Maximum Entropy Discrete Univariate Probability Distribution using Six Kapur's Measure of Entropy

Dr. Haresh R Trivedi

Associate Professor, Department of Mathematics, D B Science College, Gondia (441614) India

Abstract: ***In the present paper we consider a discrete random variate which takes a finite number of values*** $1, 2, 3 \ldots \ldots \ldots . n$ ***and find the maximum entropy probability distribution under certain conditions.***

Keywords: Kapur's measure, **Probability** Distribution, variate

1. Introduction

We shall apply Lagrange's method of undermined multipliers to maximize Kapur's entropy $(K_1, K_2, K_3, K_4, K_5, K_6)$ subject to one or more conditions of the type.

$$\sum_{l=1}^{n} P_i = 1 \text{ and } \sum_{l=1}^{m} i\, P_i l = M$$

Shannon [2] has defined measure of entropy as

$$H(p_1, p_2, p_3 \ldots \ldots \ldots) = \sum_{i=1}^{n} P_i l_n P_i \qquad (1)$$

This measure of entropy has been generalized by Kapur [1]in following manner.

$\propto > 0, \gamma > 0, M > 0, \beta > 0, \propto + \beta - 1 > 0$

$$K_1 = \frac{1 - \sum_{i=1}^{n} P_i^{\propto}}{f(\propto)}, \ldots \ldots \ldots \ldots \qquad (2)$$

$f(1) = 0 \; and f'(1) = 1$

$$K_2 = \frac{(1 - \sum_{i=1}^{n} P_i^{1/\gamma})^{\gamma}}{1 - \gamma} \ldots \ldots \ldots . \qquad (3)$$

$$K_3 = \frac{\sum_{l=1}^{n} P_i^{\propto} / \sum_{l=1}^{n} P_i^{\beta} - 1}{f(\propto)}, \ldots . \qquad (4)$$

$f(1) = 0, f'(1) = 1 \;\; \propto \neq 1$

$$K_4 = \frac{\sum_{i=1}^{n} P_i^{\propto + \beta - 1} / \sum_{i=1}^{n} P_i^{\beta}}{f(\propto)}, \ldots \qquad (5)$$

$f(1) = 0, f'(1) = 1$

$Kexp\left[logP_1^{-P_1} + logP_2^{-P_2} + \ldots \ldots \ldots \ldots . logP_n^{-P_n}\right]$

$K\left[P_1^{-P_1} P_2^{-P_2} \ldots \ldots \ldots \ldots . P_n^{-P_n} - 1\right]$

$n^{1/n} n^{1/n} \ldots \ldots . n^{1/n}$

$K[(n^1) - 1]$

$$L = K[p_i l_n p_i + p_l l_n p_2 \ldots \ldots \ldots p_n l_n p_n - 1] + (* p_1, p_2, p_3 \ldots \ldots \ldots . p_n - 1)$$

$$K_4 = (\sum p_i l_n p_i)^M \ldots \ldots \ldots \ldots \qquad (6)$$

2. Kapur [1, Chapter 2] in his famous treatise discussed various Probability Distribution for Shannon's measure of entropy, being a natural entropy under (a) No constraint and (b) when arithmetic mean alone is prescribed. The natural question arises what happens if Shannon's measure of entropy is replaced by $K_1, K_2, K_3, K_4 or K_5$? In this paper we have obtained analogous result. Kapur's results in chapter 2 becomes the particular case of own results.

3. We will discuss the case in which the discrete variate takes only a finite set of values. Let these values are $1,2,3 \ldots \ldots \ldots n$

3.1 Kapur [1] obtained the following result

If Shannon's measure $-\sum_{i=1}^{n} p_i l_n p_i$ is maximized Subject to

$$\sum_{i=1}^{n} p_i - 1 \;\; \ldots .. \qquad (7)$$

Then

$$p_1 = p_2 = p_3 = \ldots \ldots \ldots p_n = \frac{1}{n} \ldots . \qquad (8)$$

i.e. the variate follows the uniform distribution.
We will obtain following result.

If Kapur's measure K_1, K_2, K_3, K_4, K_5 are maximized subject to (7) then (8) holds, i.e. the variate follows the uniform distribution.

Proof for K_1 Our problem is to maximize

$$\frac{1 - \sum p_i^{\propto}}{f(\propto)}, \ldots \ldots \ldots \qquad .(9)$$

where $f(1) = 0 \; and f'(1) = 1$
The Lagrangian is

$$L \equiv \frac{\sum_{i=1}^{n} P_i^{\propto}}{f(\propto)} + \lambda \left[\sum_{i=1}^{n} p_i - 1\right] \ldots \ldots \qquad (10)$$

Maximizing this, $\frac{\partial L}{\partial p_1} = \frac{\partial L}{\partial p_2} = \cdots = 0$ we get

$$\frac{\propto p_i^{\propto - 1}}{f(\propto)} + \lambda = 0$$

$$\because \frac{1}{\gamma} - 1$$

$$\because p_i = \frac{[\gamma f(\propto)]^{\frac{1}{\propto} - 1}}{\propto} \ldots \ldots \ldots \qquad (11)$$

or

$$p_1 = p_2 = p_3 = \ldots\ldots\ldots = \frac{1}{n} \ldots.. \qquad (12)$$

Therefore the variate follows the uniform distribution

Proof for K_2 Our problem is to maximize $\frac{(\sum_{i=1}^{n} P_i{}^{1-r})^r}{1-\gamma}$

subject to $\sum_{i=1}^{n} p_i = 1$

The Larangian is

$$\cdot L \equiv \frac{1-(\sum P_i{}^{1-r})^r}{1-\gamma} + \gamma\left[\sum_{i=1}^{n} p_i = 1\right] \qquad \ldots(13)$$

. Maximizing this,

$$\frac{\partial L}{\partial p_1} = \frac{\partial L}{\partial p_2} = \frac{\partial L}{\partial p_3} \cdots \frac{\partial L}{\partial p_n} = 0$$

we get,

$$\frac{\frac{1}{r}(\sum_{i=1}^{n} P_i{}^{1-r})^{\frac{1}{r}-1}\frac{1}{r}P_i{}^{\frac{1}{r}-1}}{1-r} + r = 0$$

$$.P_i{}^{1/r-1} = \left[\frac{(1-r)\gamma - r^2}{\sum_{i=1}^{n} P_i{}^{1-r})^{\frac{1}{r}-1}}\right] \ldots\ldots\ldots. \qquad (14)$$

or

$$p_1 = p_2 = p_3 = \ldots\ldots\ldots p_n = \frac{1}{n} .. \qquad (15)$$

Therefore the variate follows the uniform distribution similar technique adopted for $K_3, K_4 \& K_5$.

4Kapur [4] obtained following result

Let the prescribed arithmetic mean be$m (1 < m < n)$ them if $-\sum_{i=1}^{n} p_i l_n p_i$ is maximized subject to $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{m} lp_i = m$ gives

$$p_i = ab^i i = 1,2 \ldots\ldots\ldots\ldots\ldots.n \qquad (16)$$

where$a$ and b are determined by using the constraints

$$\sum_{i=1}^{n} b^l = 1 \ldots\ldots\ldots.. \qquad (17)$$

$$\text{and } a\sum_{i=1}^{n} ib^i = m \ldots\ldots\ldots \qquad (18)$$

In this note we replace Shannon's measure of entropy by K_5. We obtain following result

If $(\sum_{i=1}^{n} p_i l_n p_i)^M$ is maximized subject to $\sum_{i=1}^{n} p_i$and$\sum_{i=1}^{n} ip_i = m$

$$\text{then} p_i = AB^i \ldots.. \qquad (19)$$

where A and B are obtained by

$$A\sum_{i=1}^{n} B^i = 1 \ldots\ldots.. \qquad (20)$$

$$\text{and} A\sum_{i=1}^{n} iB^i = m \ldots\ldots\ldots.. \qquad (21)$$

$A$and$B$ coincides with a and b defined by (17) and (18) when $M = 1$. Thus our result generalizes result of Kapur [1] by using Kapur'smeasure K_4.

Proof of our result

Subject to constraint $\sum_{i=1}^{n} p_i = 1 \& \sum_{i=1}^{n} ip_i = m$

Lagrangian is

$$.L \equiv (-\sum_{i=1}^{n} p_i l_n p_i)^M - \gamma\left[\sum_{i=1}^{n} p_i - 1\right] - \mu\left[\sum_{i=1}^{n} ip_i - m\right]$$

Maximizing this (on differentiating this practically with respect. to $p_1, p_2 \ldots$)

we get

$$M(-\sum_{i=1}^{n} p_i l_n p_i)^{M-1}\{-1 l_n p_i\} - \gamma - \mu i = 0$$

or$(l + l_n p_i) = \frac{\lambda}{F(m)} - \frac{\mu i}{F(m)}$

where$F(M) = M(\sum_{i=1}^{n} p_i l_n p_i)^{M-1}$

$$F(1) = 1$$

$$\because p_i = Exp\left[\frac{-1-\lambda}{F(m)} - \frac{\mu i}{F(m)}\right] = AB^i$$

where$A = Exp\left(\frac{-1-\lambda}{F(m)}\right)$

$$B = Exp\left(\frac{\mu i}{F(m)}\right)$$

Now under given condition $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} ip_i = m$

$$A\sum_{i=1}^{n} B^i = 1$$

and $\sum_{i=1}^{n} iB^i = m$

This establishes the result

References

[1] Kapur J.N. "Maximum Entropy Models in Science & Engineering" Wiley Eastern Limited. New Delhi ISBN: 978-0-470-21459-6 (1989)

[2] Shannon C.E. " The Mathematical Theory of Communication Bell Syatem" Tech Jour Vol. 27, 379-423

Author Profile

Haresh R. Trivedi, is an Associate Professor in Department of Mathematics completed his M.Sc and Ph.D in Mathematics, the work area is Maximum Entropy Principle, published and presented number of papers in Conferences and seminar working at D B Science College since last 32 years, Life member ISITA. (Indian Society for Information Theory and Applications)