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Application of Fractional Calculus in Euler-Darboux Equation of Convolution Type-II

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Abstract: ***In this paper, authors have established new & interesting result of Euler-Darboux equation of convolution type-II by application of fractional integrals and derivatives.***

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1. Introduction

The paper is devoted to solve a boundary value problem for the Euler-Darboux equation

$$u_{xy} - (\beta u_x - \alpha u_y)/(x-y) = 0 \ (\alpha>0,\ \beta>0, \alpha+\beta<1)$$

in the domain [(x, y) | 0 < x < y < 1] by reducing it to a dominant singular integral with Cauchy kernel. Boundary conditions are $u(x,x)=\varphi_1(x)$ and $AI_{ox}^{a,b,-a+\beta-1}u(o,x)+BJ_{x1}^{a+\alpha-\beta,c,-a+\beta-1}u(x,1)=\varphi_2(x)$, where I and J stand for generalized fractional integral operators. In the recent paper Bajpai and Gaur[2], there has been discussed a generalized Goursat problem for the Euler - Darboux equation.

$$\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\beta}{\exp x - \exp y}\frac{\partial u}{\partial y} + \frac{\alpha}{\exp x - \exp y}\frac{\partial u}{\partial y} = 0 \quad (1.1)$$

by using the generalized fractional calculus Saigo, M. [5,6]. Similar work on fractional operators did time to time by eminent mathematicians like Saxena and Sethi [8], Bajpai and Gaur [2] and Gakhov [1].

The present section is intended to solve a problem for the equation (1.1) in the domain [(x, y) | 0 < x < y < 1] assuming the value of the solution of the noncharacteristic boundary y = x and the value of a linear combination of the generalized fractional integrals or derivatives of the solution on two characteristic segments x = 0 and y = 1. Such a problem has been discussed by A.M. Nahusev, H.G. Bzihatlov, A.V. Bicalze, T.I. Lanina, S.K. Kumykova, V.A. Eleev, M.M. Smirnov, R.K. Saxena, P.L. Sethi etc. As in their problems the calculus is of the sense of Riemann-Liouville of fixed orders, in our problem the calculus is of the sense of Riemann-Liouville of fixed orders, in our problem the calculus is of the generalized sense and orders may be chosen between some positive and negative number depending on α and β.

In section (2) the problem is formulated and solved, where some of calculations will be left to sections (3). The investigation in section (2) requires various formulas of the Gauss hyper-geometric function and its properties.

2. Formulation of Problem and Its Solution

In this section, we set up our main problem and its conclusion leaving the details in the later sections. Consider the Euler-Darboux Equation (1.1) in the triangular domain Ω=OAB, where O = (0, 0), A = (0, 1) and B = (1,1). A solution u (x, y) of (1.1) under the conditions

$$u\Big|_{y=x} = \tau(x), (\exp y - \exp x)^{\alpha+\beta}\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right)\Bigg|_{y=x} = \nu(x)$$

is given in the form Darboux, (1972).

$$u(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \tau[\exp x(\exp y - \exp x)\exp t]$$

$$(\exp t)^{\beta-1}(1-\exp t)^{\alpha-1}dt + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)\,\Gamma(1-\beta)}(\exp y - \exp x)^{1-\alpha-\beta}$$

$$\int_0^1 \nu[\exp x + (\exp y - \exp x)\exp t](\exp t)^{-\alpha}(1-\exp t)^{-\beta}dt$$

Hence, it is easily seen that the values of u on the characteristics OA and AB are written as follows:

$$u^{(1)}(y) \equiv u(0,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}I_{oy}^{\alpha,0,\beta-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)}I_{oy}^{1-\beta,\alpha+\beta-1,\beta-1}\nu \quad 0<y<1 \quad (2.1)$$

and

$$u^{(2)}(x) \equiv u(x,1) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}J_{x1}^{\beta,0,\alpha-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\beta)}J_{x1}^{1-\alpha,\alpha+\beta-1,\alpha-1}\nu \quad 0<x<1 \quad (2.2)$$

respectively, by means of the generalized fractional integrals.

Let $H^k(T)$ be a class of Hölder continuous functions on a real interval T with the Hölder index k. Denote the open interval (0, 1) by U and its closure by $\overline{U}$. Now let us state our main problem.

Problem. Find a solution u(x, y) of (1.1) in Ω satisfying the boundary conditions

$$u(x,x) = \varphi_1(x), x \in \overline{U} \quad (2.3)$$

And

$$A I_{ox}^{a,b,-a+\beta-1} u^{(1)} + B J_{x1}^{a+\alpha-\beta,c,-a+\beta-1} u^{(2)} = \varphi_2(x), x \in U, \tag{2.4}$$

and having the properties that $\nu(x) \in H^k(U)$ for some $k(0<k<1)$ and $\nu(x)$ may have infinities of integrable order of end points of U, where A and B are non zero constants, a, b and c are constants such that $-\alpha<a<\beta$, $-\alpha<a+b<1-a$ and $-\alpha<a+c<\beta$, and $\varphi_1 \in H^{k_1}(\bar{U})$ and $\varphi_2 \in H^{k_2}(\bar{U})$ are given functions with $1-a-\alpha>k_1>\max(1-\alpha-\beta, c)$, $1>k_2>a-\beta+1$ and $\phi(1)=0$.

Remark. Nahusev's result [4] in the case that A and B are given function of x, $\alpha = \beta$ and $a = -b = -c = \alpha - 1$.

Substituting (2.1) and (2.2) into (2.4) and using product rules of generalized fractional calculus, we have the relation :

$$\frac{A\Gamma(\alpha+\beta)}{\Gamma(\beta)} I_{ox}^{a+\alpha,b,-a+\beta+1}\tau + \frac{A\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)} J_{ox}^{a-\beta+1,b+\alpha+\beta-1,a+\beta-1}\nu + \frac{B\Gamma(\alpha+\beta)}{\Gamma(\alpha)} J_{x1}^{a+\alpha,c,-a+\beta-1}\tau + \frac{B\Gamma(1-\alpha-\beta)}{2\Gamma(1-\beta)} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta+1}\nu \tag{2.5}$$

Then by operating $\left(J_{0x}^{a-\beta+1,b+\alpha+\beta-1,-a+\beta-1}\right)^{-1} = I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0}$ on both sides of (2.5) and using fractional integration, conditions (2.3) and (2.4) can be unified in the form:

$$\frac{A\Gamma(1-\beta)}{B\Gamma(1-\alpha)}\nu(x) + I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1}\nu = -\frac{2A\Gamma(1-\beta)\Gamma(\alpha+\beta)}{B\Gamma(\beta)\Gamma(1-\alpha-\beta)} R_{0x}^{\alpha+\beta-1}\varphi_1 - \frac{2\Gamma(1-\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a+\alpha,c,-a+\beta-1}\varphi_1 + \frac{2\Gamma(1-\beta)}{B\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0}\varphi_2 \tag{2.6}$$

Thus our problem is reduced to find $\nu(x)$ from (2.6) so let us investigate the equation (2.6).

The second term on the L.H.S. of (2.6) may be written as :

$$I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1}\nu = \frac{\sin\pi(a-\beta)}{\pi}(\exp x)^{b+\alpha+\beta-1}\int_0^1 (\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp u-\exp x)^{-1}\nu(u)\,du - \cos\pi(a-\beta)(\exp x)^{a+b+\alpha}(1-\exp x)^{-a-c-\alpha}\nu(x) \tag{2.7}$$

Which will be derived in the next section. If we set

$$\mu(x) = (\exp x)^{a-\beta+1}(1-\exp x)^{-a-c-\alpha}\nu(x) \tag{2.8}$$

and use the relation (2.7), we find that (2.6) is reduced to the dominate singular integral equation for $\mu(x)$ having the Cauchy kernel.

$$P(x)\mu(x) + \int_0^1 \frac{\mu(u)}{\exp u - \exp x}du = Q(x) \qquad 0<x<1 \tag{2.9}$$

where

$$P(x) = \pi\cos\pi(a-\beta) - \frac{\pi}{\sin\pi(a-\beta)}\frac{A\Gamma(1-\beta)}{B\Gamma(1-\alpha)}(\exp x)^{-a-b-\alpha}(1-\exp x)^{a+c+\alpha},$$

$$Q(x) = \frac{\pi}{\sin\pi(a-\beta)}(\exp x)^{b-\alpha-\beta+1}\left[\frac{2A\Gamma(1-\beta)\Gamma(\alpha+\beta)}{B\Gamma(\beta)\Gamma(1-\alpha-\beta)} R_{0x}^{\alpha+\beta-1}\varphi_1 + \frac{2\Gamma(1-\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a+\alpha,c,-a+\beta-1}\varphi_1 - \frac{2\Gamma(1-\beta)}{B\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0}\varphi_2\right]$$

Let us solve the integral equation (2.9) by applying the theory in Gakhov [1], whose results have been summarized in Saxena and Sethi [7].

In order to guarantee the Hölder continuity of coefficients, we multiply the both sides of the equation (2.9) by $(\exp x)^{a+b+\alpha}$ and set

$$a(x) = (\exp x)^{a+b+\alpha} P(x), b(x) = \pi i(\exp x)^{a+b+\alpha} \tag{2.10}$$

and $f(x) = (\exp x)^{a+b+\alpha} Q(x)$

The continuity of a(x) and b(x) is evident and that of $f(x)$ can be shown obviously, consider the function $G(x) = \frac{a(x)-b(x)}{a(x)+b(x)}$. Then we obtain the values

$G(+0) = 1$, $G(1-0) = \exp[2\pi(-a+\beta)i]$

Thus, $\theta = \arg G(+0) = 0$. If we assume Δ, the change of arg G(x) on $\bar{U}$, to be equal to $2\pi(-a+\beta-1)$ then the index k of the equation (2.9) is equal to zero in a class of solutions h(1), whose function are Hölder continuous in U, bounded at x = 0 and unbounded but having integrable singularities at x=1.

Therefore we obtain a solution of the equation (2.9) in the class h(1) represented by

$$\mu(x) = \frac{P(x)Q(x)}{P^2(x)+\pi^2} - \frac{Z(x)}{[P^2(x)+\pi^2]^{1/2}}\int_0^1 \frac{Q(u)}{Z(u)[P^2(u)+\pi^2]^{1/2}} \cdot \frac{1}{\exp u - \exp x}du \tag{2.11}$$

where

$$Z(x) = \exp\left[\frac{1}{2\pi i}\int_0^1 \frac{1}{\exp u - \exp x}\log\frac{P(u)-\pi i}{P(u)+\pi i}du\right] \tag{2.12}$$

Here the branch of the logarithm in (2.12) should be selected such that the value at u=1 is equal to $2\pi(-a+\beta-1)$ i. Thus by (2.8), $\nu(x)$ in the relation (2.6) satisfying the required conditions is determined, and then our problem is solved.

3. Derivation of (2.7).

Owing to the definitions and the fractional integrals and derivatives, we may proceed as follows :

$$I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0}\; J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1}\,\nu$$

$$= \frac{d}{dx}\; I_{0x}^{-a+\beta-1,-b-\alpha-\beta,-1}\; J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1}\,\nu \tag{3.1}$$

$$= \frac{1}{\Gamma(-a+\beta)\Gamma(a-\beta+1)}\frac{d}{dx}\Big[(\exp x)^{a+b+\alpha}$$

$$\int_0^x (\exp x-\exp t)^{-a+\beta-1} F\left(-a-b-\alpha,1;-a+\beta;1-\frac{\exp t}{\exp x}\right)$$

$$\int_t^1 (\exp u-\exp t)^{a-\beta}(1-\exp u)^{-a-c-\alpha}\,\nu(u)\,du\,dt\Big]$$

$$= \frac{1}{\Gamma(-a+\beta)\Gamma(a-\beta+1)}\frac{d}{dx}\Big[(\exp x)^{a+b+\alpha}\left(\int_0^x\int_0^u+\int_x^1\int_0^x\right)$$

$$(\exp x-\exp t)^{-a+\beta-1}.(\exp u-\exp t)^{a-\beta}$$

$$(1-\exp u)^{-a-c-\alpha} F\left(-a-b-\alpha;1;-a+\beta;1-\frac{\exp t}{\exp x}\right)\nu(u)\,dt\,du\Big]$$

$$\equiv \frac{d}{dx}\left[\int_0^x \Xi_1(x,u)\nu(u)\,du+\int_x^1 \Xi_2(x,u)\nu(u)\,du\right]$$

First we shall treat the integral $\Xi_1(x,u)$. Use of fractional integral and derivatives, we get

$$F\left(-a-b-\alpha;1;-a+\beta;1-\frac{\exp t}{\exp x}\right)=\frac{\Gamma(-a+\beta)}{\Gamma(b+\alpha+\beta)}$$

$$.\frac{\Gamma(b+\alpha+\beta-1)}{\Gamma(-a+\beta-1)}\left(1-\frac{\exp t}{\exp x}\right)^{a-\beta+1}$$

$$.F\left(a-\beta+2,-b-\alpha-\beta+1;-b-\alpha-\beta+2;\frac{\exp t}{\exp x}\right)$$

$$+\frac{\Gamma(-a+\beta)\Gamma(-b-\alpha-\beta+1)}{\Gamma(-a-b-\alpha)}\left(\frac{\exp t}{\exp x}\right)^{b+\alpha+\beta-1}\left(1-\frac{\exp t}{\exp x}\right)^{a-\beta+1}$$

Again by application of fractional integral & derivatives

$$\Xi_1(x,u)=\frac{\Gamma(b+\alpha+\beta-1)}{\Gamma(b+\alpha+\beta)\Gamma(a-\beta+2)\Gamma(a+\beta-1)}$$

$$(\exp x)^{b+\alpha+\beta-1}(\exp u)^{a-\beta+1}.(1-\exp u)^{-a-c-\alpha}$$

$$F\left(1,-b-\alpha-\beta+1;-b-\alpha-\beta+2;\frac{\exp u}{\exp x}\right)$$

$$+\frac{\Gamma(b+\alpha+\beta)\Gamma(-b-\alpha-\beta+1)}{\Gamma(a+b+\alpha+1)\Gamma(-a-b-\alpha)}(\exp u)^{a+b+\alpha}.(1-\exp u)^{-a-c-\alpha} \tag{3.2}$$

By virtue of the fractional integral and derivatives it is easy to see that $\Xi_1(x,u)$ has a logarithmic singularity at u=x.

As regards $\Xi_2(x,u)$, we can obtain :

$$\Xi_2(x,u)=\frac{1}{\Gamma(-a+\beta)\Gamma(a-\beta+1)}$$

$$(\exp x)^{b+\alpha+\beta}(1-\exp u)^{-a-c-\alpha}(\exp u-\exp x)^{\alpha-\beta}.\int_0^1 \upsilon^{-a+\beta-1}$$

$$\left(1-\frac{\exp x}{\exp x-\exp u}\upsilon\right)^{\alpha-\beta} F(-a,-b-\alpha,1-a+\beta;\exp\upsilon)\,d\upsilon \tag{3.3}$$

$$=\frac{\Gamma(b+\alpha+\beta)}{\Gamma(-a+\beta)\Gamma(a-\beta-1)\Gamma(b+\alpha+\beta+1)}$$

$$(\exp x)^{b+\alpha+\beta}(\exp u)^{a-\beta}.(1-\exp u)^{-a-c-\alpha}$$

$$F\left(1,b+\alpha+\beta;b+\alpha+\beta+1;\frac{\exp x}{\exp u}\right)$$

which has also a logarithmic singularity at u = x.

In order to compute the R.H.S. of (3.1), we have to evaluate its principal value, since the functions $\Xi_1(x,u)$ and $\Xi_2(x,u)$ have logarithmic singularities at u = x. Then we shall begin to calculate the expression

$$\psi(x;\rho)\equiv\frac{d}{dx}\left[\int_0^{x-\rho}\Xi_1(x,u)\nu(u)\,du+\int_{x+\rho}^1\Xi_2(x,u)\nu(u)\,du\right]$$

for sufficiently small ρ.

Making use of the fractional integral & derivatives formulas, we obtain

$$\frac{d}{dx}\,x^{b+\alpha+\beta-1}\int_0^{x-\rho}(\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}$$

$$F\left(1,-b-\alpha-\beta+1;-b-\alpha-\beta+2;\frac{\exp u}{\exp x}\right)\nu(u)\,du$$

$$=(\exp x)^{b+\alpha+\beta-1}(\exp x-\rho)^{a-\beta+1}(1-\exp x+\rho)^{-a-c-\alpha}$$

$$F\left(1,-b-\alpha-\beta+1;-b-\alpha-\beta+2;\frac{\exp x-\rho}{\exp x}\right)\nu(\exp x-\rho)$$

$$+(b+\alpha+\beta-1)\,(\exp x)^{b+\alpha+\beta-1}$$

$$\int_0^{x-\rho}(\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp x-\exp u)^{-1}\nu(u)du.$$

Then we have

$$\frac{d}{dx}\int_0^{x-\rho}\Xi_1(x,u)\nu(u)du=\frac{\Gamma(b+\alpha+\beta-1)}{\Gamma(b+\alpha+\beta)\Gamma(a-\beta+2)\Gamma(-a+\beta-1)}$$

$$(\exp x)^{b+\alpha+\beta-1}(\exp x-\rho)^{a-\beta+1}$$

$$(1-\exp x+\rho)^{-a-c-\alpha} F\left(1,-b-\alpha-\beta+1;-b-\alpha-\beta+2;1-\frac{\rho}{\exp x}\right)$$

$$\nu(\exp x-\rho)+\frac{(\exp x)^{b+\alpha+\beta-1}}{\Gamma(a-\beta+2)\Gamma(-a+\beta-1)}$$

$$\int_0^{x-\rho}(\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp x-\exp u)^{-1}\nu(u)du.$$

$$+\frac{\Gamma(-b-\alpha-\beta+1)\Gamma(b+\alpha+\beta)}{\Gamma(a+b+\alpha+1)\Gamma(-a-b-\alpha)}$$

$$(\exp x-\rho)^{a+b+\alpha}\ (1-\exp x+\rho)^{-a-c-\alpha}\ \nu(\exp x-\rho) \tag{3.4}$$

Similarly it follows that

$$\frac{d}{dx}\int_{x+\rho}^{1}\Xi_2(x,u)\nu(u)du=\frac{-\Gamma(b+\alpha+\beta)}{\Gamma(-\alpha+\beta)\Gamma(a-\beta+1)\Gamma(b+\alpha+\beta+1)}$$
$$(\exp x)^{b+\alpha+\beta}(\exp x+\rho)^{a-\beta}(1-\exp x-\exp\rho)^{-a-c-\alpha}$$
$$F\left(1,b+\alpha+\beta;b+\alpha+\beta+1;\frac{\exp x}{\exp x+\rho}\right)\ \nu(\exp x+\rho)$$
$$+\frac{(\exp x)^{b+\alpha+\beta+1}}{\Gamma(a-\beta+1)\Gamma(-\alpha+\beta)}$$
$$\int_{x+\rho}^{1}(\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp u-\exp x)^{-1}\nu(u)\,du \tag{3.5}$$

From the relations (3.4) and (3.5) we obtain

$$\psi(x;\rho)=\frac{(\exp x)^{b+\alpha+\beta-1}}{\Gamma(a-\beta+1)\Gamma(-a+\beta)}\left[\frac{1}{-b-\alpha-\beta+1}(\exp x-\rho)^{a-\beta+1}\right.$$
$$(1-\exp x+\rho)^{-a-c-\alpha}\ .\ F\left(1,-b-\alpha-\beta+1;-b-\alpha-\beta+2;1-\frac{\rho}{\exp x}\right)$$
$$\nu(\exp x-\rho)-\frac{1}{b+\alpha+\beta}\exp x\,(\exp x+\rho)^{a-\beta}(1-\exp x-\rho)^{-a-c-\alpha}$$
$$\left.F\left(1,b+\alpha+\beta;b+\alpha+\beta+1;\frac{\exp x}{\exp x+\rho}\right)\nu(\exp x+\rho)\right]$$
$$+\frac{\Gamma(-b-\alpha-\beta+1)\Gamma(b+\alpha+\beta)}{\Gamma(a+b+\alpha+1)\Gamma(-a-b-\alpha)}(\exp x-\rho)^{a+b+\alpha}$$
$$(1-\exp x+\rho)^{-a-c-\alpha}\ \nu(\exp x-\rho)+\frac{(\exp x)^{b+\alpha+\beta-1}}{\Gamma(a-\beta+1)\Gamma(-a+\beta)}$$
$$\left[\int_0^{x-\rho}+\int_{x+\rho}^{1}\right](\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp u-\exp x)^{-1}\nu(u)\,du \tag{3.6}$$

By virtue of fractional integral and derivatives, the terms in the first brackets on the right hand side of (3.5) can be written in the form :

$$(\exp x-\rho)^{a-\beta+1}(1-\exp x+\rho)^{-a-c-\alpha}\sum_{n=0}^{\infty}\frac{(-b-\alpha-\beta+1)_n}{n!}$$
$$.\,[\psi(n+1)-\psi(-b-\alpha-\beta+1+n)-\log\rho+\log\exp x]\left(\frac{\rho}{\exp x}\right)^n$$
$$\exp\nu(\exp x-\rho)-\exp x(\exp x+\rho)^{a-\beta}(1-\exp x-\rho)^{-a-c-\alpha}$$
$$\sum_{n=0}^{\infty}\frac{(b+\alpha+\beta)_n}{n!}.\,[\psi(n+1)-\psi(b+\alpha+\beta+n)-\log\rho+\log(\exp x+\rho)]$$
$$\left(\frac{\rho}{\exp x+\rho}\right)^n\ \nu(\exp x+\rho),$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$. Hence, by letting $\rho\to 0$ in the relation (3.6), we obtain :

$$\lim_{\rho\to 0}\psi(x;\rho)=\frac{1}{\Gamma(a-\beta+1)\Gamma(-a+\beta+\alpha)}$$
$$[\psi(b+\alpha+\beta)-\psi(-b-\alpha-\beta+1)](\exp x)^{a+b+\alpha}$$
$$(1-\exp x)^{-a-c-\alpha}\ \nu(x)+\frac{\Gamma(-b-\alpha-\beta+1)\Gamma(b+\alpha+\beta)}{\Gamma(a+b+\alpha+1)\Gamma(-a-b-\alpha)}$$
$$.\,(\exp x)^{a+b+\alpha}(1-\exp x)^{-a-c-\alpha}\nu(x)+\frac{(\exp x)^{b+\alpha+\beta+1}}{\Gamma(a-\beta+1)\Gamma(-a+\beta)}$$
$$\int_0^1(\exp u)^{a-\beta+1}(1-\exp u)^{-a-c-\alpha}(\exp u-\exp x)^{-1}\nu(u)du \tag{3.7}$$

In view of the formulas $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi\exp z$ and $\psi(z)=\psi(1-z)-\pi\cot\pi\exp z$ Magnus, Oberhettinger and soni [3], the relation (2.7) can be derived.

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