International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

Generalized Radical$_g$–Supplemented Modules

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Abstract: ***In this paper we introduce generalized Radicals-supplemented and ⊕ generalized Radical$_g$ – supplemented module as generalizations of generalized Radical- supplemented module, ⊕ generalized Radical supplemented module respectively some properties of these types of modules are investigated.***

Keywords: generalized Radicals Radical$_g$ supplemented module, supplemented module, ⊕ generalized

1. Introduction

Let R be an associative ring with identity and let M be a a unital left R-module. A submodule N of M is said to be small in M (brietly N << M), if whenever M = N + L for L ≤ M implies L = M [1]. By Rad(M) we denote the Jacobson radical of M, Rad(M) is the intersection of all maximal submodules equivalently Rad(M) is the sum of all small submodules. A submodules N of M is called essential in M denoted by $N \leq_e M$, if $N \cap K \neq 0$ for every submodule $K \neq 0$[2].A submodule L of M is called generalized small denoted by $L <<_g M$, if for essential submodule T of M with the property M = L + T implies that T = M[6]. It is clear that every small submodule is generalized small, but the converse is not true generally. The intersection of all maximal essential submodules of M is called generalized radical of M denoted by $Rad_g(M)$, If M has no maximal, then $Rad_g(M) = M$,In fact $Rad_g(M)$ is the sum of all g-small submodule of M [1].The concept of generalized supplemented module appeared in [3].Let N be any submodule of M, if there exists a submodule L of M such that M = N + L with $N \cap L \leq Rad(L)$, then L is called generalized supplement of L in M. A module is called generalized supplemented, if every submdule of M has a generalized supplement in M [3].Also in [3] a module M is called generalized⊕ supplemented, if every submodule of M has a generalized supplemented in M that is direct summand of M[5]. The concept of G- Radical supplemented module was introduced in [4] as a generalization of generalized supplemented modules, M is said to be generalized Radical supplemented (briefly) Rad_g supplemented , if $\forall N \leq M$,$\exists L \leq M$ such that M = N + L and $N \cap L \leq Rad(L)$. In this paper we will give another notation of modules as generalization of Rad_g – supplemented module. Also ⊕ G-Rad_g –supplemented will be introduced some properties of these types of modules will be proved.

2. G-Rad $_g$-supplemented Modules

In this section we will introduce G-Rad_g- supplemented modules as a generalization of generalized Radical supplemented that appeared in [4] some properties of this type of modules will be proved.

Definition(2.1) : Let N ≤ M , with Rad (M) ≤ N , N is said to has a Rad_g - supplement submodule, if $\exists L \leq M$ such that $M = N + L$ and $N \cap L \leq Rad_g(M)$.

A module M is called G- Rad_g –Supplemented, if every submodule N containing $Rad_g(M)$ has a Rad_g-supplement.

Recall that a submodule N of M is fully invariant if for every $h \in End(M)$, $h(N) \leq N$ and M is called a duo module, if every submodule of M is fully invariant.[9]

Proportion (2.2): Let N be a submodule of M such that Rad(M) ≤ N and N is a direct summand of M, then Rad N = $Rad(M) \cap N$

Proof: Let N ≤ M such that $Rad_g(M) \leq N$, since N is a direct summand of M, then $M = N \oplus K$ for some K ≤ M thus by [3],

$Rad_g(M) = Rad_g(N) \oplus Rad_g(K)$ and $Rad_g(M) \cap N = (Rad_g(N) \oplus Rad_g(K)) \cap N$,thus $Rad_g(M) \cap N = Rad_g(N)$.

Proportion (2.3): Let $M = M_1 \oplus M_2$,if M is G- Rad_g-supplemented then M_1 and M_2 are G- Rad_g-supplemented.

Proof: .Let $N_1 \leq M_1$ and let $Rad(M_1) \leq N_1$ then $Rad(M) \leq N_1 + Rad(M_1)$,since M is G- Rad_g-supplemented, then $\exists K_1 \leq M$ such that $M = N_1 + Rad_g(M) + K_1$ and $N_1 + Rad_g(M) \cap K \leq Rad_g(M)$.

Now $M_1 = N_1 + Rad_g(M) + K_1 \cap M_1$, $M_1 = N_1 + Radg(M) \cap M_1 + (K_1 \cap M_1) = N_1 + Rad_g(M_1) + (K_1 \cap M_1) = N_1 +(K_1 \cap M_1)$ by proportion (2.2) and $N_1 \cap (K_1 \cap M_1) \leq N_1 \cap K_1 \leq Rad(M) \cap M_1 = Rad(M_1)$ by proportion (2.2), similarly for M_2

Proportion (2.4): Let $M = M_1 \oplus M_2$,if M_1 ,M_2 are G- Rad_g-supplemented and M is a due modules, then M is G- Rad_g-supplemented.

Proof : Let N ≤ M such that $Rad_g (M) \leq N$, then $N \cap M_i \leq M_i$ $\forall$ i = 1,2 and $Rad_g(M) \cap M_i \leq N \cap M_i$, but $Rad_g(M_i) \leq Rad_g(M) \cap M_i \leq N \cap M_i$ $\forall$ i= 1,2,Since M_i is Rad_g-supplemented , then $\forall$ i = 1,2, $\exists K_i \leq M_i$ such that $M_i= (N \cap M_i) + K_i$ and $(N \cap M_i) \cap K_i \leq Rad_g(M_i)$

Now $M = M_1 \oplus M_2$, then $N = (N \cap M_1) \oplus (N \cap M_2)$ and $M = ((N \cap M_1) + (N \cap M_2) + (K_1 + K_2)$and $((N \cap M_1) + (N \cap M_2)) \cap (K_1 + K_2) = N \cap M_1 \cap K_1 + N \cap M_2 \cap K_2 \leq Rad_g(M_1) + Rad_g(M_2) = Rad_g(M)$.

Corollary(2.5) : Let $M = M_1 \oplus \ldots \oplus M_n$, if $\forall$ i= 1,2…n M_i is G-Rad_g –supplemented and M is duo module, then M isG-Rad_g-supplemented

Proportion(2.6) : Let M be a G-Rad_g-supplemented , then for any submodule N of M, M / N is a G- Rad_g-supplemented.

Proof: Let $K / N \leq M / N$ such that $Rad(M/N) \leq K / N$ .Since $Rad_g((M+N)/N) \leq Rad_g(M/N)$. Then $Rad_g(M) \leq K$ by hypothesis , $\exists\, L \leq M$ such that $M = K + L$ and $K \cap L \leq Rad_g(M)$. Then $M / N = (K + L) / N = K / N + (L + N) / N$ and $K / N \cap (L + N) / N = (K \cap L) + N / N) \leq (Rad_g(M) + N) / N \leq Rad_g(M / N)$.

Corollary(2.7): The homomorphic image of any G- Rad_g-Supplemented module is a G- Rad_g- supplemented module.

Proof: Clear since every homomorphic image is isomorphic to a quotient module.

Lemma (2.8): Let M be any R-module with $Rad_g(M) = M$, then M is a G- Rad_g-supplemented,

Proof: Since $Rad_g(M) = M$, then M has the travail Rad_g-supplement 0 in M Consequently M is a G-Rad_g-supplemented.

It is clear that every semi simple is a G- Rad supplemented , but the converse in general is not true for example Q as Z-module is G-Rad_g supplemented but not semi simple.

Proportion (2.9) : Let M be a module with Rad(M) = 0 then M is a G- Rad_g-supplemented if and only if M is semi simple.

Proof: $\longleftarrow$)) Clear
$\longrightarrow$)) Let $N \leq M$, then $0 \leq N$, since M is Rad_g-supplemented, then $M = N + K$ and $N \cap K \leq Rad_g(M) = 0$, hence $M = N \oplus K$ (i.e. M is semi simple) .

A ring R is called left V- ring, if every simple R-module is injective. Equivalently R is a V – ring if and only if every Rad(M)= 0 for every left R-module M [8]. Let R be a commutative ring R is regular if and only if every simple R-module is injective, [7]

Proportion (2.10): Let M be a V-ring and M is R-module, then M is G- Rad_g-supplemented if and only if M is semi simple.

Corollary (2.11): Let R be a commutative regular ring and M is a R-module, then M is G-Rad_g - supplemented if and only if M is semi simple.

§3 $\oplus$ G- Rad_g –Supplemented Modules

This section dedicated to introduces $\oplus$G-Rad_g-supplemented modules as a generalization of G-Radg-supplemented modules. The results of this section is a generalization of G-Radg-supplemented modules

Definition(3.1): A module M is called $\oplus$G- Rad_g – supplemented, if every submodule M with $Radg(M) \leq N$ has aG- Radg-supplement that is a direct summand of M.(i.e. $\forall$ $N \leq M$. with $Rad(M) \leq N$, $\exists\, L \leq M$ such that $M = N + L$, $N \cap L \leq Rad_g(M)$ and $L \leq_{\oplus} M$ i.e. $M = L \oplus K$, (for $K \leq M$).

Remark (3.2) : The submodule of G- Rad_g –supplemented need not be Rad_g-supplemented , for example Z as a submodule Q as Z-modules is not $\oplus$ G – Rad_g-supplemented.

Proportion (3.3) : If N is a direct summand of G-Rad_g supplemented module, then N is G- Rad_g supplemented.

Proof: Let $N \leq M$ such that $N \leq_{\oplus} M$ and $L \leq N$ with $Rad(N) \leq L$ then $L \leq M$, since M is G-Rad_g –supplemented , then $\exists\, L' \leq L$ such that $M = L + L'$, with $L \cap L' \leq Rad(M)$ and $M = L' \oplus L''$. Now $N \cap M = N = N \cap L' \oplus N \cap L''$ and $N = N \cap L + N \cap L''$.

$N \cap L \cap N \cap L' = N \cap (L \cap L') \leq N \cap Rad(M) = Rad(N)$ by proportion(2.2).

Proportion (3.4): Let $M = M_1 \oplus M_2$ be a $\oplus$G- Rad_g-supplemented module, then $\forall$ i = 1,2 ,M_i is a $\oplus$ G-Rad_g-supplemented module.

Proof: By using the same argument in proportion (2.5) , there exist $K_1 \leq M_1$ such that $M_1 = N_1 + K_1$ and $N_1 \cap K_1 \leq Rad(M_1)$ and since M is G-Radg-supplemented, then $\exists\, L_1 \leq M$ such that $M = (K_1 \oplus L_1) \cap M_1 = (K_1 \cap L_1) \oplus (L_1 \cap M_1)$, therefore M_1 is $\oplus$ G-Rad_g-supplemented, similarly for M_2.

Corollary (3.5): Let $M = M_1 \oplus M_2 \ldots \oplus M_i$ be a $\oplus$ G- Rad_g-supplemented module , then then $\forall$ i = 1,2,..n, M_i is a $\oplus$ G-Rad_g-supplemented module.

Proportion (3.6) : Let $M = M_1 \oplus M_2$, if M_1 andM_2 are $\oplus$ G-Rad_g-supplemented and M is a duo module then M is $\oplus$ G-Rad_g-supplemented.

Proof: Let $Rad_g(M) \leq N \leq M$, then by the same arguments in Proportion (2.2) , $N \cap M_i + K_i = M_i$, $K_i \leq M_i$, $(N \cap M_i) \cap K_i \leq Rad(M_i)$. $\forall$ i = 1,2,.. and since M_1, M_2 are $\oplus$ G-Rad_g-supplemented then $K_i \oplus L_i = M_i$, $L_i \leq M_i$. $\forall$ i = 1,2

Now $(L_1 \oplus L_2) \oplus (K_1 \oplus K_2) = M$, hence $K_1 + K_2$ is G-Rad_g-supplemented of N in M which is a direct summand of M.

Corollary (3.7): Let $M = M_1 \oplus M_2 \oplus \ldots M_i$, . *If* $\forall$ i = 1,2,.. n, is $\oplus$ G- Rad_g-supplemented and M is a duo module , then M is$\oplus$ G- Rad_g-supplemented

Proportion (3.8): Let M be a $\oplus$ G-Rad_g-supplemented module, then for any submodule N of M, M / N is $\oplus$ G-Rad_g-supplemented.

Proof: Let $K / N \leq M / N$ such that $Rad(M / N) \leq K / N$. Since $(Rad(M)+ N) / N \leq Rad(M / N) \leq K / N$ then $Rad(M) \leq K$, by assumption, there exists $L \leq M$ such that $M = K + L$ with $K \cap L \leq Rad(M)$ and $M = L \oplus L'$, $L' \leq M$.

Now $M / N = (K + L) / N = K / N + (L + N / N)$ and $K / N \cap (L + N) / N = (L \cap K + N) / N \leq (Rad(M) + N) / N \leq Rad(M / N)$. And $M / N = (L \oplus L') / N =(L + N) / N \oplus (L' + N) / N$.thus M /N is a $\oplus$ G-Rad_g-supplemented.

Corollary (3.9): The homorphic image of $\oplus$ G-Rad_g-supplemented module is $\oplus$ G- Rad_g-supplemented module.

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