad}_g$ - supplemented modules as a generalization of generalized Radical supplemented that appeared in [4] some properties of this type of modules will be proved.

Definition(2.1) : Let $N \leq M$, with $\text{Rad}(M) \leq N$, N is said to has a Rad_g - supplement submodule, if $\exists L \leq M$ such that $M = N + L$ and $N \cap L \leq \text{Rad}_g(M)$.

A module M is called $G\text{-Rad}_g$ –Supplemented, if every submodule N containing $\text{Rad}_g(M)$ has a Rad_g -supplement.

Recall that a submodule N of M is fully invariant if for every $h \in \text{End}(M)$, $h(N) \leq N$ and M is called a duo module, if every submodule of M is fully invariant.[9]

Proportion (2.2): Let N be a submodule of M such that $\text{Rad}(M) \leq N$ and N is a direct summand of M , then $\text{Rad} N = \text{Rad}(M) \cap N$

Proof: Let $N \leq M$ such that $\text{Rad}_g(M) \leq N$, since N is a direct summand of M , then $M = N \oplus K$ for some $K \leq M$ thus by [3],

$$\text{Rad}_g(M) = \text{Rad}_g(N) \oplus \text{Rad}_g(K) \text{ and } \text{Rad}_g(M) \cap N = (\text{Rad}_g(N) \oplus \text{Rad}_g(K)) \cap N, \text{ thus } \text{Rad}_g(M) \cap N = \text{Rad}_g(N).$$

Proportion (2.3): Let $M = M_1 \oplus M_2$, if M is $G\text{-Rad}_g$ -supplemented then M_1 and M_2 are $G\text{-Rad}_g$ -supplemented.

Proof: Let $N_1 \leq M_1$ and let $\text{Rad}(M_1) \leq N_1$ then $\text{Rad}(M) \leq N_1 + \text{Rad}(M_1)$, since M is $G\text{-Rad}_g$ -supplemented, then $\exists K_1 \leq M$ such that $M = N_1 + \text{Rad}_g(M) + K_1$ and $N_1 + \text{Rad}_g(M) \cap K_1 \leq \text{Rad}_g(M)$.

Now $M_1 = N_1 + \text{Rad}_g(M) + K_1 \cap M_1$, $M_1 = N_1 + \text{Rad}_g(M) \cap M_1 + (K_1 \cap M_1) = N_1 + \text{Rad}_g(M_1) + (K_1 \cap M_1) = N_1 + (K_1 \cap M_1)$ by proportion (2.2) and $N_1 \cap (K_1 \cap M_1) \leq N_1 \cap K_1 \leq \text{Rad}(M) \cap M_1 = \text{Rad}(M_1)$ by proportion (2.2), similarly for M_2

Proportion (2.4): Let $M = M_1 \oplus M_2$, if M_1, M_2 are $G\text{-Rad}_g$ -supplemented and M is a duo modules, then M is $G\text{-Rad}_g$ -supplemented.

Proof : Let $N \leq M$ such that $\text{Rad}_g(M) \leq N$, then $N \cap M_i \leq M_i \forall i = 1, 2$ and $\text{Rad}_g(M) \cap M_i \leq N \cap M_i$, but $\text{Rad}_g(M_i) \leq \text{Rad}_g(M) \cap M_i \leq N \cap M_i \forall i = 1, 2$, Since M_i is Rad_g -supplemented, then $\forall i = 1, 2, \exists K_i \leq M_i$ such that $M_i = (N \cap M_i) + K_i$ and $(N \cap M_i) \cap K_i \leq \text{Rad}_g(M_i)$

Now $M = M_1 \oplus M_2$, then $N = (N \cap M_1) \oplus (N \cap M_2)$ and $M = ((N \cap M_1) + (N \cap M_2) + (K_1 + K_2)) \cap ((N \cap M_1) + (N \cap M_2)) \cap (K_1 + K_2) = N \cap M_1 \cap K_1 + N \cap M_2 \cap K_2 \leq \text{Rad}_g(M_1) + \text{Rad}_g(M_2) = \text{Rad}_g(M)$.

Corollary(2.5) : Let $M = M_1 \oplus \dots \oplus M_n$, if $\forall i = 1, 2, \dots, n$ M_i is $G\text{-Rad}_g$ -supplemented and M is duo module, then M is $G\text{-Rad}_g$ -supplemented

Proportion(2.6) : Let M be a $G\text{-Rad}_g$ -supplemented, then for any submodule N of M , M/N is a $G\text{-Rad}_g$ -supplemented.

Proof: Let $K/N \leq M/N$ such that $\text{Rad}(M/N) \leq K/N$. Since $\text{Rad}_g((M+N)/N) \leq \text{Rad}_g(M/N)$. Then $\text{Rad}_g(M) \leq K$ by hypothesis, $\exists L \leq M$ such that $M = K + L$ and $K \cap L \leq \text{Rad}_g(M)$. Then $M/N = (K + L)/N = K/N + (L + N)/N$ and $K/N \cap (L + N)/N = (K \cap L) + N/N \leq (\text{Rad}_g(M) + N)/N \leq \text{Rad}_g(M/N)$.

Corollary(2.7): The homomorphic image of any $G\text{-Rad}_g$ -Supplemented module is a $G\text{-Rad}_g$ -supplemented module.

Proof: Clear since every homomorphic image is isomorphic to a quotient module.

Lemma (2.8): Let M be any R -module with $\text{Rad}_g(M) = M$, then M is a $G\text{-Rad}_g$ -supplemented,

Proof: Since $\text{Rad}_g(M) = M$, then M has the trivial Rad_g -supplement 0 in M . Consequently M is a $G\text{-Rad}_g$ -supplemented.

It is clear that every semi simple is a $G\text{-Rad}$ supplemented, but the converse in general is not true for example Q as Z -module is $G\text{-Rad}_g$ supplemented but not semi simple.

Proportion (2.9) : Let M be a module with $\text{Rad}(M) = 0$ then M is a $G\text{-Rad}_g$ -supplemented if and only if M is semi simple.

Proof: (\leftarrow) Clear
 \rightarrow) Let $N \leq M$, then $0 \leq N$, since M is Rad_g -supplemented, then $M = N + K$ and $N \cap K \leq \text{Rad}_g(M) = 0$, hence $M = N \oplus K$ (i.e. M is semi simple).

A ring R is called left V - ring, if every simple R -module is injective. Equivalently R is a V - ring if and only if every $\text{Rad}(M) = 0$ for every left R -module M [8]. Let R be a commutative ring R is regular if and only if every simple R -module is injective, [7]

Proportion (2.10): Let M be a V -ring and M is R -module, then M is $G\text{-Rad}_g$ -supplemented if and only if M is semi simple.

Corollary (2.11): Let R be a commutative regular ring and M is a R -module, then M is $G\text{-Rad}_g$ -supplemented if and only if M is semi simple.

§3 $\oplus G\text{-Rad}_g$ -Supplemented Modules

This section dedicated to introduces $\oplus G\text{-Rad}_g$ -supplemented modules as a generalization of $G\text{-Rad}_g$ -

supplemented modules. The results of this section is a generalization of $G\text{-Rad}_g$ -supplemented modules

Definition(3.1): A module M is called $\oplus G\text{-Rad}_g$ -supplemented, if every submodule M with $\text{Rad}_g(M) \leq N$ has a $G\text{-Rad}_g$ -supplement that is a direct summand of M . (i.e. $\forall N \leq M$ with $\text{Rad}(M) \leq N$, $\exists L \leq M$ such that $M = N + L$, $N \cap L \leq \text{Rad}_g(M)$ and $L \leq \oplus M$ i.e. $M = L \oplus K$, (for $K \leq M$).

Remark (3.2) : The submodule of $G\text{-Rad}_g$ -supplemented need not be Rad_g -supplemented, for example Z as a submodule Q as Z -modules is not $\oplus G\text{-Rad}_g$ -supplemented.

Proportion (3.3) : If N is a direct summand of $G\text{-Rad}_g$ supplemented module, then N is $G\text{-Rad}_g$ supplemented.

Proof: Let $N \leq M$ such that $N \leq \oplus M$ and $L \leq N$ with $\text{Rad}(N) \leq L$ then $L \leq M$, since M is $G\text{-Rad}_g$ -supplemented, then $\exists L' \leq L$ such that $M = L + L'$, with $L \cap L' \leq \text{Rad}(M)$ and $M = L' \oplus L''$. Now $N \cap M = N = N \cap L' \oplus N \cap L''$ and $N = N \cap L + N \cap L''$.

$N \cap L \cap N \cap L' = N \cap (L \cap L') \leq N \cap \text{Rad}(M) = \text{Rad}(N)$ by proportion(2.2).

Proportion (3.4): Let $M = M_1 \oplus M_2$ be a $\oplus G\text{-Rad}_g$ -supplemented module, then $\forall i = 1, 2$, M_i is a $\oplus G\text{-Rad}_g$ -supplemented module.

Proof: By using the same argument in proportion (2.5), there exist $K_1 \leq M_1$ such that $M_1 = N_1 + K_1$ and $N_1 \cap K_1 \leq \text{Rad}(M_1)$ and since M is $G\text{-Rad}_g$ -supplemented, then $\exists L_1 \leq M$ such that $M = (K_1 \oplus L_1) \cap M_1 = (K_1 \cap L_1) \oplus (L_1 \cap M_1)$, therefore M_1 is $\oplus G\text{-Rad}_g$ -supplemented, similarly for M_2 .

Corollary (3.5): Let $M = M_1 \oplus M_2 \dots \oplus M_i$ be a $\oplus G\text{-Rad}_g$ -supplemented module, then $\forall i = 1, 2, \dots, n$, M_i is a $\oplus G\text{-Rad}_g$ -supplemented module.

Proportion (3.6) : Let $M = M_1 \oplus M_2$, if M_1 and M_2 are $\oplus G\text{-Rad}_g$ -supplemented and M is a duo module then M is $\oplus G\text{-Rad}_g$ -supplemented.

Proof: Let $\text{Rad}_g(M) \leq N \leq M$, then by the same arguments in Proportion (2.2), $N \cap M_1 + K_1 = M_1$, $K_1 \leq M_1$, $(N \cap M_1) \cap K_1 \leq \text{Rad}(M_1)$. $\forall i = 1, 2, \dots$ and since M_1, M_2 are $\oplus G\text{-Rad}_g$ -supplemented then $K_i \oplus L_i = M_i$, $L_i \leq M_i$. $\forall i = 1, 2$

Now $(L_1 \oplus L_2) \oplus (K_1 \oplus K_2) = M$, hence $K_1 + K_2$ is $G\text{-Rad}_g$ -supplemented of N in M which is a direct summand of M .

Corollary (3.7): Let $M = M_1 \oplus M_2 \oplus \dots \oplus M_i$, . If $\forall i = 1, 2, \dots, n$, is $\oplus G\text{-Rad}_g$ -supplemented and M is a duo module, then M is $\oplus G\text{-Rad}_g$ -supplemented

Proportion (3.8): Let M be a $\oplus G\text{-Rad}_g$ -supplemented module, then for any submodule N of M , M/N is $\oplus G\text{-Rad}_g$ -supplemented.

Proof: Let $K/N \leq M/N$ such that $\text{Rad}(M/N) \leq K/N$. Since $(\text{Rad}(M) + N)/N \leq \text{Rad}(M/N) \leq K/N$ then $\text{Rad}(M) \leq K$, by assumption, there exists $L \leq M$ such that $M = K + L$ with $K \cap L \leq \text{Rad}(M)$ and $M = L \oplus L'$, $L' \leq M$.

Now $M/N = (K + L)/N = K/N + (L + N)/N$ and $K/N \cap (L + N)/N = (L \cap K + N)/N \leq (\text{Rad}(M) + N)/N \leq \text{Rad}(M/N)$. And $M/N = (L \oplus L')/N = (L + N)/N \oplus (L' + N)/N$. thus M/N is a \oplus G-Rad_g-supplemented.

Corollary (3.9): The homomorphic image of \oplus G-Rad_g-supplemented module is \oplus G-Rad_g-supplemented module.

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