# Centroidal Mean Derivative - Based Closed Newton Cotes Quadrature

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***Abstract :In this paper, a new scheme of the evaluation of numerical integration by using Centroidal mean derivative - based closed Newton cotes quadrature rule (CMDCNC) is presented in which the centroidal mean is used for the computation of function derivative. The accuracy of these numerical formulas are higher than the existing closed Newton cotes quadrature (CNC) fromula. The error terms are also obtained by using the concept of precision. Comparisions are made between the existing closed Newton cotes formula and the centroidal mean derivative - based closed Newton cotes quadrature formula by using the numerical examples.***

**Keyword:** Closed Newton-Cotes formula, Error terms, Centroidal mean derivative, Numerical examples, Numerical integration.

## 1. Introduction

Numerical integration is used to find the numerical value of a definite integral. The general form of a numerical integration formula is

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) \quad (1)$$

subdividing the finite interval [a , b] into a large number of subintervals, by defining (n+1) intermediate points $x_0$ , $x_1$ , … , $x_n$ , $x_i=x_0+ih$, i=0,1,2,..n, where $h=\frac{b-a}{n}$ and $w_i$ are weights i=0,1,,2,...,n. In closed Newton Cotes formula, function evaluation at the end points of the interval is included in the quadrature rule.

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) \quad (2)$$

Select the values for $w_i$, i=0,1,...n. so that the error of approximation for the quadrature formula is zero, that is

$$E_n[f] = \int_a^b f(x)\,dx - \sum_{i=0}^{n} w_i f(x_i) = 0,$$
$$\text{for } f(x) = x^j\ j = 0,1,\dots n \quad (3)$$

**Definition 1.1.[9]** An integration method of the form ( 1) is said to be of order P, if it produces exact results ($E_n[f]$ = 0) for all polynomials of degree less than or equal to P.

Following are some of the existing closed Newton cotes quadrature rules.

When n=1 : Trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right) - \frac{(b-a)^3}{12} f''(\xi),$$
$$where\ \xi \in (a,b) \quad (4)$$

When n =2: Simpson's 1/3rd rule

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi), where\ \xi \in (a,b) \quad (5)$$

When n =3: Simpson's 3/8th rule

$$\int_a^b f(x)\,dx = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}(\xi), where\ \xi \in (a,b) \quad (6)$$

When n=4 : Boole's rule

$$\int_a^b f(x)\,dx = \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}(\xi), Where\ \xi \in (a,b) \quad (7)$$

It is known that the degree of precision is (n+1) for even value of n and n for odd value of n.

In the closed Newton Cotes formula, function evaluations are uniformly spaced. So that the weights are the only parameters to be determined. There are several works has been done for the improvement of closed Newton cotes quadrature formula. In 2005, Dehghan et al. have focused on increasing the order of accuracy of the existing numerical integration formula [3,4,5] by order two by including the location of boundaries of the interval as two additional parameter and rescaling the original integral to fit the optimal boundary locations. In 2006, these authors introduced the improvements of first and second kind Chebyshev - Newton cotes quadrature rules [7,8].Clarence O.E Burg introduced a new family of derivative based - closed ,open and Midpoint quadrature rules[1,6,2].Also, Weijing Zhao and Hongxing Li [13] took a different approach by introducing a Midpoint derivative - based closed Newton-Cotes quadrature rules. Considering the arithmetic mean of end points as midpoint, we proposed

Geometric mean[10] and Harmonic mean[11] derivative - based closed Newton cotes quadrature rule. Also, we Compared the arithmetic mean ,geometric mean and harmonic mean derivative - based closed Newton Cotes uadrature rules[12].

In this paper, centroidal mean derivative-based closed Newton cotes quadrature formulas are presented where derivative values are used in addition to the existing formula to increase the order of accuracy. The precision of the new scheme is higher than the existing closed Newton cotes quadrature formulas. The error terms are also obtained by using the concept of precision. Finally, the comparisons are made between the closed Newton cotes quadrature and centroidal mean derivative-based closed Newton cotes quadrature and are discussed in detail. The degree of precision of the proposed rule is (n+2) for even n and is (n+1) for odd n.

## 2. Centroidal mean derivative - based closed Newton cotes quadrature rule

In this section a new set of centroidal mean derivative - based closed Newton cotes quadrature rule is derived for the computation of definite integral over [a,b].

**Theorem 2.1.** Closed Trapezoidal rule (n=1) using centroidal mean derivative is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^3}{12} f''\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right), (8)$$

The precision of this method is 2.

**Proof:** For $f(x) = x^2$

$$\text{Exact value of } \int_a^b x^2\,dx = \frac{1}{3}(b^3-a^3);$$

$$(8) \Rightarrow \frac{b-a}{2}(a^2+b^2) - \frac{2(b-a)^3}{12} = \frac{1}{3}(b^3-a^3).$$

It shows that the solution is exact. Therefore, the precision of closed Trapezoidal rule with centroidal mean derivative is 2 where as the precision of the existing Trapezoidal rule (4) is 1.

**Theorem 2.2.** Closed Simpson's1/3rd rule with centroidal Mean derivative (n=2) is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right), (9)$$

The precision of this method is 4.

**Proof:** For $f(x) = x^4$

$$\text{Exact value of } \int_a^b x^4\,dx = \frac{1}{5}(b^5-a^5);$$

$$(9) \Rightarrow \left(\frac{b-a}{6}\right)\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right] - \frac{24(b-a)^5}{2880} = \frac{1}{5}(b^5-a^5).$$

It shows that the solution is exact. Therefore, the precision of closed Simpson's1/3rd rule with centroidal mean derivative is 4 where as the precision of the existing Simpson's1/3rd rule (5) is 3.

**Theorem 2.3.** Closed Simpson's3/8rd rule with centroidal mean derivative (n=3) is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right), (10)$$

The precision of this method is 4.

**Proof:** For $f(x) = x^4$

$$\text{Exact value of } \int_a^b x^4\,dx = \frac{1}{5}(b^5-a^5);$$

$$(10) \Rightarrow \left(\frac{b-a}{8}\right)\left[a^4 + 3\left(\frac{2a+b}{3}\right)^4 + 3\left(\frac{a+2b}{3}\right)^4 + b^4\right] - \frac{24(b-a)^5}{6480} = \frac{1}{5}(b^5-a^5).$$

It shows that the solution is exact. Therefore, the precision of closed Simpson's3/8rd rule with centroidal mean derivative is 4 where as the precision of the existing Simpson's3/8th rule (6) is 3.

**Theorem 2.4.** Closed Boole's rule with centroidal mean derivative (n=4) is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right]$$

$$-\frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right) \quad (11)$$

The precision of this method is 6.

**Proof:** For $f(x) = x^6$

$$\text{Exact value of } \int_a^b x^6\,dx = \frac{1}{7}(b^7 - a^7);$$

$$(11) \Rightarrow \left(\frac{b-a}{90}\right)\left[7a^6 + 32\left(\frac{3a+b}{4}\right)^6 + 12\left(\frac{a+b}{2}\right)^6 + 32\left(\frac{a+3b}{4}\right)^6 + 7b^6\right] + \frac{720(b-a)^7}{1935360} = \frac{1}{7}(b^7 - a^7).$$

It shows that the solution is exact. Therefore, the precision of closed Boole's rule with centroidal mean derivative is 6 where as the precision of the existing Boole's rule (7) is 5.

## 3. The error terms of centroidal mean derivative - based closed Newton cotes quadrature rule

The error terms of centroidal mean derivative -based closed Newton cotes quadrature rules are obtained by using the difference between the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$ and the exact result $\frac{1}{(p+1)!}\int_a^b x^{p+1}\,dx$ where p is the precision of the quadrature formula.

**Theorem 3.1.** Centroidal mean derivative-based closed Trapezoidal rule (n=1)with the error term is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^3}{12} f''\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$
$$+\frac{(b-a)^5}{72(a+b)} f^{(4)}(\xi), \quad (12)$$

where $\xi \in (a, b)$.The order of accuracy is 5 with the error term

$$E_1[f] = \frac{(b-a)^5}{72(a+b)} f^{(4)}(\xi).$$

**Proof:**

$$\text{Let } f(x) = \frac{x^3}{3!}, \frac{1}{3!}\int_a^b x^3\,dx = \frac{1}{24}(b^4 - a^4);$$

$$\frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^3}{12} f''\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$
$$= \frac{b-a}{3!.2}\left(b^3 + a^3 - (b-a)^2\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)\right).$$

Therefore,

$$\frac{1}{24}(b^4 - a^4) - \frac{b-a}{3!.2}\left(b^3 + a^3 - (b-a)^2\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)\right) = \frac{(b-a)^5}{72(a+b)}.$$

Therefore the error term is,

$$E_1[f] = \frac{(b-a)^5}{72(a+b)} f^{(4)}(\xi).$$

**Theorem 3.2.** Centroidal mean derivative-based closed Simpson's1/3rd rule (n=2)with the error term is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$-\frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$
$$+\frac{(b-a)^7}{17280(a+b)} f^{(6)}(\xi), \quad (13)$$

where $\xi \in (a, b)$. The order of accuracy is 7 with the error term

$$E_2[f] = \frac{(b-a)^7}{17280\,(a+b)} f^{(6)}(\xi).$$

**Proof:**

$$\text{Let } f(x) = \frac{x^5}{5!}, \frac{1}{5!}\int_a^b x^5\,dx = \frac{1}{720}(b^6 - a^6);$$

$$\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$

$$= \frac{b-a}{5!.48}\left(8a^5 + (a+b)^5 + 8b^5 - 2(b-a)^4\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)\right).$$

Therefore,

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{5!.48}\left(8a^5 + (a+b)^5 + 8b^5 - 2(b-a)^4\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)\right) = \frac{(b-a)^7}{17280(a+b)}.$$

Therefore the error term is,

$$E_2[f] = \frac{(b-a)^7}{17280\,(a+b)} f^{(6)}(\xi).$$

**Theorem 3.3.** Centroidal mean derivative-based closed Simpson's 3/8th rule (n=3) with the error term is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right) + \frac{(b-a)^7}{38880(a+b)} f^{(6)}(\xi), \quad (14)$$

Where $\xi \in (a, b)$. The order of accuracy is 7 with the error term

$$E_3[f] = \frac{(b-a)^7}{38880\,(a+b)} f^{(6)}(\xi).$$

**Proof:**

$$\text{Let } f(x) = \frac{x^5}{5!}, \frac{1}{5!}\int_a^b x^5\,dx = \frac{1}{720}(b^6 - a^6);$$

$$\frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$

$$= \frac{b-a}{5!.648}\left(\begin{array}{c} 81a^5 + (2a+b)^5 + (a+2b)^5 + 81b^5 \\ -12(b-a)^4\left(\dfrac{2(a^2+ab+b^2)}{3(a+b)}\right)\end{array}\right),$$

Therefore,

$$\frac{1}{720}(b^6 - a^6) - \frac{b-a}{5!.648}\left(\begin{array}{c} 81a^5 + (2a+b)^5 + (a+2b)^5 + 81b^5 \\ -12(b-a)^4\left(\dfrac{2(a^2+ab+b^2)}{3(a+b)}\right)\end{array}\right)$$

$$= \frac{(b-a)^7}{38880\,(a+b)}.$$

Therefore the error term is,

$$E_3[f] = \frac{(b-a)^7}{38880\,(a+b)} f^{(6)}(\xi).$$

**Theorem 3.4**. Centroidal mean derivative-based closed Boole's rule (n=4) with the error term is

$$\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] - \frac{(b-a)^7}{1935360} f^{(6)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$

$$+\frac{(b-a)^9}{11612160(a+b)}f^{(8)}(\xi),(15)$$

where $\xi \in (a, b)$. The order of accuracy is 9 with the error term

$$E_4[f]=\frac{(b-a)^9}{11612160(a+b)}f^{(8)}(\xi).$$

**Proof:**

$$\text{Let } f(x)=\frac{x^7}{7!},\frac{1}{7!}\int_a^b x^7\,dx=\frac{1}{40320}(b^6-a^6);$$

$$\frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)+7f(b)\right]-\frac{(b-a)^7}{11612160}f^{(6)}\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)$$

$$=\frac{b-a}{7!.768}(97a^7+91a^6b+105a^5b^2+91a^4b^3+91a^3b^4+105a^2b^5+91ab^6+97b^7-2(b-a)^4\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)),$$

Therefore

$$\frac{1}{720}(b^6-a^6)-\frac{b-a}{7!.768}(97a^7+91a^6b+105a^5b^2+91a^4b^3+91a^3b^4+105a^2b^5+91ab^6+97b^7-2(b-a)^4\left(\frac{2(a^2+ab+b^2)}{3(a+b)}\right)),$$

$$=\frac{(b-a)^9}{11612160(a+b)}.$$

Therefore the error term is,

$$E_4[f]=\frac{(b-a)^9}{11612160(a+b)}f^{(8)}(\xi).$$

The summary of precision, the orders and the error terms for Centroidal mean derivative based closed Newton- Cotes quadrature are shown in Table 1.

**Table 1:** Comparison of error terms

| Rules | Precision | Order | *Error terms* |
|---|---|---|---|
| Trapezoidal rule (n=1) | 2 | 5 | $\frac{(b-a)^5}{72(a+b)}f^{(4)}(\xi)$ |
| Simpson's 1/3rd rule (n=2) | 4 | 7 | $\frac{(b-a)^7}{17280(a+b)}f^{(6)}(\xi)$ |
| Simpson's 3/8th rule (n=3) | 4 | 7 | $\frac{(b-a)^7}{38880(a+b)}f^{(6)}(\xi)$ |
| Boole's rule (n=4) | 6 | 9 | $\frac{(b-a)^9}{11612160(a+b)}f^{(8)}(\xi)$ |

## 4. Numerical Examples

To compare the effectiveness of the closed Newton cotes formula and the centroidal mean derivative - based closed Newton cotes formula , the values of the following integrals: $\int_1^2 3^x\,dx$ and $\int_1^2 x^2e^x\,dx$ are computed using these formulas and are compared with error term (Table 2 and 3).

**Example 4.1:**Solve $\int_1^2 3^x\,dx$ and compare the solutions with the CNC and CMDCNC rules.

**Solution:**

Exact value of $\int_1^2 3^x\,dx$ =5.46143536.

**Table 2:** Comparison of CNC and CMDCNC rules

| *Value of n* | *CNC* | | *CMDCNC* | |
|---|---|---|---|---|
| | *App. value* | *Error* | *App. value* | *Error* |
| n=1 | 6.000000000 | 0.538564640 | 5.444484377 | 0.016950983 |
| n=2 | 5.464101615 | 0.002666255 | 5.461307952 | 0.000127408 |
| n=3 | 5.462625068 | 0.001189708 | 5.461383440 | 0.000051920 |
| n=4 | 5.461440131 | 0.000004771 | 5.461434619 | 0.000000741 |

**Example 4.2:** Solve $\int_1^2 x^2e^x\,dx$ and compare the solutions with the CNC and CMDCNC rules.

**Solution:**

Exact value of $\int_1^2 x^2e^x\,dx$=12.05983037.

**Table 3:** Comparison of CNC and CMDCNC rules

| *Value of n* | *CNC* | | *CMDCNC* | |
|---|---|---|---|---|
| | *App. value* | *Error* | *App. value* | *Error* |
| n=1 | 16.13725311 | 4.07742274 | 11.93569674 | 0.12413363 |
| n=2 | 12.10161798 | 0.04178761 | 12.05742528 | 0.00240509 |
| n=3 | 12.07851901 | 0.01868864 | 12.05887781 | 0.00095256 |
| n=4 | 12.05994843 | 0.00011806 | 12.05978638 | 0.00004399 |

## 5. Conclusion

In this paper, centroidal mean derivative - based closed Newton - Cotes quadrature formulas were presented, that include the use of centroidal mean value at the derivative to increase the order of accuracy of the computation of definite integrals. The error bounds for the quadrature formula were derived by using the difference between the quadrature formula for the monomials and the exact results. Finally, numerical examples demonstrate the accuracy of the proposed formula.

## References

[1] Clarence O.E.Burg,"Derivative-based closed Newton-cotes numerical quadrature", *Applied Mathematics and Computations, vol.218,* pp. 7052-7065, 2012.+
[2] Clarence O.E.Burg, and Ezechiel Degny, "Derivative-based midpoint quadrature rule, *Applied Mathematics and Computations, vol.4*, pp. 228-234 , 2013.
[3] M.Dehghan, M.Masjed-Jamei, and M.R.Eslahchi, "On numerical improvement of closed Newton-Cotes quadrature rules", *Applied Mathematics and Computations, vol.165*, pp. 251-260 , 2005.
[4] M.Dehghan, M.Masjed-Jamei, and M.R.Eslahchi, "On numerical improvement of open Newton-Cotes quadrature rules", *Applied Mathematics and Computations, vol.175*, pp. 618-627 , 2006.
[5] M.Dehghan, M.Masjed-Jamei, and M.R.Eslahchi, "The semi-open Newton-Cotes quadrature rule and its numerical improvement", *Applied Mathematics and Computations, vol.171*, pp. 1129-1140 , 2005.
[6] Fiza Zafar,Saira Saleem and Clarence O.E.Burg, "New Derivaive based open Newton- cotes quadrature rules ",*Abstract and Applied Analysis,Volume 2014 ,* Article ID 109138, 16 pages,2014.
[7] S.M.Hashemiparast, M.R.Eslahchi, M.Dehghan and M.Masjed-Jamei, "On numerical improvement of the first kind Chebyshev - Newton - Cotes quadrature rules ", *Applied Mathematics and Computations,vol.174*, pp. 1 0 2 0 -1032 , 2006.
[8] S.M.Hashemiparast, M.R.Eslahchi, M.Dehghan and M.Masjed-Jamei, "On numerical improvement of the second kind Chebyshev - Newton - Cotes quadrature rules(open type) ", *Applied Mathematics and Computations,vol.180*, pp. 6 0 5 -613 , 2006.
[9] M.K.Jain,S.R.K.Iyengar and R.K.Jain, *Numerical methods for Scientific and Computation*, New Age International (P) limited, Fifth Edition ,2007.
[10] T.Ramachandran,D.Udayakumar and R.Parimala,"Geometric mean derivative -based closed Newton cotes quadrature ", *International Journal of Pure & Engineering. Mathematics,Vol.4,* pp. 107-116,2016.
[11] T.Ramachandran,D.Udayakumar and R.Parimala,"Harmonic mean derivative -based closed Newton cotes quadrature ", *IOSR - Journal of Mathematics",Vol.12,* pp. 36 - 41,2016.
[12] T.Ramachandran,D.Udayakumar and R.Parimala," Comparison of Arithmetic Mean, Geometric Mean and Harmonic Mean Derivative-Based Closed Newton Cotes Quadrature", *Progress in Nonlinear dynamics and Chaos,Vol.4,* pp. 35 - 43,2016.
[13] Weijing Zhao and Hongxing,"Midpoint Derivative-Based Closed Newton-Cotes Quadrature", *Abstract and Applied Analysis, vol.2013*, Article ID 492507,10 pages,2013.