# A Survey on Subordination

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**Abstract:** ***In this paper, we discuss subordination for certain classes of univalent functions. Under this subordination, we discuss the various properties and results of subordination for the different subclass of analytic and univalent functions.***

**Keywords:** Analytic function, Univalent function, Subordination, Superordinate, Differential subordination.

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## 1.Introduction and Definition

The concept of subordination is introduces by Lindelöf [1], J.E.Littlewood [2] and W.Rogosinski [3]. Over the years a substantial theory has been developed and subordination now plays an important role in theory of analytic function.

Encouraged by wide applications of subordination and differential subordination in the study of univalent functions, many complex analysts attempted to apply this technique to univalent function and brought to daylight many new facts of this field. Sanford S. Miller & Petru T. Mocanu [4], Maslina Darus [5], S. Owa *et. al.* [6] are the few who contributed significantly to the study of univalent function using subordination tools. The purpose of this paper is to take review of development in subordination of analytic & univalent functions.

Let $A_n$ denotes the class of univalent functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k , \quad (a_k \ge 0 , n = 1, 2, 3 ...) \quad (1.1)$$

analytic in the unit disc $U = \{z : z \in \mathbb{C}; |z| < 1\}$.

A function $f(z) \in A_n$ is said be starlike function of order $\alpha$ $(0 \le \alpha < 1)$ if it satisfies, for $z \in U$, the condition

$$Re\left(\frac{z f'(z)}{f(z)}\right) > 0$$

A function $f(z) \in A_n$ is said to be convex function of order $\alpha$ $(0 \le \alpha < 1)$ if it satisfies, for $z \in U$, the condition

$$Re\left(1 + \frac{z f''(z)}{f'(z)}\right) > 0$$

**Definition 1:** *The functions $f(z)$, $g(z) \in A_n$ where $f(z)$ defined by (1.1) and $g(z)$ is defined by*

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k ,$$

*the Hardmard product (or Convolution) $f(z) * g(z)$ defined as,*

$$h(z) = f(z) * g(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k , \; a_k, b_k \ge 0 \quad (1.4)$$

**Definition 2:** *The function $f(z)$ is said to be subordinate to $g(z)$ if there exists Schwarz function $w(z)$, analytic in $U$ with*
*$w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$*
*such that*
*$f(z) = g(w(z))$ for $z \in U$.*

We denote this subordination as $f \prec g$ or $f(z) \prec g(z)$.

In particular if $f(z)$ is univalent in $U$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$. Some of the important consequences of subordination are as follows,

(i) $|f'(0)| \le |g'(0)|$ ,

(ii) $\max_{|z| \le r} |f(z)| \le \max_{|z| \le r} |g(z)|$ , $|z| = r$, (1.2)

(iii) $\Re(f(z)) \le \max_{|z| \le r} \Re(g(z))$

(iv) $\Re(f(z)) \ge \min_{|z| \le r} \Re(g(z))$

## 2.Differential Subordination (1.3)

In 1935 Goluzin [7] consider the first order differential subordination

$$z p'(z) \prec h(z).$$


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He showed for any convex h

$$p(z) \prec q(z)=\int_0^z h(t)t^{-1}dt,$$

and this q is the best dominate.

Let $\phi(r,s,t;z): \mathbb{C}^2 \times U \to \mathbb{C}$. If p and $\phi(p(z), zp'(z); z)$ are univalent and if p satisfies the first order superordination

$$h(z) \prec \phi(p(z), zp'(z); z), h(0) = \phi(p(0), 0, 0) \quad (1.5)$$

then p is a solution of the differential superordination (1.5).

Let $\phi(r,s,t;z): \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \quad (1.6)$$

then p is a solution of the differential superordination (1.5). If *f is* superordination of F is called to be superordinate to *f*. An analytic function q is called a subordinate if q $\prec$ p for all p satisfying (1.5).

Millar and Mocanu [8] considered the second order linear differential subordination

$$A(z)z^2 p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \quad (1.7)$$

where A, B, C and D are complex-valued functions defined on U and and h(z) is any convex function.

## 3.Subordination for Analytic functions

**Lemma 1:** (cf. Jack [9]): let the non constant function w (z) be analytic in U with w (0) = 0. If |w(z)| attains its maximum value on the circle |z|= r < 1 at a point $z_0 \in U$, then

$$z_0 w'(z_0) = c\, w(z_0)$$

where c is a real number and c ≥ 1.

**Lemma 2**: [8, p132, Theorem 3.4h] Let q (z) be univalent in the unit disk U and θ and ϕ be analytic in a domains D containing q ( U ) with ϕ (w) ≠ 0 when w∈q (w). Set

$$Q(z) := z q'(z) \phi(q(z)), \ h(z) := \theta(q(z)) + Q(z)$$

Suppose that either h (z) is convex, or Q (z) is starlike univalent in U. In addition, assume that

$$\Re\left(\frac{z h'(z)}{Q(z)}\right) > 0 \qquad (z \in U).$$

If p (z) is analytic in U, with p (0) = q (0), p ( U ) ⊆ D and

$$\theta(p(z)) + z'p'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \quad (1.8)$$

Then p (z) $\prec$ q (z) and q (z) is best dominant.

**Theorem 1:** [8, page 188, Theorem 4.1a] Let n be a positive integer and A ( z ) = A ≥ 0. Suppose that the function B ( z ), C ( z ), D ( z ) : U → ℂ satisfy $\Re B(z) \geq A$ and

$$|\Im C(z)|^2 \leq n\, |\Re B(z) - nA - 2D(z)| > 0,$$

If p ∈ H [1,n] and if

$$\Re[A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)] > 0,$$

then

$$\Re p(z) > 0.$$

where H [a,n] is the subclass of the form H [a,n] = $a + a_n z^n + \ldots$ .for positive integer a ∈ ℂ which is analytic in U.

**Theorem 2:** [8, page 195, Theorem 4.1e] Let h be convex univalent in U with h (0) = 0 and let A ≥ 0. Suppose that $k > 4/|h'(0)|$ and that B ( z ), C ( z ) and D ( z ) are analytic in U and satisfy

$$\Re B(z) \geq A + |C(z) - 1| - \Re(C(z) - 1) + k|D(z)|.$$

If p ∈ H [1, n] satisfies the differential subordination

$$A(z)\, z^2p''(z) + B(z)\, z\, p'(z) + C(z)\, p(z) + D(z) \prec h(z)$$

then

p (z) $\prec$ h (z)

## 4.Conclusion

The aim of this paper is to study the importance of subordination in analytic and univalent functions. In this paper, we focus on subordination and various results of differential subordination.

## 5.Future Scope

In future we will introduce fractional differential operator to study subordination and Superordinate.

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