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A New Class of Quasi-Cubic Trigonometric Bezier Curve and Surfaces

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Abstract: *A kind of quasi-cubic Bézier curves by the blending of algebraic polynomials and trigonometric polynomials using weight method is presented, named WAT Bézier curves. Here weight coefficients are also shape parameters, which are called weight parameters. The interval [0, 1] of weight parameter values can be extended to* $[-2, \pi^2/(\pi^2-6)]$ *and the corresponding WAT Bézier curves and surfaces are defined by the introduced base functions. The WAT Bézier curves inherit most of properties similar to those of c Bézier curves, and can be adjusted easily by using the shape parameter λ. The jointing conditions of two pieces of curves with* G^2 *and* C^4 *continuity are discussed. With the shape parameter chosen properly, the defined curves can express exactly any plane curves or space curves defined by parametric equation based on{1, sint, cost, sint2t, cos2t} and circular helix with high degree of accuracy without using rational form. Examples are given to illustrate that the curves and surfaces can be used as an efficient new model for geometric design in the fields of CAGD. Unlike the existing techniques based on C-Bézier methods which can approximate the Bézier curves only from single side, the WAT Bézier curves can approximate the Bézier curve from the both sides, and the change range of shape of the curves is wider than that of C-Bézier curves. The geometric effect of the alteration of this weight parameter is discussed.*

Keywords: Bézier curves and surfaces, trigonometric polynomial, quasi-quartic, shape parameter, G^2 and C^4 continuity

1. Introduction

Computer aided geometric design (CAGD) studies the construction and manipulation of curves and surfaces using polynomial, rational, piecewise polynomial or piecewise rational methods. Among many generalizations of polynomial splines, the trigonometric splines are of particular theoretical interest and practical importance. In recent years, trigonometric splines with shape parameters have gained wide spread application in particular in curve design. Bézier form of parametric curve is frequently used in CAD and CAGD applications like data fitting and font designing, because it has a concise and geometrically significant presentation. Smooth curve representation of scientific data is also of great interest in the field of data visualization. Key idea of data visualization is the graphical representation of information in a clear and effective manner. When data arises from a physical experiment, prerequisite for the interpolating curve is to incorporate the inherit feature of the data like positivity, monotonicity, and convexity. Various authors have worked in the area of shape preserving using ordinary and trigonometric rational splines [7-9].

In many problems of industrial design and manufacturing, the given data often have some special shape properties, such as positivity, monotonicity and convexity, it is usually needed to generate a smooth function, which passes through the given set of data and preserves those certain geometric shape properties of the data. In the recent past, a number of authors and references have contributed to the shape-preserving interpolation. In [1, 3, 6, 7, 12], different polynomial methods, which are used to generate the shape-preserving interpolant, have been considered. In this paper, we present a class of new different trigonometric polynomial basis functions with a parameter based on the space Ω=span {1, sin*t*, cos*t*, sint2*t*, cos2*t*}, and the corresponding curves and tensor product surfaces named quasi-quartic trigonometric Bézier curves and surfaces are constructed based on the introduced basis functions. The quasi-quartic trigonometric Bézier curves not only inherit most of the similar properties to quartic Bézier curves, but also can express any plane curves or space curves defined by parametric equation based on {1, sin*t*, cos*t*, sint2*t*, cos2*t*} including some quadratic curves such as the circular arcs, parabolas, cardioid exactly and circular helix with high degree of accuracy under the appropriate conditions.

In this paper, we present a class of quasi-cubic Bézier curves with weight parameter based on the blending space span. Also the change range of the curves is wider than that of C-Bézier curves. The paths of the given curves are line segments. Some transcendental curves can be represented by the WAT with the shape parameters and control points chosen properly. The rest of this paper is organized as follows. Section 2 defines the WAT-Bezier Base Functions and the corresponding curves and surfaces, theirs propositions are discussed. In section 3, we discussed the continuity conditions of WAT-Bezier curves. In section 4, we show the representations of some curves. Besides, some examples of shape modeling by using the WAT-Bezier Bézier surfaces are presented also. The conclusions are given in section 5.

2. WAT-Bezier Base Functions, WAT-Bezier Curves and Surfaces

2.1 The Construction of the WAT-Bézier Base Functions

Definition 2.1.1: For $0 \le \lambda \le 1$, the following four functions of $t \in [0, 1]$, are defined as WAT- Bézier basis functions

$$WAT_0(t,\lambda)=\lambda(1-t)^3+(1-\lambda)\frac{\pi(1-t)-\sin\pi t}{\pi},$$
$$WAT_1(t,\lambda)=3\lambda(1-t)^2t+(1-\lambda)\left(\frac{1}{2}+t+\frac{1}{2}\cos\pi t+\frac{\sin\pi t}{\pi}\right),$$
$$WAT_2(t,\lambda)=3\lambda(1-t)t^2+(1-\lambda)\left(\frac{1}{2}-t-\frac{1}{2}\cos\pi t+\frac{\sin\pi t}{\pi}\right),$$
$$WAT_4(t,\lambda)=\lambda t^3+(1-\lambda)\frac{\pi t-\sin\pi t}{\pi}, \quad (1)$$

Obviously, WAT-Bézier basis functions are cubic Bernstein bases when $\lambda=1$. And, when $\lambda=0$, WAT-Bézier basis functions are C-Bézier bases associated to $\alpha=\pi$.

2.1.2 The Properties of the Basis Functions

Theorem 1: The basis functions (2.1) have the following properties:

Straight calculation testifies that these WAT-Bézier bases have the properties similar to the cubic Bernstein basis as follows.

1) Properties at the endpoints:

$$WAT_0(0,\lambda)=1, WAT_i^{(j)}(0,\lambda)=0,$$
$$WAT_3(1,\lambda)=1, WAT_{i-3}^{(j)}(1,\lambda)=0$$

Where j = 0, 1, 2…i-1, i = 1, 2, 3 and

$$WAT_i^{(0)}(t,\lambda)=WAT_i(t,\lambda)$$

2) Symmetry:

$$WAT_1(t,\lambda)=WAT_2(1-t,\lambda)$$
$$WAT_0(t,\lambda)=WAT_3(1-t,\lambda)$$

3) Partition of unity:

$$\sum_{i=0}^{3}WAT_i(t,\lambda)=1$$

4) Nonnegativity:

$$WAT_i(t,\lambda)\geq 0; i=0,1,2,3.$$

According to the method of extending definition interval of C-curves in Ref., the interval [0, 1] of weight parameter values can be extended to $\left[-2,\frac{\pi^2}{\pi^2-6}\right]$

where $\frac{\pi^2}{\pi^2-6}=2.55055.$

2.2 WAT-Bézier Curves

2.2.1 The Construction of the WAT-Bézier Curves

Definition 2.2.1 Given points P_k (k = 0, 1, 2, 3) in R^2 or R^3, then

$$R(t,\lambda)=\sum_{i=0}^{3}P_i WAT_i(t,\lambda),\ t\in[0,1] \text{ for } i = 0, 1, 2, 3.$$
$$\lambda\in[-2, 2.5505], \quad (2)$$

This $R(\lambda,t)$ is called WAT-Bezier curve and $WAT_i(t,\lambda)\geq 0; i=0,1,2,3.$ are the WAT-Bezier basis.

2.2.2 The Properties of the WAT-Bezier Curve

From the definition of the basis function some properties of the WAT- Bezier curve can be obtained as follows:

Theorem 2: The WAT-Bezier curve curves (2.2..1) have the following properties:

- **Terminal Properties**

$$R(0,\lambda)=P_{0,}\ \ R(1,\lambda)=P_{4,}$$
$$R'(0,\lambda)=(\lambda+2)(P_1-P_0) \quad (3)$$
$$R'(1,\lambda)=(\lambda+2)(P_3-P_2)$$

- **Symmetry**: Assume we keep the location of control points P_i (i = 0,1,2,3,4) fixed, invert their orders, and then the obtained curve coincides with the former one with opposite directions. In fact, from the symmetry of WAT-Bezier base functions, we have

$$\mathbf{R(1-t,\lambda)}=\sum_{i=0}^{3}P_i WAT_i(1-t,\lambda)$$

$$=\sum_{i=0}^{3}P_{3-i}WAT_i(t,\lambda)=\mathbf{R(t,\lambda)};\ \mathbf{t\in[0,1]},\ \boldsymbol{\lambda}\in[-2, 2.5505],$$

- **Geometric Invariance:** The shape of a WAT-Bezier curve is independent of the choice of coordinates, i.e. (2.2.1) satisfies the following two equations:

$R(t;\lambda;P_0+q,P_1+q,P_2+q,,P_3+q)=R(1-t;\lambda;P_3,P_2,P_1,P_0)+q;$

$R(t;\lambda;P_0*T,P_1*T,P_2*T,,P_3*T)=R(1-t;\lambda;P_3,P_2,P_1,P_0)*T;$

Where q is arbitrary vector in R^2 or R^3 and T is an arbitrary d * d matrix, d = 2 or 3.

- **Convex Hull Property:** The entire WAT-Bezier curve segment lies inside its control polygon spanned by P_0,P_1,P_2,P_3.

2.3 WAT Bézier Surfaces

Definition 2.3 Given the control mesh $[\mathbf{P}_{rs}]$ (r = i ….,i+2; s = j…,j+2),(i = 0,1,…, n – 1; j = 0,1,…, m −1) , Tensor product WAT-Bézier surfaces can be defined as

$$R_{i,j}(u,v)=\sum_{r=i}^{3}\sum_{s=j}^{3} WAT_i(\lambda_1,u)WAT_i(\lambda_2,v)P_{rs}(u,v);(u,v)\in\left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{2}\right] \quad (4)$$

Where $WAT_i(\lambda_1,u)$ and $WAT_i(\lambda_2,v)$ are WAT-Bezier base function.

2.4 Shape Control Of The WAT-Bezier Curve

Due to the interval [0, 1] of weight parameter values can be extended to[−2, 2.55055], the change range of the WAT-Bézier curve is wider than that of C-Bézier. From the Figure 1, it can be seen that when the control polygon is fixed, by adjusting the weight parameter from -2 to 2.55055, the WAT-Bézier curves can cross the cubic Bézier curves and reach the both sides of cubic Bézier curves, in other words, the WAT-Bézier curves can range from below the C-Bézier curve to above the cubic Bézier curve. The weight parameters have the property of geometry. The larger the shape parameter is, and the more approach the curves to the control polygon is. Also, these WAT-Bézier curves we defined include C-Bézier curve ($\alpha=\pi$) as special cases.

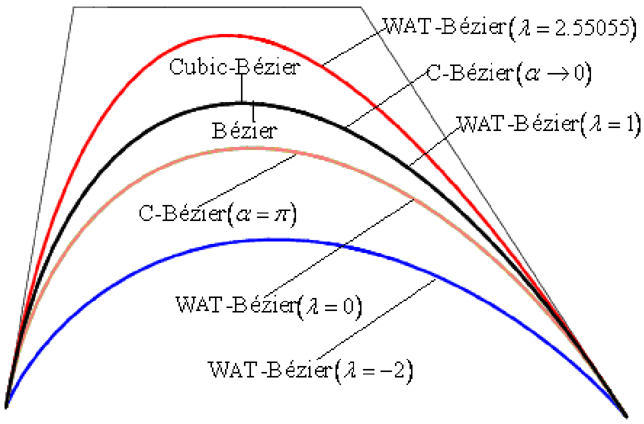

Figure 1: Adjusting WAT Bezier Curve

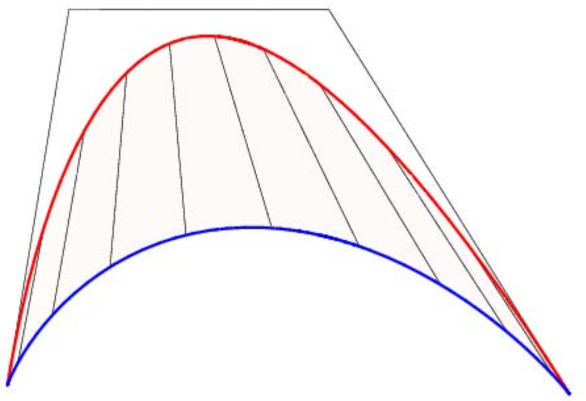

Figure 2: Paths of WAT Bezier Curve

3. Jointing of WAT- Bézier Curves

Suppose there are two segment of WAT- Bezier curves $R(t,\lambda)=\sum_{i=0}^{3} P_i WAT_i(t,\lambda)$ and $Q(t,\lambda)=\sum_{i=0}^{3} Q_i WAT_i(t,\lambda)$; where $P_3=Q_0$, parameters of $P_i(t)$ and $Q_i(t)$ are λ_1 and λ_2 respectively.

To achieve G^1 continuity of the two curve segments, it is required that not only the last control point of $\mathbf{P}_i(t)$ and the first control point of $\mathbf{Q}_i(t)$ must be the same, but also the direction of the first order derivative at jointing point should be the same, namely

$$P'(1)=kQ'(0) \quad ;(k\ge 1)$$

Substituting Eq. (3) into the above equitation, one can get

$$(2+\lambda_1)(P_3-P_2)=k(2+\lambda_2)(Q_1-Q_0)$$

Let $\delta=\dfrac{k(1+\lambda_2)}{1+\lambda_1}$, substituting it into the above equitation, then

$$(\mathbf{P}_3-\mathbf{P}_2)=\delta(\mathbf{Q}_1-\mathbf{Q}_0)\ (\delta>0)$$

Especially, for k=1, namely, $\delta=\dfrac{1+\lambda_2}{1+\lambda_1}$, the first order derivative of two segment of curves is equal .Thus,

G^1 continuity has transformed into C^1 continuity. Then we can get following theorem 3.

Theorem 3 If P_2 ,P_3 and Q_O, Q_1 is collinear and have the same directions, i.e.

$$(P_3 - P_2) = \delta (Q_1 - Q_0) \ (\delta > 0) \quad (5)$$

Then curves of $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ will reach G^1 continuity at a jointing point and when $\delta = \frac{1+\lambda_2}{1+\lambda 1}$ they will get C^1 continuity.

Then we will discuss continuity conditions of G^2 when $\lambda_1 = \lambda_2 = 1$.

First, we'll discuss conditions of G^2 continuity which is required to have common curvature, namely

$$\frac{|P'(1)\times P''(1)|}{|P''(1)|^3} = \frac{|Q'(0)\times Q''(0)|}{|Q''(0)|^3} \quad (6)$$

Let $\lambda_1 = \lambda_2 = 1$, second derivatives of two segments of curves can be get

$$P''(1) = 6P_1 - 12P_2 + 6P_3$$
$$Q''0) = 6Q_0 - 12Q_1 + 6Q_2 \quad (7)$$

Substituting Eq. (3) and (7) into Eq. (6), simplifying it, then

$$\frac{|(P_3 - P_2)\times(P_2 - P_1)|}{|P_3 - P_2|^3} = \frac{|(Q_1 - Q_0)\times(Q_2 - Q_1)|}{|Q_1 - Q_0|^3} \quad (9)$$

Substituting Eq. (5) into the above equitation, one can get

$\boldsymbol{h_1 = \delta^2 h_2}$

where h_1 is the distance from $\boldsymbol{P_1}$ to $\boldsymbol{P_2}$ $\boldsymbol{P_3}$ and h_2 is the distance from $\boldsymbol{Q_2}$ to $\boldsymbol{Q_0}$ $\boldsymbol{Q_1}$. Hence we can get theorem 4.

Theorem 4 Let parameters λ_1 , λ_2 are all equal one, if they satisfy Eq. (5) and (9), five points $\boldsymbol{P_1}$, $\boldsymbol{P_2}$, $\boldsymbol{P_3}$,$\boldsymbol{Q_1}$,$\boldsymbol{Q_2}$ are coplanar and $\boldsymbol{P_1}$, $\boldsymbol{Q_2}$ are in the same side of the common tangent, then jointing of curves $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ reach G^2 continuity.

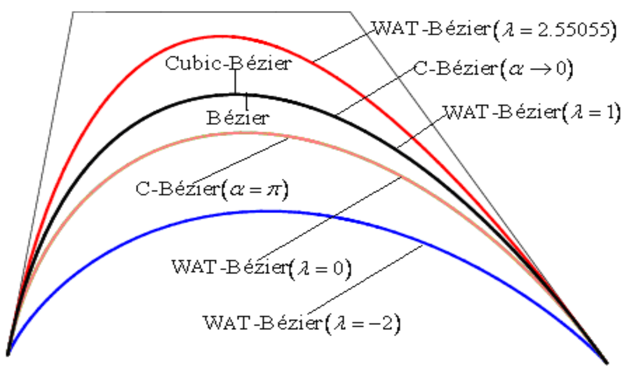

Figure 3: WAT Bezier Curves with Different Values of Shape Parameter

These curves are generated by setting $\boldsymbol{\lambda}$ =-2 in (a), $\boldsymbol{\lambda}$ =0 in (b), $\boldsymbol{\lambda}$ =-1 in (c) and $\boldsymbol{\lambda}$ =2.5505 in (d).

Next, we will discuss the conditions of C^4 continuity. When curves **P**(t) and **Q**(t) reach C^1 continuity at the linked points, i.e. $\mathbf{Q_1} - \mathbf{Q_0} = \mathbf{P_3} - \mathbf{P_2}$ and $\mathbf{Q_0} = \mathbf{P_3}$ Under such circumstances, if $P''(1) = Q''(0); P'''(1) = Q'''(0); P''''(1) = Q''''(0)$, then two curves will become C^4 continuity. Combine all the above conditions, we get

$$Q_1 = 2P_3 - P_2; Q_2 = 2P_3 - 2P_2 + P_1$$
$$Q_3 = 2P_3 - 2P_2 + 2P_1 - P_0 \quad (10)$$

Theorem 5 Let parameters λ_1 , λ_2 are all equal one and satisfy (10) in theorem 3, curves $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ will reach C^4 continuity at the linked point.

4. Applications of WAT- Bézier Curves and Surfaces

Proposition 4.1 Let $\mathbf{P_0}$, $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$ be four control points. By proper selection of coordinates, their coordinates can be written in the form

$$P_0 = (0,0), P_1 = \left(\frac{1-\pi}{2}a, o\right), P_2 = \left(\frac{1-\pi}{2}a, 2a\right),$$
$$P_3 = (a, 2a)(a \neq 0)$$

Then the corresponding WAT-Bézier curve with the weight parameters $\lambda = 0$ and $t \in [0,1]$ represents an arc of cycloid.

Proof: If we take $\mathbf{P_0}$, $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$ into (2), then the coordinates of the WAT-Bézier curve are

$x(t) = a(t - \sin \pi t)$,

$y(t) = a(1 - \cos \pi t)$.

It is a cycloid in parametric form, see Figure 4.

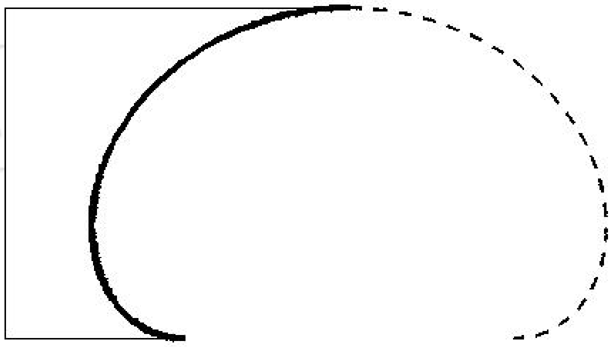

Figure 4: The Representation of Cycloid With WAT-Bézier Curve

Proposition 4.2 Let $\mathbf{P_0}$, $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$ be four properly chosen control points such that

$$P_0 = (a,0,0), P_1 = \left(0, a, \frac{\pi}{2}b\right), P_2 = \left(-a, a, \frac{\pi}{2}b\right),$$
$$P_3 = (-a, 0, b)(a \neq 0, b \neq 0)$$

Then the corresponding WAT-Bézier curve with the weight parameters $\lambda = 0$ and $t \in [0,1]$ represents an arc of a helix.

Proof: Substituting $\mathbf{P_0}$, $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$ into (2) yields the coordinates of the WAT-Bézier curve $x(t) = a\cos\pi t$, $y(t) = a\sin\pi t$, $z(t) = bt$, which is parameter equation of a helix, see Figure 5.

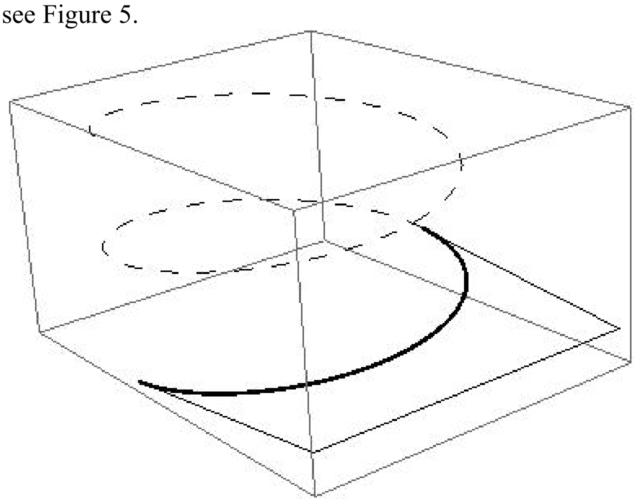

Figure 5: The Representation of Helix With WAT-Bézier Curve

Proposition 4.3 Given the following four control points, $\mathbf{P_0} = (0,0)$, $\mathbf{P_1} = \mathbf{P_2} = (a, \frac{\pi}{2}b)$, $\mathbf{P_3} = (2a, 0)$ $(ab \neq 0)$.Then the corresponding WAT-Bézier curve with the weight parameters $\lambda = 0$ and $t \in [0, 1]$ represents a segment of sine curve.

Proof: Substituting $\mathbf{P_0}$, $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$ into (2), we get the coordinates of the WAT-Bézier curve, $x(t) = at$ $y(t) = b\sin\pi t$,
which implies that the corresponding WAT-Bézier curve represents a segment of sine curve, see Figure5.

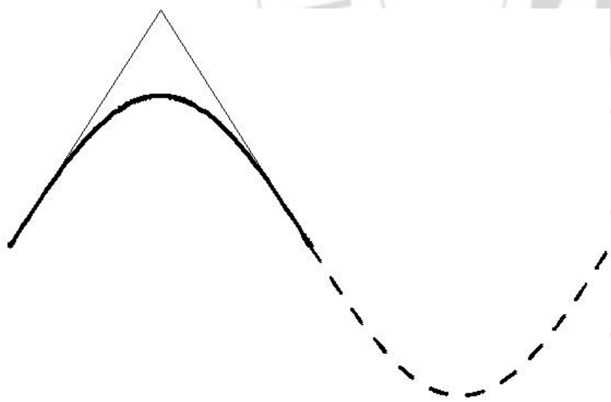

Figure 5: The Representation of Sine Curve with WAT-Bézier Curves

5. Conclusions

In this paper, the WAT-Bézier curves have the similar properties that cubic Bézier curves have. The jointing of two pieces of curves can reach G^2 and C^4 continuity under the appropriate conditions. The given curves can represent some special transcendental curves. What is more, the paths of the curves are linear, the WAT-Bézier curves have more advantages in shape adjusting than that C-Bézier curves.Both rational methods (NURBS or Rational Bézier curves) and WAT-Bézier curves can deal with both free form curves and the most important analytical shapes for the engineering. However, WAT-Bézier curves are simpler in structure and more stable in calculation. The weight parameters of WAT-Bézier curves have geometric meaning and are easier to determine than the rational weights in rational methods. Furthermore, some complex surfaces can be constructed by these basic surfaces exactly. While the method of traditional quartic Bézier curves needs joining with many patches of surface in order to satisfy the precision of users for designing. Therefore the method presented by this paper can raise the efficient of constituting surfaces and precision of representation in a large extent Meanwhile, WAT-Bézier curves can represent the helix and the cycloid precisely, but NURBS can't. Therefore, WAT-Bézier curves would be useful for engineering

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