International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2015): 6.391

Fixed Point Results in Fuzzy Menger Space

Manoj Kumar Shukla[1], Surendra Kumar Garg[2]

[1]Department of Mathematics, Institute for Excellence in Higher Education, Bhopal, (MP)

[2]Department of Mathematics, Shri Ram Institute of Technology, Jabalpur, (MP), India

Abstract: ***This paper presents some common fixed point theorems for occasionally weakly compatible mappings in Fuzzy menger spaces.***

Keywords: Occasionally weakly compatible mappings, Fuzzy Menger space, Weak compatible mapping, Semi-compatible mapping, Implicit function, common fixed point.

Subject Classification: AMS (2000) 47H25

1. Introduction

The study of fixed point theorems in probabilistic metric spaces is an active area of research. The theory of probabilistic metric spaces was introduced by Menger [5] in 1942 and since then the theory of probabilistic metric spaces has developed in many directions, especially in nonlinear analysis and applications. In 1966, Sehgal [9] initiated the study of contraction mapping theorems in probabilistic metric spaces. Since then several generalizations of fixed point theorems in probabilistic metric space have been obtained by several authors including Sehgal and Bharucha-Reid [10], Sherwood [11], and Istratescu and Roventa [2]. The study of fixed point theorems in probabilistic metric spaces is useful in the study of existence of solutions of operator equations in probabilistic metric space and probabilistic functional analysis. The development of fixed point theory in probabilistic metric spaces was due to Schweizer and Sklar [8]. Singh, Pant and Talwar [14] introduced the concept of weakly commuting mappings in probabilistic metric spaces. Using the ideas of point wise R-weak commutativity and reciprocal continuity of mappings [1,4,7,13],, Kumar and Chugh [3] established some common fixed point theorems in metric spaces. In 2005, Mihet [6] proved a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. Shrivastav et al [12] proved fixed point result in fuzzy probabilistic metric space. The purpose of the present paper is to prove a common fixed point theorem for six mappings via pointwise R-weakly commuting mappings in fuzzy probabilistic metric spaces satisfying contractive type implicit relations. Further, as a bi-product we obtain several fixed point results as corollaries of our main result.

2. Preliminary Notes

Let us define and recall some definitions:

Definition 2.1: A fuzzy probabilistic metric space (FPM space) is an ordered pair (X, F_α) consisting of a nonempty set X and a mapping F_α from XxX into the collections of all distribution functions $F_\alpha \in R$ for all $\alpha \in [0, 1]$. For x, y $\in$ X we denote the distribution function $F_\alpha(x,y)$ by $F_{\alpha(x,y)}$ and $F_{\alpha(x,y)}(u)$ is the value of $F_{\alpha(x,y)}$ at u in R.

The functions $F_{\alpha(x,y)}$ for all $\alpha \in [0,1]$ assumed to satisfy the following conditions:

a) $F_{\alpha(x,y)}(u) = 1\ \forall\ u > 0$ iff $x = y$,
b) $F_{\alpha(x,y)}(0) = 0\ \forall\ x, y$ in X,
c) $F_{\alpha(x,y)} = F_{\alpha(y,x)}\ \forall\ x, y$ in X,
d) If $F_{\alpha(x,y)}(u) = 1$ and $F_{\alpha(y,z)}(v) = 1$ then $F_{\alpha(x,z)}(u+v) = 1\ \forall$ x, y, z in X and u, v > 0.

Definition 2.2: A commutative, associative and non-decreasing mapping t: $[0,1] \times [0,1] \to [0,1]$ is a t-norm if and only if $t(a,1)= a\ \forall a\in[0,1]$, $t(0,0) = 0$ and $t(c,d) \ge t(a,b)$ for $c \ge a$, $d \ge b$.

Definition 2.3: A Fuzzy F-Menger space is a triplet (X, F^2_α, t), where (X, F^2_α) is a FPM-space, t is a t-norm and the generalized triangle inequality
$F^2_{\alpha(x,z)}(u+v) \ge t(F^2_{\alpha(x,z)}(u), F^2_{\alpha(y,z)}(v))$ holds for all x, y, z in X u, v > 0 and $\alpha \in [0, 1]$

The concept of neighbourhoods in Fuzzy F-Menger space is introduced as

Definition 2.4: Let (X, F^2_α, t) be a Fuzzy Menger space. If $x\in X$, $\varepsilon > 0$ and $\lambda\in(0,1)$, then (ε,λ) - neighbourhood of x, called $U_x(\varepsilon,\lambda)$, is defined by
$U_x(\varepsilon,\lambda) = \{y\in X: F^2_{\alpha(x,y)}(\varepsilon)>(1-\lambda)\}$
An (ε,λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon,\lambda): x \in X, \varepsilon > 0, \alpha \in [0, 1]$ and $\lambda\in(0,1)\}$ of neighbourhood.

Remark: If t is continuous, then Fuzzy F-Menger space (X, F^2_α, t) is a Housdorff space in (ε,λ)-topology.
Let (X, F^2_α, t) be a complete Fuzzy F-Menger space and $A\subset X$. Then A is called a bounded set if
$$\lim_{u\to\infty} \inf_{x,y\in A} F^2_{\alpha(x,y)}(u) = 1$$

Definition 2.5: A sequence $\{x_n\}$ in (X, F^2_α, t) is said to be convergent to a point x in X if for every $\varepsilon>0$and $\lambda>0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $x_n \in U_x(\varepsilon,\lambda)$ for all $n \ge N$ or equivalently $F^2_\alpha(x_n, x; \varepsilon) > 1-\lambda$ for all $n \ge N$ and $\alpha\in[0,1]$.

Definition 2.6: A sequence $\{x_n\}$ in (X, F^2_α, t) is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $F^2_\alpha(x_n,x_m; \varepsilon) > 1-\lambda$ $\forall$ n, m $\geq$ N for all $\alpha\in[0,1]$.

Definition 2.7: A Fuzzy F-Menger space (X, F^2_α, t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all $\alpha\in[0,1]$.

Definition 2.8: Let (X, F^2_α,t) be a Fuzzy F-Menger space. Two mappings f, g :$X\to X$ are said to be weakly compatible if they commute at coincidence point for all $\alpha\in[0,1]$.

Lemma 2.1: Let $\{x_n\}$ be a sequence in a Fuzzy F-Menger space (X, F^2_α,t), where t is continuous and $t(p, p) \geq p$ for all $p\in[0,1]$, if there exists a constant k(0,1) such that for all p > 0 and $n\in N$
$F^2_\alpha (x_n,x_{n+1}; kp) \geq F^2_\alpha(x_{n-1},x_n; p)$,
for all $\alpha\in[0,1]$ then $\{x_n\}$ is cauchy sequence.

Lemma 2.2: If (X, d) is a metric space, then the metric d induces, a mapping F^2_α: $X\text{x}X\to L$ defined by F^2_α (p, q) = $H_\alpha(x- d(p, q))$, p, q $\in$ R for all $\alpha\in[0,1]$. Further if t: $[0,1] \times [0,1]\to [0,1]$ is defined by t(a,b) = min{a,b}, then (X, F^2_α, t) is a Fuzzy F-Menger space. It is complete if (X, d) is complete.

Definition 2.9: Let (X, F^2_α, t) be a Fuzzy F-Menger space. Two mappings f, g: $X\to X$ are said to be compatible if and only if $F^2\,\alpha(fgxn,gfxn)(t)\to1$ for all t > 0 whenever $\{x_n\}$ in X such that fxn,gxn$\to$ z for some z $\in$ X.

Definition 2.10: Two self mappings f and g of a Fuzzy F-Menger space (X, F^2_α) are said to be pointwise R-weakly commuting if given x $\in$ X, there exists R > 0 such that
$F^2_{\alpha\,(fgx,gfx)}(t) \geq F^2_{\alpha\,(fx,gx)}(t/R)$ for t > 0 and $\alpha\in[0,1]$.

Definition 2.11: Let X be a set, f, g be self maps of X. A point x in X is called a coincidence point of f and g iff fx = gx. We shall call w = fx = gx a point of coincidence of f and g.

Definition 2.12: A pair of maps f and g is called weakly compatible pair if they commute at coincidence points.

Definition 2.13: Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition2.14. A function $\phi:[0,\infty)\to[0,\infty)$ is said to be a ϕ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if t = 0,

(ii) $\phi(t)$ is strictly increasing and $\phi(t) \to\infty$ as $t \to \infty$

(iii) ϕ is left continuous in $(0,\infty)$ and

(iv) ϕ is continuous at 0.

An altering distance function with the additional property that $h(t) \to\infty$ as $t \to\infty$, generates a Φ function in the following way.

$$\phi(t) = \begin{cases} \sup\{s: h(s) < t \text{ if } t > 0 \\ 0 \quad \text{if } t = 0 \end{cases}$$

It can be easily seen that ϕ is a Φ-function.

Lemma 2.3: Let $\{x_n\}$ be a sequence in F-Menger space (X, F^2_α, t) where t is continuous If there exists a constant h ε (0, 1) such that $F^2_\alpha (x_n, x_{n+1}; kt) \geq F^2_\alpha (x_{n-1}, x_n;t)$, n ε **N**, then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.4: Let X be a set, f, g be owc self maps of X. If f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

3. Main Results

Theorem 3.1: Let (X, F^2_α, t) be a complete F-Menger space and let p, q, f and g be self mappings of X. Let pairs {p, f} and {q, g} be owc. If there exists h ε (0, 1) such that

$$F^2_{\alpha\,(px,qy)}(ht) \geq \min\{F^2_{\alpha\,(fx,gy)}(t),\ F^2_{\alpha\,(fx,px)}(t),\ F^2_{\alpha\,(qy,gy)}(t),\ F^2_{\alpha\,(px,gy)}(t),\ F^2_{\alpha\,(qy,fx)}(t)\} \quad \ldots..(1)$$

for all x,y $\in$ X and for all t > 0, then there exists a unique point w $\in$ X such that pw = fw = w and a unique point z $\in$ X such that qz = gz =z .Moreover, z = w, so that there is a unique common fixed point of p, f, q and g.

Proof: Let the pairs { p,f } and {q,g } be owc, so there are points x,y $\in$ X such that px =fx and qy = gy. We claim that px = qy. If not, by inequality (1)

$$F^2_{\alpha\,(px,qy)}(ht)\geq\min\{F^2_{\alpha\,(fx,gy)}(t),\ F^2_{\alpha\,(fx,px)}(t),\ F^2_{\alpha\,(qy,gy)}(t),\ F^2_{\alpha\,(px,gy)}(t),\ F^2_{\alpha\,(qy,fx)}(t)\}$$

$$= \min\{F^2_{\alpha\,(px,qy)}(t),\ F^2_{\alpha\,(px,px)}(t),\ F^2_{\alpha\,(qy,qy)}(t),\ F^2_{\alpha\,(px,qy)}(t),\ F^2_{\alpha\,(qy,px)}(t)\}$$

$$= F^2_{\alpha\,(px,qy)}(t)$$

Therefore px = qy, i.e. px = fx = qy = gy. Suppose that there is an another point z such that pz = fz then by (1) we have pz =fz = qy = gy, so px = pz and w = px = fx is the unique point of coincidence of p and f. By Lemma 2.4 w is the only common fixed point of p and f. Similarly there is a unique point z$\in$ X such that z =qz = gz. Assume that w $\neq$ z. We have

$$F^2_{\alpha\,(w,z)}(ht) = F^2_{\alpha\,(pw,fz)}(ht)$$
$$\geq \min\{F^2_{\alpha\,(fw,gz)}(t),\ F^2_{\alpha\,(fw,pw)}(t),\ F^2_{\alpha\,(qz,gz)}(t),\ F^2_{\alpha\,(pw,gz)}(t),\ F^2_{\alpha\,(qz,fw)}(t)\}$$
$$\geq \min\{F^2_{\alpha\,(w,z)}(t),\ F^2_{\alpha\,(w,w)}(t),\ F^2_{\alpha\,(z,z)}(t),\ F^2_{\alpha\,(w,z)}(t),\ F^2_{\alpha\,(z,w)}(t)\}$$
$$= F^2_{\alpha\,(w,z)}(t)$$

Therefore we have z = w by lemma 2.4 and z is a common fixed point of p, f, q and g. The uniqueness of the fixed point holds from (1)

Theorem3.2: Let (X, F^2_α, t) be a complete Fuzzy F-Menger space and let p, q, f and g be self mappings of X. Let pairs {p, f} and {q, g} be owc. If there exists h ε (0, 1) such that

$$F^2_{\alpha\,(px,qy)}(ht) \geq \phi\,\{F^2_{\alpha\,(fx,gy)}(t),\ F^2_{\alpha\,(fx,px)}(t), F^2_{\alpha\,(qy,gy)}(t),\ F^2_{\alpha\,(px,gy)}(t),\ F^2_{\alpha\,(qy,fx)}(t)\}\ \ldots\ldots..(2)$$

for all x,y ∈ X and $\phi:[0, 1]^5 \to [0, 1]$ such that $\phi(t,1,1,t,t) > t$ for all 0< t <1, then there exists a unique common fixed point of p, f, q and g.

Proof: Let the pairs {p, f } and {q, g} be owc, so there are points x, y ∈ X such that px = fx and qy = gy. We claim that px = qy. By inequality (2) we have

$$F^2_{\alpha\,(px,qy)}(ht) \geq \phi\,(\{F^2_{\alpha\,(fx,gy)}(t),\ F^2_{\alpha\,(fx,px)}(t),\ F^2_{\alpha\,(qy,gy)}(t),\ F^2_{\alpha\,(px,gy)}(t),\ F^2_{\alpha\,(qy,fx)}(t)\})$$

$$= \phi\,(\{F^2_{\alpha\,(px,qy)}(t),\ F^2_{\alpha\,(px,px)}(t).\ F^2_{\alpha\,(qy,qy)}(t),\ F^2_{\alpha\,(px,qy)}(t),\ F^2_{\alpha\,(qy,px)}(t)\})$$

$$= \phi(\{F^2_{\alpha\,(px,qy)}(t),\ 1,\ 1,\ F^2_{\alpha\,(px,qy)}(t),\ F^2_{\alpha\,(px,qy)}(t)\})$$

$$> F^2_{\alpha\,(px,qy)}(t)$$

This is a contradiction, therefore px = qy, i.e. px = fx = qy = gy. Suppose that there is a another point z such that pz = fz then by (2) we have pz =fz = qy = gy, so px = pz and w = px = fx is the unique point of coincidence of p and f. By Lemma 2.4 w is the only common fixed point of p and f. Similarly there is a unique point z∈ X such that z =qz = gz. Thus z is a common fixed point of p, f, q and g. The uniqueness of the fixed point holds from (2).

Corollary 3.3: Let (X, F^2_α, t) be a complete Fuzzy F-Menger and let p, q, f and g be self mappings of X. Let pairs {p, f} and {q, g} be owc. If there exists h ε (0, 1) such that

$$F^2_{\alpha\ px,qy}(ht) \geq \min\{F^2_{\alpha(fx,gy)}(t),\ F^2_{\alpha(px,fx)}(t),\ F^2_{\alpha(qy,gy)}(t),\ F^2_{\alpha\ px,gy}(t)\}\ \ldots\ldots\ldots\ldots\ldots..(3)$$

for all x,y ∈ X and t > 0, then there exists a unique common fixed point of p, f, q and g.

Proof: Let the pairs {p, f} and {q, g} be owc, so there are points x, y ∈ X such that px = fx and qy = gy. We claim that px = qy. By inequality (3) we have

$$F^2_{\alpha(\ px,qy)}(ht) \geq \min\{F^2_{\alpha\,(fx,gy)}(t),\ F^2_{\alpha(\ px,fx)}(t),\ F^2_{\alpha\,(qy,gy)}(t),\ F^2_{\alpha(\ px,gy)}(t)\}$$

$$= \min\{F^2_{\alpha\,(px,qy)}(t),\ F^2_{\alpha\,(px,px)}(t),\ F^2_{\alpha(\ qy,qy)}(t),\ F^2_{\alpha(\ px,qy)}(t)\}$$

$$\geq \min\{F^2_{\alpha\,(px,qy)}(t),\ 1,\ 1,\ F^2_{\alpha(\ px,qy)}(t)\}$$

$$\geq F^2_{\alpha\,(px,qy)}(t)$$

Thus we have px = qy, i.e. px = fx = qy = gy. Suppose that there is an another point z such that pz = fz then by (3) we have pz =fz = qy = gy, so px = pz and w = px = fx is the unique point of coincidence of p and f. By Lemma 2.4 w is the only common fixed point of p and f. Similarly there is a unique point z∈ X such that z =qz = gz. Thus w is a common fixed point of p, f, q and g. The uniqueness of the fixed point holds from (3).

Corollary 3.4 Let (X, F^2_α, t) be a complete Fuzzy F-Menger space and let p, q, f and g be self mappings of X. Let pairs {p, f} and {q, g} be owc. If there exists h ε (0, 1) such that

$$F^2_{\alpha\,(px,qy)}(ht) \geq F^2_{\alpha\,(fx,gy)}(t)\ldots\ldots\ldots\ldots\ldots..(4)$$

For all x,y ∈ X and t > 0 , then there exists a unique common fixed point of p, f, q and g.

Proof: The proof follows from corollary 3.3

Theorem 3.5: Let (X, F^2_α, t) be a complete Fuzzy F-Menger space for all t∈ [0, 1]. Then self mappings f and g of X have a common fixed point in X if and only if there exists a self mapping p of X such that the following conditions are satisfied

(i) pX ⊂ gX ∩ f X,

(ii) the pairs {p, f} and {p, g} are weakly compatible,

(iii) there exists a point h ε (0, 1) such that for every x,y ∈ X and t > 0

$$F^2_{\alpha\,(px,py)}(ht) \geq \min\{F^2_{\alpha\,(px,gy)}(t),\ F^2_{\alpha\,(px,fx)}(t),\ F^2_{\alpha\,(py,gy)}(t),\ F^2_{\alpha\,(px,gy)}(t),\ F^2_{\alpha\,(fx,gy)}(t)\}\quad \ldots\ldots\ldots.(5)$$

Then p, f and g have a unique common fixed point.

Proof: Since compatible implies owc, the result follows from corollary3.3.

Theorem 3.6: Let (X, F^2_α, t) be a complete F- Menger space and let p and f be self mappings of X. Let the p and f be owc. If there exists h ε (0, 1) for all x,y ∈ X and t > 0

$$F^2_{\alpha\,(fx,fy)}(ht) \geq \alpha F^2_{\alpha\,(px,py)}(t) + \beta \min\{F^2_{\alpha\,(px,py)}(t),\ F^2_{\alpha\,(fx,px)}(t),\ F^2_{\alpha\,(fy,py)}(t),\ F^2_{\alpha\,(fy,qy)}(t)\}\ \ldots\ldots\ldots.(6)$$

for all x,y ∈ X where α,β >0, α + β >1. Then there exists a unique common fixed point of p and f.

Proof: Let the pairs { p,f } be owc, so there are points x ∈ X such that px =fx. Suppose that there exists y ∈ X for which py = fy. We claim that px = fy. By inequality (6) we have

$$F^2_{\alpha\,(fx,fy)}(ht) \geq \alpha\ F^2_{\alpha\,(px,py)}(t) + \beta \min\{F^2_{\alpha\,(px,py)}(t),\ F^2_{\alpha\,(fx,px)}(t), F^2_{\alpha\,(fy,py)}(t)\}$$

$$= \alpha\ F^2_{\alpha\,(fx,fy)}(t) + \beta \min\{F^2_{\alpha\,(fx,fy)}(t),\ F^2_{\alpha\,(fx,fx)}(t),\ F^2_{\alpha\,(fy,fy)}(t)\}$$

$$= (\alpha + \beta)\ F^2_{\alpha\,(fx,fy)}(t)$$

A contradiction, since α + β >1. Therefore fx = fy. Therefore px = py and px is unique. From lemma 2.4, p and f have a unique fixed point.

References

[1] Lj. B. ´Ciri´c and M.M. Milovanovi´c-Arandjelovi´c, Common fixed point theorem for R-weak commuting mappings in Menger spaces, J. Indian Acad. Math. **22** (2000), 199–210.

[2] V.I. Istr˘atescu and I. S˘acuiu, Fixed point theorem for contraction mappings on probabilistic metric spaces,

Rev. Roumaine Math. Pures. Appl.**18** (1973), 1375–1380.

[3] S. Kumar and R. Chugh, Common fixed point theorems using minimal commutativity and reciprocal continuity conditions in metric spaces, Sci. Math. Japan **56** (2002), 269–275.

[4] S. Kumar and B.D. Pant, A common fixed point theorem in probabilistic metric space using implicit relation, Filomat **22** : **2** (2008), 43–52.

[5] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 535–537.

[6] D. Mihet, A generalization of a contraction principle in probabilistic metric spaces, Part II, Int. J. Math. Math. Sci. **2005** (2005), 729–736.

[7] S.N. Mishra, Common fixed points of compatible mappings in PM-spaces, Math. Japan **36** (1991), 283–289.

[8] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North Holland (Amsterdam, 1983).

[9] V.M. Sehgal, Some fixed point theorems in function analysis and probability,Ph.D dissertation, Wayne State Univ. Michigan (1966).

[10] V.M. Sehgal and A.T. Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, Math. Systems Theory **6** (1972), 97–102.

[11] H. Sherwood, Complete probabilistic metric spaces, Z. wahrscheinlichkeits theorie and verw. Grebiete **20** (1971), 117–128.

[12] R. Shrivastav,S. Nath , V. Patel and V. Dhagat,weak and semi compatible maps in Fuzzy Probabilistic metric space using implicit relation, IJMA 2(6), 2011, 958-963.

[13] S.L. Singh and B.D. Pant, Common fixed point theorems in probabilistic metric spaces and extension to uniform spaces, Honam. Math. J. **6** (1984), 1–12.

[14] S.L. Singh, B.D. Pant and R. Talwar, Fixed points of weakly commuting mappings on Menger spaces, Jnanabha **23** (1993), 115–122.