Common Fixed Point Theorem in L-Space with Rational Contraction

Raghu Nandan Patel\(^1\), Damyanti Patel\(^2\)

\(^1\)Department of Mathematics, Government Mukut Dhar College, Katghora, (CG)
\(^2\)Department of Mathematics, Government Engineering College, Bilaspur, (CG)

Abstract: Many authors are prove several theorems in L-space, using various type of mappings. In this paper, we prove common fixed point theorem in L-space with rational contraction

Keywords: Fixed point, Common Fixed point, L-space, Continuous Mapping, Self Mapping, commutative Mappings

2000 Mathematics Subject Classification: 47H10, 54H25

1. Introduction

It was shown by S. Kasahara \([7]\) in 1976, that several known generalization of the Banach Contraction Theorem can be derived easily from a Fixed Point Theorem in an L-space. Iseki \([10]\) has used the fundamental idea of Kasahara to investigate the generalization of some known Fixed Point Theorem in L-space.

Let \(N\) be the set of natural numbers and \(X\) be a nonempty set. Then L-space is defined to be the pair \((X, \mathcal{L})\) of the set \(X\) and a subset \(\mathcal{L}\) of \(X\) × \(X\), satisfying the following conditions:

1. If \(x = y\) in \(X\), then \((x, y) \in \mathcal{L}\).
2. If \((x, y) \in \mathcal{L}\), then \((y, x) \in \mathcal{L}\).
3. If \((x, y) \in \mathcal{L}\) and \((y, z) \in \mathcal{L}\), then \((x, z) \in \mathcal{L}\).

For every subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) in \(X\), we write \(\{x_{n_i}\}\) converges to \(x\) if for each \(\varepsilon > 0\), there exists \(N_0 \in \mathbb{N}\) such that \(d(x_{n_i}, x) < \varepsilon\) for all \(i \geq N_0\).

Definition 1. Let \((X, \mathcal{L})\) be an L-space. It is said to be separated if each sequence in \(X\) converges to at most one point of \(X\).

Definition 2. A mapping \(f\) on \((X, \mathcal{L})\) into an L-space \((X', \mathcal{L}')\) is said to be 'continuous' if \(f(x) \in \mathcal{L}'\) for some subsequence \(\{x_{n_i}\}\) for \(\{x_n\}\).

\[
d(Ax, By) \leq \alpha d(Tx, Sy) + \beta d(Ty, Ax) + \gamma d(Tx, Bx) + \delta d(Sx, Ay)
\]

For all \(x, y \in X\), where non negative \(\alpha, \beta, \gamma, \delta\) such that \(0 < \alpha + 2\beta + 2\gamma + \delta < 1\), and \(0 < 2\beta + \gamma + \delta < 1\) with \(TX \neq SY\). Then \(A, B, S\) and \(T\) have unique common fixed point.

Proof: Let \(x_0 \in X\) be an arbitrary point. Then, since \(A(X) \subset T(X), B(X) \subset S(X)\), there exists \(x_1, x_2 \in X\) such that \(AX_0 = TX_1\) and \(BY_0 = SX_2\). Inductively, we construct the sequences \(\{y_{n}\}\) and \(\{x_{n}\}\) in \(X\) such that \(y_{2n} = Ax_{2n} = TX_{2n+1}\) and \(y_{2n+1} = Bx_{2n+1} = SX_{2n+2}\), for \(n = 0, 1, 2, \ldots\).

Now, by \([1.1]\), we have

Volume 5 Issue 9, September 2016

www.ijsr.net

Licensed Under Creative Commons Attribution CC BY
\begin{align*}
d(Ax_{2n}, Bx_{2n+1}) & \leq \alpha \left[ \frac{d(Tx_{2n}, Ax_{2n})}{d(Tx_{2n}, Bx_{2n})} + \frac{d(Sx_{2n+1}, Bx_{2n+1})}{d(Sx_{2n+1}, Ax_{2n})} \right] + \beta [d(Tx_{2n}, Ax_{2n}) + d(Sx_{2n+1}, Bx_{2n+1})] \\
& + \gamma [d(Tx_{2n}, Bx_{2n+1}) + d(Sx_{2n+1}, Ax_{2n})] + \delta d(Tx_{2n}, Sx_{2n+1}) \\
d(Ax_{2n}, Bx_{2n+1}) & \leq \alpha \left[ \frac{d(Ax_{2n-1}, Bx_{2n})}{d(Ax_{2n-1}, Bx_{2n})} + \frac{d(Bx_{2n-1}, Bx_{2n})}{d(Bx_{2n-1}, Ax_{2n})} \right] + \beta [d(Ax_{2n-1}, Ax_{2n}) + d(Bx_{2n}, Bx_{2n})] \\
& + \gamma [d(Ax_{2n-1}, Bx_{2n+1}) + d(Bx_{2n}, Ax_{2n})] + \delta d(Ax_{2n-1}, Bx_{2n+1}) \\
d(y_{2n}, y_{2n+1}) & \leq \alpha \left[ \frac{d(y_{2n-1}, y_{2n})}{d(y_{2n-1}, y_{2n})} + \frac{d(y_{2n+1}, y_{2n})}{d(y_{2n+1}, y_{2n})} \right] + \beta [d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n})] \\
& + \gamma [d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})] + \delta d(y_{2n-1}, y_{2n}) \\
d(y_{2n}, y_{2n+1}) & \leq \left[ \frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma} \right] d(y_{2n-1}, y_{2n})
\end{align*}

Where \( q = \left[ \frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma} \right] < 1 \) for \( n = 1, 2, 3, \ldots \) 

Similarly, we have 

d(y_{2n+1}, y_{2n}) \leq q^2 d(y_{2n-1}, y_{2n})

for every positive integer \( n \), this means 

\[
\sum_{n=0}^{\infty} d(y_{2n+1}, y_{2n}) < \infty
\]

Thus the \( d \)-completeness of the space, the sequence \( \{y_n\} \) converges to some point \( u \) in \( X \). 

So by [1.2] and [1.2] sequences \( \{Ax_{2n}\}, \{Sx_{2n}\}, \{Tx_{2n+1}\} \) and \( \{Bx_{2n+1}\} \) also converges to \( u \).

\[
d(Au, Bx_{2n+1}) \leq \alpha \left[ d(Tu, Sx_{2n+1}) + \frac{d(Tu, Ax_{2n}) + d(Sx_{2n+1}, Bx_{2n+1})}{d(Tu, Ax_{2n}) + d(Sx_{2n+1}, Bx_{2n+1})} \right] + \beta d(Tu, Ax_{2n}) + d(Sx_{2n+1}, Bx_{2n+1}) \\
+ \gamma d(Tu, Bx_{2n+1}) + d(Sx_{2n+1}, Ax_{2n}) + \delta d(Tu, Sx_{2n+1})
\]

\[
d(Au, u) \leq \alpha \left[ d(Tu, u) + \frac{d(Tu, Au) + d(Sx_{2n+1}, Bx_{2n+1})}{d(Tu, Au) + d(Sx_{2n+1}, Bx_{2n+1})} \right] + \beta d(Tu, Au) + d(Sx_{2n+1}, Bx_{2n+1}) \\
+ \gamma d(Tu, Bx_{2n+1}) + d(Sx_{2n+1}, Ax_{2n}) + \delta d(Tu, Sx_{2n+1})
\]

\[
d(Au, u) \leq (2\gamma + \delta) d(Au, u)
\]

Which is contradiction. Hence [1.5] \( Au = u \)

From [1.3] and [1.5] we get \( Au = Tu = u \) Which is contradiction. Hence [1.5] \( Au = u \)

Similarly setting \( x = x_{2n} \) and \( y = u \) in contractive condition [1.2], then

\[
d(u, w) = d(Au, Bw) \leq \alpha \left[ d(Tu, Sw) + \frac{d(Tu, Au) + d(Sw, Bw)}{d(Tu, Au) + d(Sw, Bw)} \right] + \beta d(Tu, Au) + d(Sw, Bw) \\
+ \gamma d(Tu, Bw) + d(Sw, Au) + \delta d(Tu, Sw)
\]

This implies that [1.6] \( Bu = u \). From [1.3] and [1.6] we get \( Bu = Su = u \). Therefore, we get \( u = Au = Bu = Su = Tu \). Hence \( u \) is a common fixed point of \( A, B, S \) and \( T \).

Uniqueness

The uniqueness of a common fixed point of the mappings \( A, B, S \) and \( T \) be easily verified by using [1.2]. In fact, if \( w \) be another fixed point for mappings \( A, B, S \) and \( T \). Then, we have

\[
d(u, w) = d(Au, Bw) \leq \alpha \left[ d(Tu, Sw) + \frac{d(Tu, Au) + d(Sw, Bw)}{d(Tu, Au) + d(Sw, Bw)} \right] + \beta d(Tu, Au) + d(Sw, Bw) \\
+ \gamma d(Tu, Bw) + d(Sw, Au) + \delta d(Tu, Sw)
\]
\[ d(u, w) \leq (2\gamma + \delta)d(u, w) \]

Which is contradiction. Hence \( u = v \).

Hence \( u \) is a unique common fixed point of \( A, B, S, T \) in \( X \).

This complete the proof of the theorem.

References


Author Profile

Raghu Nandan Patel is in Department of Mathematics, Government Mukut Dhar College, Katghora, (CG)

Damyanti Patel is in Department of Mathematics, Government Engineering College, Bilaspur, (CG)