On P-ρ-Connected Space in a Topological Space

M. Priyadarshini[1], R. Selvi[2]

[1]Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, India
[2]Department of Mathematics, Sri Parasakthi College for Women, Courtallam, India

Abstract: ***The authors introduced P-L-open sets, P-M-closed sets, P-R-open sets and P-S-closed sets and established their relationships with some generalized sets in topological spaces. Connected spaces constitute the most important classes of topological spaces. In this paper we introduce the concept "P- ρ -connected" in a topological space.***

Keywords: P-L-open sets, P-M-closed sets, P-R-open sets, P-S-closed sets, P-L-connected space, P-M- connected space, P-R- connected space, P-S- connected space, P-L-irresolute, P-M- irresolute, P-R-resolute, P-S-resolute

1. Introduction

In topology and related branches of mathematics a connected space is a topological spaces that cannot be represented as the union of two disjoint non empty open subsets. Connectedness is one of the principal topological properties that are used to distinguish topological spaces. In this paper we introduce P-ρ-connected spaces. A topological space X is said to be P-ρ-connected if X cannot be written as the disjoint union of two non-empty P-ρ-open (P-ρ-closed) sets in X.

2. Preliminaries

Throughout this paper $f^{-1}(f(A))$ is denoted by A^* and $f(f^{-1}(\mathrm{B}))$ is denoted by B^*.

Definition 2.1

Let A be a subset of a topological space (X, τ). Then A is called pre-open if A$\subseteq$int(cl(A)) and pre-closed if cl(int(A))$\subseteq$A; [6].

Definition 2.2

Let f: (X,τ)$\rightarrow$(Y,σ) be a function. Then f is pre-continuous if f^{-1}(B) is open in X for every pre-open set B in Y. [6]

Definition: 2.3

Let f: (X,τ) $\rightarrow$(Y,σ) be a function. Then f is pre-open (resp. pre-closed) if f(A) is pre-open(resp. pre-closed) in Y for every pre-open(resp. pre-closed) set A in X. [6]

Definition: 2.4

Let f: (X,τ)$\rightarrow$Y be a function. Then f is

1)p-L-Continuous if A^* is open in X for every pre-open set A in X.

2)p-M-Continuous if A^* is closed in X for every pre-closed set A in X. [10]

Definition: 2.5

Let f: X$\rightarrow$(Y,σ) be a function. Then f is

1)p-R-Continuous if B^* is open in Y for every pre-open set B in Y.

2)p-S-Continuous if B^* is closed in Y for every pre-closed set B in Y. [10]

Definition: 2.6

Let f: (X,τ) $\rightarrow$(Y,σ) be a function, then f is said to be

1)P-irresolute if $f^{-1}(V)$ is pre-open in X, whenever V is pre-open in Y.

2)P-resolute if $f(V)$ is pre-open in Y, whenever V is pre-open in X. [6]

Definition: 2.7

Let (X,τ) is said to be 1) finitely p-additive if finite union of pre-closed set is pre-closed. 2) Countably p-additive if countable union of pre-closed set is pre-closed. 3) p-additive if arbitrary union of pre-closed set is pre-closed. [6]

Definition: 2.8 Let (X,τ) be a topological space and x$\in$ X .Every pre-open set containing x is said to be a p-neighbourhood of x.[8]

Definition: 2.9 Let A be a subset of X. A point x$\in$X is said to be pre-limit point of A if every pre-neighbourhood of x contains a point of A other than x. [5]

Definition: 2.10
X=A$\cup$B is said to be a pre-separation of X if A and B are non empty disjoint and pre-open sets .If there is no pre-separation of X, then X is said to be p-connected. Otherwise it is said to be p-disconnected. [11]

3. P- ρ -Open(Closed) Sets

Definition: 3.1

Let f: $(X,\tau)\rightarrow Y$ be a function and A be a subset of a topological space (X, τ). Then A is called

1)P-L-open if $A^* \subseteq \text{int}(\text{cl}(A^*))$

2)P-M-closed if $A^* \supseteq \text{cl}(\text{int}(A^*))$

Definition: 3.2

Let f: $X\rightarrow(Y,\sigma)$ be a function and B be a subset of a topological space (Y,σ). Then B is called

1)P-R-open if $B^* \subseteq \text{int}(\text{cl}(B^*))$

2)P-S-closed if $B^* \supseteq \text{cl}(\text{int}(B^*))$

Example: 3.3

Let X = {a, b, c, d} and Y = {1, 2, 3, 4}. Let τ = { Φ, X ,{a}, {b}, {a, b}, {a, b, c} }. Let f: $(X,\tau)\rightarrow Y$ defined by f(a)=1, f(b)=2, f(c)=3, f(d)=4. Then f is p-L-open and p-M-Closed.

Example: 3.4

Let X = {a, b, c, d} and Y = {1, 2, 3, 4}. Let σ ={ Φ,Y, {1},{2},{1,2},{1,2,3} }. Let g : $X\rightarrow(Y,\sigma)$ defined by g(a)=1, g(b)=2, g(c)=3, g(d)=4. Then g is p-R-open and p-S-Closed.

Definition: 3.5

Let f: $(X,\tau)\rightarrow(Y,\sigma)$ be a function, then f is said to be

1)P-L-irresolute if $f^{-1}(f(A))$ is pre-L-open in X, whenever A is pre-L-open in X.
2)P-M-irresolute if $f^{-1}(f(A))$ is pre-M-open in X, whenever A is pre-M-open in X.
3)P-R-resolute if $f(f^{-1}(B))$ is pre-L-open in Y, whenever B is pre-R-open in Y.
4)P-S-resolute if $f(f^{-1}(B))$ is pre-M-open in Y, whenever B is pre-S-open in Y.

Definition: 3.6

Let (X,τ) is said to be 1) finitely p-L-additive if finite union of p-L--closed set is p-L--closed. 2) Countably p-L-additive if countable union of pre-L-closed set is pre-L-closed. 3) p-L-additive if arbitrary union of pre-L-closed set is pre-L-closed.

4. P- ρ -Connected Spaces

Definition: 4.1

X=A$\cup$B is said to be a p-L-separation of X if A and B are non empty disjoint and p-L-open sets .If there is no p-L-separation of X, then X is said to be p-L-connected. Otherwise it is said to be p-L-disconnected.

Definition: 4.2

X=A$\cup$B is said to be a p-M-separation of X if A and B are non empty disjoint and p-M-closed sets. If there is no p-M-separation of X, then X is said to be p-M-connected. Otherwise it is said to be p-M-disconnected.

Example: 4.3

Let X = {a, b, c, d} and Y = {1, 2, 3, 4}. Let τ = { Φ, X ,{a}, {b}, {a, b}, {a, b, c} }. Let f: $(X,\tau)\rightarrow Y$ defined by f(a)=1, f(b)=1, f(c)=2, f(d)=2. Then X is P-L-connected space and P-M-connected space.

Definition: 4.4

X=A$\cup$B is said to be a p-R-separation of X if A and B are non empty disjoint and p-R-open sets .If there is no p-R-separation of X, then X is said to be p-R-connected. Otherwise it is said to be p-R-disconnected.

Definition: 4.5

X=A$\cup$B is said to be a p-S-separation of X if A and B are non empty disjoint and p-S-closed sets. If there is no p-S-separation of X, then X is said to be p-S-connected. Otherwise it is said to be p-S-disconnected.

Example: 4.6

Let X = {a, b, c, d} and Y = {1, 2, 3, 4}. Let σ ={ Φ,Y, {1},{2},{1,2},{1,2,3} }. Let g : $X\rightarrow(Y,\sigma)$ defined by g(a)=1, g(b)=1, g(c)=2, g(d)=2. Then Y is p-R-connected space and p-S-connected space.

Note: 4.7

i) If X=A$\cup$B is a p-L-separation then A^C = B and B^C = A. Hence A and B are p-M-closed.
ii) If X=A$\cup$B is a p-M-separation then A^C = B and B^C = A. Hence A and B are p-L-open.
iii) If X=A$\cup$B is a p-R-separation then A^C = B and B^C = A. Hence A and B are p-S-closed.
iv) If X=A$\cup$B is a p-S-separation then A^C = B and B^C = A. Hence A and B are p-R-open.

Remark: 4.8

(X,τ) is p-L-connected (or) p-M-connected if and only if the only subsets which are both P-L-open and P-M-closed are X and ϕ.

Proof:
Let (X,τ) is p-L-connected space (or) p-M-connected space. Suppose that A is a proper subset which is both P-L-open and P-M-closed, then X= $A \cup A^c$ is a P-L-separation (P-M-separation) of X which is contradiction. Conversely, let ϕ be the only subsets which is both P-L-open and P-M-closed. Suppose X is not p-L-connected (not p-M-connected), then X=$A \cup B$ where A and B are non empty disjoint p-L-open (P-M-closed) subsets which is contradiction.

Example: 4.9

Any indiscrete topological space (X,τ) with more than one point is not P-L-connected (not P-M-connected). X= $\{a\} \cup \{a\}^c$ is a P-L-separation(P-M-separation), since every subset is P-L-open(P-M-closed).

Remark: 4.10

(Y,σ) is p-R-connected (or) p-S-connected if and only if the only subsets which are both P-R-open and P-S-closed are Y and ϕ.

Proof:

Let (Y,σ) is p-R-connected space (or) p-S-connected space. Suppose that A is a proper subset which is both P-R-open and P-S-closed, then Y= $A \cup A^c$ is a P-R-separation (P-S-separation) of Y which is contradiction. Conversely, let ϕ be the only subsets which is both P-R-open and P-S-closed. Suppose Y is not p-R-connected (not p-S-connected), then Y=$A \cup B$ where A and B are non empty disjoint p-R-open (P-S-closed) subsets which is contradiction.

Example: 4.11

Any indiscrete topological space (Y,σ) with more than one point is not P-R-connected (not P-S-connected). Y= $\{a\} \cup \{a\}^c$ is a P-R-separation(P-S-separation), since every subset is P-R-open(P-S-closed).

Theorem: 4.12

Every p-L-connected space is connected.

Proof:

Let X be an p-L-connected space. Suppose X is not connected. Then X= $A \cup B$ is a separation then it is a P-L-separation. Since every open set is P-L-open set in X. This is contradiction to fact that (X,τ) is p-L-connected. Therefore hence (X,τ) is connected.

Remark: 4.13

The converse of the theorem (4.12) is not true as seen from example (4.14).

Example: 4.14

Any indiscrete topological space (X,τ) with more than one point is not P-L-connected. Since every subset is P-L-open. But it is connected, since the only open sets are X and ϕ.

Theorem: 4.15

Every p-M-connected space is connected.

Proof:

Let X be an p-M-connected space. Suppose X is not connected. Then X= $A \cup B$ is a separation then it is a P-M-separation. Since every closed set is P-M-closed set in X. This is contradiction to fact that (X,τ) is p-M-connected. Therefore hence (X,τ) is connected.

Definition: 4.16

Let Y be a subset of X. Then Y= $A \cup B$ is said to be a P-L-separation (P-M-separation) of Y if A and B are non empty disjoint P-L-open (P-M-closed) sets in X. If there is no P-L-separation (P-M-separation) of Y then Y is said to be P-L-connected (P-M-connected) subset of X.

Theorem: 4.17

Every p-R-connected space is connected.

Proof:

Let Y be an p-R-connected space. Suppose X is not connected. Then Y= $A \cup B$ is a separation then it is a P-R-separation. Since every open set is P-R-open set in Y. This is contradiction to fact that (Y,σ) is p-R-connected. Therefore hence (Y,σ) is connected.

Theorem: 4.18

Every p-S-connected space is connected.

Proof:

Let Y be an p-S-connected space. Suppose X is not connected. Then Y= $A \cup B$ is a separation then it is a P-S-separation. Since every closed set is P-S-closed set in Y. This is contradiction to fact that (Y,σ) is p-S-connected. Therefore hence (Y,σ) is connected.

Definition: 4.19

Let Y be a subset of X. Then Y= A$\bigcup$B is said to be a P-R-separation (P-S-separation) of Y if A and B are non empty disjoint P-R-open (P-S-closed) sets in X. If there is no P-R-separation (P-S-separation) of Y then Y is said to be P-R-connected (P-S-connected) subset of X.

Theorem: 4.20 Let (X,τ) be a finitely P-L-additive topological space and X= A$\bigcup$B be a P-L-separation of X. if Y is P-L-open and P-L-connected subset of X, then Y is completely contained in either A or B.

Proof:

X= A$\bigcup$B is a P-L-separation of X. Suppose Y intersects both A and B then $Y=(A\bigcap Y)\bigcup(B\bigcap Y)$ is a P-L-separation of Y, Since X is finitely P-L-additive. This is a contradiction.

Theorem: 4.21 Let (X,τ) and (Y,σ) be bijection. Then

1)f is P-L-continuous(P-M-continuous) and X is P-L-connected(P-M-connected) $\Rightarrow$Y is connected.
2)f is continuous and X is P-L-connected(P-M-connected) $\Rightarrow$Y is connected.
3)f is P-L-open(P-M-closed) and Y is P-L-connected(P-M-connected) $\Rightarrow$ X is connected.
4)f is open then X is connected$\Rightarrow$ Y is P-L-connected(P-L-connected).
5)f is P-L-irresolute(P-M-irresolute) then X is P-L-connected(P-M-connected) $\Rightarrow$ Y is P-L-connected(P-M-connected).
6)f is P-R-resolute(P-S-resolute) then Y is P-R-connected(P-S-connected) $\Rightarrow$ X is P-R-connected(P-S-connected).

Proof:

1) Suppose Y= A$\bigcup$B is a separation of Y, then $X=f^{-1}(Y)=f^{-1}(A)\bigcup f^{-1}(B)$ is a P-L-separation of X which is contradiction. Therefore Y is connected. Proof for (2) to (7) is similar to the above proof.

Theorem: 4.22 A topological space (X,τ) is P-L-disconnected if and only if there is exists a P-L-continuous map of X onto discrete two point space Y = {0, 1}.

Proof:

Let (X,τ) be P-L-disconnected and Y = {0, 1} is a space with discrete topology. **X** = A$\cup$B be a P-L-separation of X. Define f: X$\rightarrow$Y such that f (A) = 0 and f(B) = 1. Obviously f is onto, P-L-continuous map. Conversely, let f: X$\rightarrow$Y be P-L-continuous onto map. Then X = $f^{1}(0)\cup f^{1}(1)$ is a P-L-separation of X.

Theorem: 4.23 Let (X,τ) be a finitely P-L-additive topological space. If {A$_\alpha$} is an arbitrary family of P-L-open P-L-connected subset of X with the common point p then $\cup$A$_\alpha$ is P-L-connected. **Proof:** Let $\cup$A$_\alpha$ = B$\cup$C be a P-L-separation of $\cup$ A$_\alpha$. Then B and C are disjoint non empty P-L-open sets in X. P$\in\cup$A$_\alpha$ $\Rightarrow$P$\in$B or P$\in$C. Assume that P$\in$B. Then by theorem (4.20) A$_\alpha$ is completely contained in B for all α (since P$\in$B). Therefore C is empty which is a contradiction.

Corollary: 4.24 Let (X,τ) be a finitely P-L-additive topological space. If {A_n} is a sequence of P-L-open P-L-connected subsets of X such that $A_n\cap A_{n+1}=\Phi$, for all n. Then $\cup A_n$ is P-L-connected. **Proof:** This can be proved by induction on n. By theorem (4.23), the result is true for n=2. Assume that the result to be true when n=k. Now to prove the result when n=k+1. By the hypothesis $\bigcup_{i=1}^{n}A_i$ is P-L-connected. Now $(\bigcup_{i=1}^{n}A_i)\bigcap A_{k+1}\neq\Phi$. Therefore by theorem (4.23) $\bigcup_{i=1}^{k+1}A_i$ is P-L-connected. By induction hypothesis the result is true for all n.

Corollary: 4.25 Let (X,τ) be a finitely P-L-additive topological space. Let $\{A_\alpha\}_{\alpha\in\Omega}$ be an arbitrary collection of P-L-open P-L-connected subsets of X. Let A be a P-L-open P-L-connected subsets of X. If A$\cap$A$_\alpha\neq\Phi$ for allα then A$\cup$($\cup$ A$_\alpha$) is P-L-connected. **Proof:** Suppose that A$\cup$($\cup$ A$_\alpha$) =B$\cup$C be a P-L-separation of the subset A$\cup$($\cup$ A$_\alpha$) since A$\subseteq$ B$\cup$C by theorem (4.23), A$\subseteq$B or A$\subseteq$C. Without loss of generality assume that A$\subseteq$B. Let $\alpha\in\Omega$ be arbitrary. A$_\alpha\subseteq$B$\cup$C, by theorem (4.20), A$_\alpha\subseteq$B or A$\subseteq$C. But A$\cap$A$_\alpha\neq\Phi\Rightarrow$ A$_\alpha\subseteq$ B. Since α is arbitrary, A$_\alpha$ $\subseteq$ B for all α. Therefore A$\cup$ ($\cup$ A$_\alpha$)$\subseteq$B which implies C=Φ. Which is a contradiction. Therefore A$\cup$($\cup$ A$_\alpha$) is P-L-connected.

Definition: 4.26 A space (X,τ) is said to be totally P-L-disconnected if its only P-L-connected subsets are one point sets.

Example: 4.27 Let (X,τ) be an indiscrete topological space with more than one point. Here all subsets are P-L-open. If A={X_1,X_2} then A={ X_1}$\cup${X_2} is P-L-separation of A. Therefore any subset with more than one point is P-L-disconnected. Hence (X,τ) is totally P-L-disconnected

Remark: 4.28 Totally P-L-disconnectedness implies P-L-disconnectedness.

Definition: 4.29 In a topological space (X, τ) a point x$\in$X is said to be in P-L-boundary of A, (P-L-Bd(A)) if every P-L-open set containing x intersects both A and X-A.

Theorem: 4.30 Let (X, τ) be a finitely P-L-additive topological space and let A be a subset of X. If C is P-L-open P-L-connected subset of X that intersects both A and X-A then C intersects P-L-Bd(A).

Proof: It is given that $C \cap A \neq \Phi$ and $C \cap A^c \neq \Phi$. Now $C = (C \cap A) \cup (C \cap A^c)$ is a non empty disjoint union. Suppose both are P-L-open then it is a contradiction to the fact that C is P-L-connected. Hence either $C \cap A$ or $C \cap A^c$ is not P-L-open. Suppose that $C \cap A$ is not P-L-open. Then there exists $x \in C \cap A$ which is not a P-L-interior point of $C \cap A$. Let U be a P-L-open set containing x. Then $U \cap C$ is a P-L-open set containing x. Hence $(U \cap C) \cap (C \cap A^c) \neq \Phi$. This implies U intersects both A and A^c and therefore $x \in$ P-L-Bd(A). Hence $C \cap$ P-L-Bd(A) $\neq \Phi$.

Theorem: 4.31 (Generalisation of intermediate Value theorem)
Let f: $X \to R$ be a P-L-continuous map where X is a P-L-connected space and R with usual topology. If x, y are two points of X, a=f(x) and b=f(y) then every real number r between a and b is attained at a point in X.

Proof: Assume the hypothesis of the theorem. Suppose there is no point $c \in X$ such that f(c) =r. Then A= $(-\infty, r)$ and B= (r, ∞) are disjoint open sets in R, since f is P-L-continuous, f^1 (B) are P-L-open in X. $X = f^{-1}(A) \cup f^{-1}(B)$ which is a P-L-separation of X. This is a contradiction to the fact that X is P-L-connected. Therefore there exists $c \in X$ such that f(c)= r.

Definition 4.32
Let X be a topological space and $A \subseteq X$. The P-M-closure of A is defined as the intersection of all P-M-closed sets in X containing A, and is denoted by P-M-(cl(A)). It is clear that P-M-(cl(A)) is P-M-closed set for any subset A of X.

Proposition 4.33 Let X be a topological space and $A \subseteq B \subseteq X$.

Then:

(i) P-M-(cl(A)) $\subseteq$ P-M-(cl(B))
(ii) $A \subseteq$ P-M-(cl(A))
(iii) A is P-M-closed iff A= P-M-(cl(A))

Definition 4.34

Let X be a topological space and $x \in X$, $A \subseteq X$. The Point x is called a P-L-limit point of A if each P-L-open set containing x, contains a point of A distinct from x.
We shall call the set of all P-L-limit points of A the P-L-derived set of A and denoted it by P-L-(A'). Therefore $x \in$ P-L-(A') iff for every P-L-open set G in X such that $x \in G$ implies that $(G \cap A) - \{x\} \neq \phi$.

Proposition 4.35

Let X be a topological space and $A \subseteq B \subseteq X$.

Then:

(i) P-M-(cl(A))= $A \cup$ P-L-(A')
(ii) A is P-M-closed set iff P-L-(A') $\subseteq A$
(iii) P-L-(A') $\subseteq$ P-L-(B')
(iv) P-L-(A') $\subseteq A'$ **(v)** P-M-(cl(A)) $\subseteq cl(A)$

Proof

(i) If $x \notin$ P-M-(cl(A)), then there exists a P-M-closed set F in X such that $A \subseteq F$ and $x \notin F$. Hence $G = X - F$ is a P-L-open set such that $x \in G$ and $G \cap A = \phi$.
Therefore $x \notin A$ and $x \notin$ P-L-(A'), then $x \notin A \cup$ P-L-(A'). Thus $A \cup$ P-L-(A') $\subseteq$ P-M-($cl(A)$. On the other hand, $x \notin A \cup$ P-L-(A') implies that there exists a P-L-open set G in X such that $x \in G$ and $G \cap A = \phi$. Hence $F = X - G$ is a P-M-closed set in X such that $A \subseteq F$ and $x \notin F$. Hence $x \notin$ P-M-($cl(A)$). Thus P-M-($cl(A)$) $\subseteq A \cup$ P-L-(A').
Therefore P-M-($cl(A)$) $\subseteq A \cup$ P-L-(A').
For (ii), (iii), (iv) and (v) the proof is similar.

Definition 4.36

Let X be a topological space. Two non- empty subsets A and B of X are called P-M-separated iff P-M-($cl(A)$) $\cap B = A \cap$ P-M-($cl(B)$) $= \phi$.

Remark 4.37

A set A is called P-L-M-clopen iff it is P-L-open and P-M-closed.

Theorem 4.38

Let X be a topological space, then the following statements are equivalent:

(i) X is a P-M-connected space.
(ii) X cannot be expressed as the union of two disjoint, non- empty and P-M-closed sets.
(iii) The only P-L-M-clopen sets in the space are X and ϕ.

Example 4.39

In this example we show that P-L-connectivity is not a hereditary property. Let $X = \{a, b, c, d\}$ and $T_x = \{\{a\},\{a, b\},\{a, c\},\{a, b, c\}, \phi, X\}$ be a topology on X.

The P-L-open sets are: $\{a\},\{a, b\},\{a, c\},\{a, b, c\},\{a, d\},\{a, b, d\},\{a, c, d\}, \phi, X$. It is clear that X is P-L-connected space since the only P-L-clopen sets are X and ϕ. Let $Y = \{b, c\}$, then $T_y = \{\{b\},\{c\}, Y, \phi\}$. It is clear that Y is not P-L-connected space since $\{b\} \neq \phi$, $\{b\} \neq Y$ and

$\{b\}$ is P-L-M-clopen set in Y. Thus a P-L-connectivity is not a hereditary property.

Proposition 4.40

Let A be a P-M-connected set and H, K are P-M-separated sets. If $A \subseteq H \bigcup K$ then
Either $A \subseteq H$ or $A \subseteq K$.

Proof

Suppose A is P-M-connected set and H, K are P-M-separated sets such that $A \subseteq H \bigcup K$. Let $A \not\subset H$ or $A \not\subset K$. Suppose $A_1 = H \bigcap A = \phi$ and $A_2 = K \bigcap A \neq \phi$. Then $A = A_1 \bigcup A_2$. Since $A_1 \subseteq H$, hence P-M-$(cl(A_1)) \subseteq$ P-M-$(cl(H))$. Since P-M-$(cl(H)) \bigcap K = \phi$, then P-M- $(cl(A_1)) \bigcap A_2 = \phi$. Since $A_2 \subseteq K$, hence P-M-$(cl(A_2)) \subseteq$ P-M-$(cl(K))$.

Since P-M-$(cl(K)) \bigcap H = \phi$ then P-M-$(cl(A_2)) \bigcap A_1 = \phi$
But $A = A_1 \bigcup A_2$, therefore A is not P-M-connected space which is a contradiction. Then either $A \subseteq H$ or $A \subseteq K$.

Proposition 4.41

If H is P-M-connected set and $H \subseteq E \subseteq$ P-M-$(cl\ (H))$ then E is P-M-connected.

Proof

If E is not P-M-connected, then there exists two sets A, B such that P-M-$(cl(A)) \bigcap B = A \bigcap$ P-M-$(cl(B)) = \phi$ and $E = A \bigcup B$. Since $H \subseteq E$, thus either $H \subseteq A$ or $H \subseteq B$. Suppose $H \subseteq A$, then P-M-$(cl\ (H)) \subseteq$ P-M-$(cl(A))$, thus P-M-$(cl\ (H)) \bigcap B =$ P-M-$(cl(A)) \bigcap B = \phi$. But $B \subseteq E \subseteq$ P-M-$(cl\ (H))$, thus P-M-$(cl\ (H)) \bigcap B = B$. Therefore $B = \phi$ which is a contradiction. Thus E is P-M-connected set. If $H \subseteq E$, then by the same way we can prove that $A = \phi$ which is a contradiction. Then E is P-M-connected.

Corollary 4.42

If a space X contains a P-M-connected subspace A such that P-M-$(cl(A)) = X$, then X is P-M-connected.

Proof

Suppose A is a P-M-connected subspace of X such that P-M-$(cl(A)) = X$. Since $A \subseteq X =$ P-M-$(cl(A))$, then by proposition 4.41, X is P-M-connected.

Proposition 4.43

If A is P-M-connected set then P-M-$(cl(A))$ is P-M-connected.

Proof

Suppose A is P-M-connected set and P-M-$(cl(A))$ is not. Then there exist two P-M-separated sets H, K such that P-M-$(cl(A)) = H \bigcup K$. But $A \subseteq$ P-M-$(cl(A))$,, then $A \subseteq H \bigcup K$ and since A is P-M-connected set then either $A \subseteq H$ or $A \subseteq K$ (by proposition 4.40)

(1) If $A \subseteq H$, then P-M-$(cl(A)) \subseteq$ P-M-$(cl(H))$. But P-M-$(cl(H)) \bigcap K = \phi$, hence P-M-$(cl(A)) \bigcap K = \phi$. Since $K \subseteq$ P-M-$(cl(A))$, then $K = \phi$ which is a contradiction.
(2) If $A \subseteq K$, then the same way we can prove that $H = \phi$ which is a contradiction.

Therefore P-M-$(cl(A))$ is P-M-connected set.

Proposition 4.44

Let X be a topological space such that any two elements a and b of X are contained in some P-M-connected subspace of X. Then X is P-M-connected.

Proof

Suppose X is not P-M-connected space. Then X is the union of two P-M-separated sets A, B. Since A, B are non-empty sets, thus there exist a, b such that $a \in A$, $b \in B$. Let H be a P-M-connected subspace of X which contains a and b. Therefore by proposition 4.40 either $H \subseteq A$ or $H \subseteq B$ which is a contradiction since $A \bigcap B = \phi$.

Then X is P-M-connected space.

Proposition 4.45

If A and B are P-L-connected subspace of a space X such that $A \bigcap B = \phi$, then
$A \bigcup B$ is P-L-connected subspace.

Proof

Suppose that $A \bigcup B$ is not P-L-connected. Then there exist two P-L-separated sets
H and K such that $A \bigcup B = H \bigcup K$. Since $A \subseteq A \bigcup B = H \bigcup K$ and A is P-L-connected, then either $A \subseteq H$ or $A \subseteq K$. Since $B \subseteq A \bigcup B = H \bigcup K$ and B is P-L-connected, then either $B \subseteq H$ or $B \subseteq K$.

(1) If $A \subseteq H$ and $B \subseteq H$, then $A \bigcup B \subseteq H$. Hence $K = \phi$ which is a contradiction.
(2) If $A \subseteq H$ and $B \subseteq K$, then $A \bigcap B \subseteq H \bigcap K = \phi$. Therefore $A \bigcap B = \phi$ which is a contradiction.

By the same way we can get a contradiction if $A \subseteq K$ and $B \subseteq H$ or if $A \subseteq K$ and $B \subseteq K$. Therefore $A \bigcup B$ is P-L-connected subspace of a space X.

Theorem 4.46

If X and Y are P-L-connected spaces, then $X \times Y$ is P-L-connected space.

Proof

For any points (x_1, y_1) and (x_2, y_2) of the space $X \times Y$, the subspace

$X \times \{y_1\} \cup \{x_2\} \times Y$ contains the two points and this subspace is P-L-connected, since it is the union of two P-L-connected subspaces with a point in common (by proposition 4.45).

Thus $X \times Y$ is P-L-connected (by proposition 4.44).

References

[1] M. E. Abd EI Morsef. E. F. Lashinen and A. a. Nasef, Some topological operators Via Ideals, Kyungpook Mathematical Journal, Vol-322(1992).
[2] Erdal. Ekici and Migual calder "on generalized pre-closure spaces and separation for some special types of functions" Italian journal of pure and applied mathematics-N 27-2010(81-90).
[3] K. Kuratowski, Topology I. Warrzawa 1933.
[4] N. Levine, Rend. Cire. Math. Palermo, 19(1970), 89-96.
[5] S. R. Malghan, S. S. Benchall and Navalagi " S-Completely regular spaces".
[6] A. S Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On Pre-Continuous and Weak Pre-continuous mappings, Proc. Math and Phys. Soc. Egypt, 53(1982), 47-53.
[7] James. R. Munkers, Topology. Ed. 2, PHI Learning Pvt. Ltd., New Delhi. 2010.
[8] V. Popa, "Characterization of H-almost continuous functions" Glasnik Mat.
[9] R. Vaidyanathaswamy, The Localization theory in set topology. Proc-Indian Acad. Sci, 20(1945), 51-61.
[10] R. Selvi and M. Priyadarshini, "on pre- ρ -continuity where $\rho \in$ {L, M, R, S}", International journal of science and research. Volume 4, Issue 4, April 2015.
[11] V. Leelavathy, L. Elvina mary, "Pre-connectedness modulu an ideal", international journal of computer application. Issue 3, volume 6 (Nov-Dec 2013)