Singular Two-Point Boundary Value Problems of Finite Element method

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Abstract: ***In this paper, a class of singular two-point boundary value problems using the Galerkin method, using hierarchical grid method in discrete space, using one yuan for discrete function respectively, is obtained the general grid on the norm of the error convergence.***

Keywords: singular, two-point boundary value problems, Galerkin method, super convergence, Finite Element method

1. Introduction

In this paper a yuan in hierarchical grid numerical solving the following singular two-point boundary value problems.

$$\begin{cases} -u''(x) - \dfrac{b}{x}u'(x) + q(x)u(x) = f(x), \quad 0 < x < 1 \\ u'(0) = 0, \qquad u(1) = 0 \end{cases} \tag{1.1}$$

Here is the constant, $b > 0$ is responsible $q(x)$ for the function, $f(x)$ for a given function. [1], [2], [3] know if there is the only solution of this problem and smooth.

The singular boundary value problems can be caused by high dimensional non simplified spherically symmetric problems, the first side value problem of three dimensional elliptic equations:

$$\begin{cases} -\nabla(a\nabla u) = f \qquad in\Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

Through coordinate transformation into this problem.

This equation numerical Xie Ben Wen from the following several aspects: one, layered grid; Second, the equation of variational form; Three, equations of discrete; Four, error estimate.

Duran [7]in 2006 for the first time put forward the hierarchical grid, the grid to adjacent cell changes slowly, without considering the change of the thickness of the grid. Chuan-miao Chen[13]on special grid discussed the convergence of the differential error norm and superconvergence. Zhang xu, [1],[12] on the basis of chuan-miao Chen made a general grid model and the error of the maximum modulus difference of convergence.
In this paper, on the basis of zhang xu, do the general layered grid on the grid on the norm of the error convergence and convergence.

2. The Discrete

By the Green points male, can get the variational equation

$$a(u,v) = \int_\Omega x^b fvdx \quad , \forall v \in H^1(\Omega) \tag{2.1}$$

there

$$a(u,v) = \int_0^1 x^b u'(x)v'(x) + x^b u(x)v(x)dx \quad v \in H^1$$

$$\langle f, v \rangle = \int_0^1 x^b fvdx \qquad v \in H^1$$

By [5] the problem (1) and (2.3), and other equivalent. The interval are partitioned into coarse grid and fine grid:

$\Omega_1 = [0, \frac{1}{n}], \Omega_2 = [\frac{1}{n}, 1]$.The fine grid Ω_2 for subdivision: average composition , $\frac{1}{n} = x_1 < \cdots < x_n = 1$.

The i remember as a unit $I_i = [x_{i-1}, x_i]$, $h_i = \frac{1}{n}$.The same as Ω_1, $0 = x_{00} < x_{01} < \cdots < x_{0n} = \frac{1}{n}$.The 1i remember as a unit $I_{1i} = [x_{1i-1}, x_{1i}]$, $h_i = \frac{1}{n^2}$.

Define the finite dimensional test space V_h is piecewise polynomial in hierarchical grid space,

$$V_h = \{v \in H_0^1(\Omega) : v|_{\Omega_i} \in P_{k-1}(I_i), 1 \le i \le n\}$$

Here $P_{k-1}(I_i)$ is polynomial representative the i remember as a unit I_i, $V_h \subset H^1$.

The equation can be turned into

$$a(u_h, v) = \int_0^1 x^b fvdx \quad v \in H_0^1 \tag{2.2}$$

$$\sum_{i=1}^n a(u_i, v_i) = \int_0^1 x^b fv_i dx, \qquad i = 1, 2, \cdots, n$$

Norm: $\|v\|_{m,p,\Omega} = \left\{ \sum_{|\alpha| \le m} \|D^\alpha v\|_{0,p,\Omega}^p \right\}^{1/p}$

Semi-norm: $|v|_{m,p,\Omega} = \left\{ \sum_{|\alpha| = m} \|D^\alpha v\|_{0,p,\Omega}^p \right\}^{1/p}$

Weight norm: $\|v\|_{m,p,\Omega} = \left\{ \sum_{|\alpha|\le m} x^{\alpha} \|D^{\alpha} v\|_{0,p,\Omega}^{p} \right\}^{1/p}$

Weight semi- norm: $|v|_{m,p,\Omega} = \left\{ \sum_{|\alpha|=m} x^{\alpha} \|D^{\alpha} v\|_{0,p,\Omega}^{p} \right\}^{1/p}$

3. The Convergence of Solution

3.1 Preliminary Knowledge

Lemma1[5] We assumption $u(x)$ and $u_h(x)$ are the solution of vanational problem and the solution of the discrete problem. Here, $\exists C = const > 0$,can be used

$$\|u-u_h\| \le C \inf_{v_h \in V_h} \|u - v_h\|$$

Lemma2[8],[9],[10] Construct the interpolation postprocessing operator $\prod_{2h}^{k}$,here $u_I \in V_h$ is the finite element interpolation for u , k is the number of interpolation polynomia,so

$$C = const > 0,$$

$$\|\prod_{2h}^{k} u - u\|_l \le C h^{r+1-l} \|u\|$$

$$\|\prod_{2h}^{k} v\|_l \le C \|v\|_l , \quad \forall v \in V_h$$

$$\prod_{2h}^{k} u_I = \prod_{2h}^{k} u$$

When $l = 0,1, 0 \le r \le k$.

Lemma3[6] Assumption $u(x)$ and $u_I(x)$ is the solution of variational problem and linear interpolation ,

$$a(u-u_I, v) \le O(h^2)|u|_3 \|v\|_1 , \qquad v \in V_h$$

$\forall p \ge 2和b > 0$.

3.2 Error estimation for $u - u_I$

Theorem1 Assumption $u(x)$ and $u_I(x)$ is the solution of variational problem and linear interpolation ,

$$|u-u_I|_1 \le Ch^2 \|u\|_3$$

$\forall p \ge 2 \text{ and } b > 0$.

proof: In order to discuss on local unit, we introduce the following auxiliary function

$$\phi(x) = \begin{cases} \int_{x_j}^{x_{j+1}} v'(t)dt & 0 \le x \le x_{j-1} \\ \int_{x_j}^{x} v'(t)dt & x_{j-1} \le x \le x_j \\ 0 & x_j \le x \le 1 \end{cases}$$

The auxiliary function $\phi(x) \in V_h$,

Use the affine transformation $x = x_{i-1} + h_i \xi$,here $\xi \in [0,1]$ in I_i .order $v_h = \phi(x)$,then

$$x^b = (x_{i-1} + h\xi_i)^b = (x_i\xi + (1-\xi)x_{i-1})^b \ge x_i^b \xi^b$$

so

$$\int_0^1 x^b (u-u_I)' v_h' = \int_0^1 x^b (u-u_I)' \phi' dx$$
$$= \int_{x_i}^{x_{i+1}} (v'(x))^2 dx$$
$$\ge h_i x_i^b \int_0^1 t^b (v'(x))^2 dx$$
$$\ge C h_i x_i^b \max |v'|_1^2$$

and because

$$\int_0^1 x^b (u-u_I)' v'(x) dx$$
$$= \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} x^b (u-u_I)' v'(x) dx$$
$$= \sum_{i=0}^{n} [x^b (u-u_I) v(x) \Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} b x^{b-1} (u-u_I)' v'(x) dx]$$
$$= \sum_{i=0}^{n} -b x^{b-1} \int_{x_i}^{x_{i+1}} (u-u_I)' v'(x) dx$$
$$\le C x^{-1} \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (u-u_I)' v'(x) dx$$
$$\le C x^{-1} O(h^2) \|u\|_3 \; |v|_1$$

From Lemma3 :

$$a(u-u_I, v) = \int_0^1 [x^b (u-u_I)' v'(x) + x^b (u-u_I) v(x)] dx \quad v \in H^1$$

$$\int_0^1 x^b (u-u_I)' v'(x) dx < \int_0^1 [x^b (u-u_I)' v'(x) + x^b (u-u_I) v(x)] dx \quad v \in H^1$$

$$a(u-u_I, v) \le CO(h^2) \|u\|_3 |v|_1$$

so:

$$|u-u_I|_1 \le Ch^2 \|u\|_3 .$$

3.3 Error estimation for $u_h - u_I$

Theorem2 Assumption $u_h(x)$ and $u_I(x)$ are discrete problems and the solution of a linear interpolation, $\forall p \ge 2 \text{ and } b > 0$,then

$$|u_h - u_I|_1 \le Ch^2 \|u\|_3$$

proof: Take coefficient of variable $x = x_e$,because (2.1), (2.2)

$$a(u,v) = \int_0^1 x^b u'(x)v'(x) + x^b u(x)v(x)dx \quad v \in V_h$$

$$a(u_h,v) = \int_0^1 x^b u_h'(x)v'(x) + x^b u_h(x)v(x)dx \quad v \in V_h$$

Two type is presupposed regularity,

$$a(u-u_h,v) = 0, \quad \forall v \in V^h$$

so

$$a(u-u_I+u_I-u_h,v) = 0, \quad \forall v \in V^h$$

$$a(u_h-u_I,v) = a(u-u_I,v) \quad \forall v \in V_h$$

Because theorem1 then

$$a(u-u_I,v) = O(h^2)\|u\|_3 |v|_1, \qquad \forall v \in V_h$$

order $v = u_h - u_I \in V_h$

$$\begin{aligned} & C\|u-u_I\|^2 \\ & \le a(c,u_h-u_I) \\ & \le a(u-u_I,u_h-u_I) \\ & = O(h^2)\|u\|_3 |u_h-u_I|_1, \quad \forall v \in V_h \end{aligned}$$

Then

$$|u_h-u_I|_1 \le Ch^2\|u\|_3.$$

3.4 The postprocessing for the Finite Element Method

Theorem3 Assumption u_h 和 $u_I(x)$ the solution of discrete problem and the solution of the linear interpolation, and meet the superclose property theorem2,then

$$\left\|\Pi_{2h}^2 u_h - u\right\|_1 \le Ch^2\|u\|_3$$

proof: Through the variable type for $\Pi_{2h}^2 u_h - u$

$$\Pi_{2h}^2 u_h - u = (\Pi_{2h}^2 u_h - \Pi_{2h}^2 u_I) + (\Pi_{2h}^2 u_I - u)$$

Fist

$$\left\|\Pi_{2h}^2 u_h - \Pi_{2h}^2 u_I\right\|_1 \le \left\|\Pi_{2h}^2 (u_h - u_I)\right\|_1$$

From Lemma2 , $\Pi_{2h}^k u_I = \Pi_{2h}^k u$ so

$$\left\|\Pi_{2h}^2 u_h - \Pi_{2h}^2 u_I\right\|_1 \le \left\|\Pi_{2h}^2 (u_h - u_I)\right\|_1 \le C\|u_h - u_I\|$$

From theorem3

$$\left\|\Pi_{2h}^2 u_h - \Pi_{2h}^2 u_I\right\|_1 \le \left\|\Pi_{2h}^2 (u_h - u_I)\right\|_1 \le C\|u_h - u_I\| \le Ch^2\|u\|_3$$

Second,from Lemma2

$$\left\|\Pi_{2h}^k u - u\right\|_l \le Ch^{r+1-l}\|u\|,$$

$$\left\|\Pi_{2h}^k v\right\|_l \le C\|v\|_l, \qquad \forall v \in V_h \text{ ,then}$$

$$\left\|\Pi_{2h}^2 u_h - u\right\|_1 \le \left\|\Pi_{2h}^2 u - u\right\|_1 \le Ch^2\|u\|_3.$$

3.5 Error estimation for $u - u_h$

Theorem4 Assumption $u(x)$ and $u_I(x)$ is the solution of variational problem and the solution of discrete problem , $\forall p \ge 2$ and $b > 0$,

$$|u-u_h|_1 \le Ch^2\|u\|_3$$

proof: From theorem1,3

$$|u-u_I| \le O(h^2)|u_h|_3 \|v\|_1, \qquad v \in V_h$$

From theorem2and the triangle inequality

$$\begin{aligned} |u-u_h|_1 &= |u-u_I+u_I-u_h|_1 \\ &\le |u-u_I|_1 + |u_I-u_h|_1 \le C(h^2)\|u\|_3 \end{aligned}.$$

4. Conclusion

Chuan-miao Chen and zhang xu, is presented in this paper on the basis of the numerical solution of one dimensional singular pertueb- ation problems do the global superconvergence analysis, and got the expected results.

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