Common Fixed Point Theorem in L-Space with Rational Contraction

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Abstract: ***Many authors are prove several theorems in L-space, using various type of mappings. In this paper, we prove common fixed point theorem in L-Space with rational contraction***

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1. Introduction

It was shown by S. Kasahara [7] in 1976, that several known generalization of the Banach Contraction Theorem can be derived easily from a Fixed Point Theorem in an L-space. Iseki [10] has used the fundamental idea of Kasahara to investigate the generalization of some known Fixed Point Theorem in L-space.

Let N be the set of natural numbers and X be a nonempty set. Then L-space is defined to be the pair (X, $\rightarrow$) of the set X and a subset $\rightarrow$ of the set $X^N \times X$, satisfying the following conditions:

L1 . if $x_n = x \in X$ for all n□ N, then $(\{x_n\}_{n\in N}, x) \in \rightarrow$

L2 if $(\{x_n\}_{n\in N}, x) \in \rightarrow$, then $(\{x_{ni}\}_{i\in N}$

For every subsequence $\{x_{ni}\}_{i\in N}$ of $\{x_n\}_{n\in N}$

In what follows instead of writing $(\{x_n\})_{n\in N}, x) \in \rightarrow$,we write $\{x_n\}_{n\in N} \rightarrow x$ or $x_n \rightarrow x$ and read $\{x_n\}_{n\in N}$ converges to x. Further we give some definitions regarding L-space.

Definition 1. Let (X, $\rightarrow$) be an L-space. It is said to be .separated. if each sequence in x converges to at most one point of X.

Definition 2. A mapping f on (X, $\rightarrow$) into an L-space (X', $\rightarrow$')is said to be 'continuous' if $x_n \rightarrow x$ implies $f(x_n) \rightarrow' f(x)$ for some subsequence $\{x_n\}_{i\in N}$ for $\{x_n\}_{n\in N}$.

Definition 3. Let d- be a non negative extended real valued function on X × X: $0 \le d(x, y) \le \infty_i$ for all x, y $\in$ X. The L-space is said to be d- complete if each sequence $\{x_n\}_{n\in N}$ in X with $\sum_{i=0}^{\infty} d(x_i, x_{i+1}) < \infty$ converges to the at most one point of X.

In this context Kasahara S. proved a lemma, which as follows:

Lemma (S. Kasahara):

Let (X, $\rightarrow$) be an L-space which is d- complete for a non negative real valued function d on X × X. If (X, $\rightarrow$) is separated then:

$d(x, y) = d(y, x) = 0$ implies, $x = y$ for all $x, y \in X$

During the past few years many great mathematicians Yeh [13], Singh [12], Pathak and Dubey [8], Sharma and Agrawal [11], Patel, Sahu and Sao [9], Patel and Patel [10], worked for L-space. In this chapter, we similar investigation for the study of Fixed Point Theorems in L-space are worked out. We find Common Fixed Point Theorem in L-space with rational contraction

Theorem 1:

Let *(X, →)* be a separated L-space, which is *d*- complete for a non negative extended real valued function *d* on $X \times X$ with $d(x, x) = 0$, for each *x* in X. Let *A, B, S* and *T* be continuous self mapping satisfying:

[1.1] : $A(X) \subset T(X)$ & $B(X) \subset S(X)$, and $AS = SA$, $BT = ST$

[1.2] :

$$d(Ax, By) \le \alpha\left[d(Tx, Sy)\frac{d(Tx, Ax) + d(Sy, By)}{d(Tx, By) + d(Sy, Ax)}\right] + \beta\left[d(Tx, Ax) + d(Sy, By)\right]$$
$$+ \gamma\left[d(Tx, By) + d(Sy, Ax)\right] + \delta d(Tx, Sy)$$

For all *x, y* in *X*, where non negative *α, β, γ, δ* such that $0 < \alpha + 2\beta + 2\gamma + \delta < 1$, and $0 < 2\gamma + \delta < 1$ with $Tx \ne Sy$. Then *A, B, S* and *T* have unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point. Then, since $A(X) \subset T(X)$, $B(X) \subset S(X)$, there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, ...$

Now, by [1.1], we have

$$d(Ax_{2n},Bx_{2n+1})\le\alpha\left[d(Tx_{2n},Sx_{2n+1})\frac{d(Tx_{2n},Ax_{2n})+d(Sx_{2n+1},Bx_{2n+1})}{d(Tx_{2n},Bx_{2n+1})+d(Sx_{2n+1},Ax_{2n})}\right]+\beta[d(Tx_{2n},Ax_{2n})+d(Sx_{2n+1},Bx_{2n+1})]$$
$$+\gamma[d(Tx_{2n},Bx_{2n+1})+d(Sx_{2n+1},Ax_{2n})]+\delta d(Tx_{2n},Sx_{2n+1})$$

$$d(Ax_{2n},Bx_{2n+1})\le\alpha\left[d(Ax_{2n-1},Bx_{2n})\frac{d(Ax_{2n-1},Ax_{2n})+d(Bx_{2n},Bx_{2n+1})}{d(Ax_{2n-1},Bx_{2n+1})+d(Bx_{2n},Ax_{2n})}\right]+\beta[d(Ax_{2n-1},Ax_{2n})+d(Bx_{2n},Bx_{2n+1})]$$
$$+\gamma[d(Ax_{2n-1},Bx_{2n+1})+d(Bx_{2n},Ax_{2n})]+\delta d(Ax_{2n-1},Bx_{2n})$$

$$d(y_{2n},y_{2n+1})\le\alpha\left[d(y_{2n-1},y_{2n})\frac{d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})}{d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})}\right]+\beta[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})]$$
$$+\gamma[d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})]+\delta d(y_{2n-1},y_{2n})$$

$$d(y_{2n},y_{2n+1})\le\left[\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma}\right][d(y_{2n-1},y_{2n})]$$

$d(y_{2n}, y_{2n+1}) \le q\, d(y_{2n-1}, y_{2n})$

Where $q=\left[\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma}\right]<1$ for n = 1, 2, 3............

Similarly, we have

$d(y_{2n+1}, y_{2n+2}) \le q^n\, d(y_0, y_1)$

for every positive integer n, this means

$\sum_{i=0}^{\infty} d(y_{2i+1},y_{2i+2})<\infty$

Thus the d – completeness of the space, the sequence $\{y_n\}$ converges to some point u in X

So by [1.2] and [1.2] sequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converges to u.

Since A, B, S and T be continuous, there is a subsequence t of $\{y_n\}$ such that

$A(T(t)) \to A(u)$, $T(A(u)) \to T(u)$, $B(S(t)) \to B(u)$ and $S(B(t)) \to S(u)$

By [1.1], we get

[1.3] $A(u) = T(u)$ and $B(u) = S(u)$

Thus we can write

[1.4] $T(T(u)) = T(A(u)) = A(T(u)) = A(A(u))$ and $S(S(u)) = S(B(u)) = B(S(u)) = B(B(u))$

We claim that $Au = u$. For this, suppose that $Au \ne u$.

Then, setting $x = u$ and $y = x_{2n+1}$ in contractive condition By [1.2], [1.3], and [1.4] we have,

$$d(Au,Bx_{2n+1})\le\alpha\left[d(Tu,Sx_{2n+1})\frac{d(Tu,Au)+d(Sx_{2n+1},Bx_{2n+1})}{d(Tu,Bx_{2n+1})+d(Sx_{2n+1},Au)}\right]+\beta[d(Tu,Au)+d(Sx_{2n+1},Bx_{2n+1})]$$
$$+\gamma[d(Tu,Bx_{2n+1})+d(Sx_{2n+1},Au)]+\delta d(Tu,Sx_{2n+1})$$
$$d(Au,u)\le\alpha\left[d(Tu,u)\frac{d(Tu,Au)+d(u,u)}{d(Tu,u)+d(u,Au)}\right]+\beta[d(Tu,Au)+d(u,u)]$$
$$+\gamma[d(Tu,u)+d(u,Au)]+\delta d(Tu,u)$$

$$d(Au,u)\le(2\gamma+\delta)d(Au,u)$$

Which is contradiction. Hence [1.5] $Au = u$

From [1.3] and [1.5] we get $Au = Tu = u$

Which is contradiction. Hence [1.5] $Au = u$

Similarly setting $x = x_{2n}$ and $y = u$ in contractive condition [1.2], then

This implies that [1.6] $Bu = u$.

From [1.3] and [1.6] we get $Bu = Su = u$. Therefore , we get $u = Au = Bu = Su = Tu$. Hence u is a common fixed point of A, B, S and T.

Uniqueness

The uniqueness of a common fixed point of the mappings A, B, S and T be easily verified by using [1.2]. In fact, if w be another fixed point for mappings A, B, S and T. Then, , we have

$$d(u,w)=d(Au,Bw)\le\alpha\left[d(Tu,Sw)\frac{d(Tu,Au)+d(Sw,Bw)}{d(Tu,Bw)+d(Sw,Au)}\right]+\beta[d(Tu,Au)+d(Sw,Bw)]$$
$$+\gamma[d(Tu,Bw)+d(Sw,Au)]+\delta d(Tu,Sw)$$

$$d(u,w) \le (2\gamma + \delta) d(u,w)$$

Which is contradiction. Hence $u = v$.

Hence u is a unique common fixed point of A, B, S, T in X. This complete the proof of the theorem.

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