

Degree of Approximation of a Function Belonging To Lip $(\xi(t), p)$ And $W(L_r, \xi(t))$ Class By $(C,2)$ $(E,1)$ Means of Its Fourier Series And Conjugate Fourier Series

Shobha Shukla¹, U. K. Shrivastav²

¹Department of Mathematics, Dr.C.V.Raman University, Kota, Bilaspur(C.G.), India

²Department of Mathematics, Govt.Bilasa Girls P.G.College, Bilaspur(C.G), India

Abstract: This paper introduces the concept of $(C,2)(E,1)$ product operators and establishes two new theorems on $(C,2)$ $(E,1)$ product summability of Fourier series and its conjugate Fourier series.

Keywords: $(C,2)$ operators, $(E,1)$ operators, $(C,2)(E,1)$ product operators, Lip $(\xi(t), p)$ class of functions, $W(L_r, \xi(t))$ class of function, Fourier series, conjugate Fourier series, Lebesgue integral

1. Introduction

The degree of approximation of function belonging to Lip α by Cesaro means and Norlund means has been discussed by a number of researchers like Alexits[4], Sahney and Goel[3], Chandra[7], Qureshi[6], Rhoades[2].

H.K.Nigam[5] obtained the degree of approximation of the function on $(C,2)$ $(E,1)$ product means of its Fourier series and conjugate Fourier series. In the present we have obtained the degree of approximation of function f belonging to a lip $(\xi(t), p)$ and $W(L_r, \xi(t))$ class of fourier series and its conjugate fourier series.

2. Definition and Notation

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with s_n for its n^{th} partial sum.

Let $\{t_n^{(E,1)}\}$ denote the sequence of $(E,1)$ mean of the sequence $\{s_n\}$. If the $(E,1)$ transform of s_n is defined as $t_n^{(E,1)}(f; x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} s_k(f; x) \rightarrow s$ as $n \rightarrow \infty$. (1) the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by the $(E,1)$ method.

Let $\{t_n^{(C,2)}\}$ denote the sequence of $(C,2)$ mean of the sequence $\{s_n\}$. If the $(C,2)$ transform of s_n is defined as $t_n^{(C,2)}(f; x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^n (n-k+1) s_k(f; x) \rightarrow s$ as $n \rightarrow \infty$. (2) the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by $(C,2)$ method.

Thus if $t_n^{(C,2)(E,1)}(f; x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^n (n-k+1) \frac{1}{2^k} \sum_{v=0}^k \binom{k}{v} s_v(f; x) \rightarrow s$ as $n \rightarrow \infty$. (3)

where $\{t_n^{(C,2)(E,1)}\}$ denote the sequence of $(C,2)(E,1)$ product means of the sequence s_n , the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by $(C,2)$ $(E,1)$ method.

Let f be a periodic function with period 2π and integrable in the Lebesgue sense. The Fourier series be given by $f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n(x)$ (4) with n^{th} partial sums $s_n(f; x)$.

The conjugate series of Fourier series (4) is given by $\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x)$ (5) with n^{th} partial sums $\tilde{s}_n(f; x)$.

A 2π - periodic function $f(x)$ is said to belong to the class Lip $(\psi(t), p)$, $p > 1$ if $|f(x+t) - f(x)| \leq M(\psi(t)t^{-1/p})$, $0 < t < \pi$. (6) where $\psi(t)$ is a positive increasing function and M is a positive number independent of x and t .

The degree of approximation of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ by a trigonometric polynomial t_n of degree n under super norm $\|\cdot\|_{\infty}$ is defined by $\|t_n - f\|_{\infty} = \sup\{|t_n(x) - f(x)| : x \in \mathbb{R}\}$ (7)

A function $f \in \text{Lip}\alpha$ if $|f(x+t) - f(x)| = O(|t|^\alpha)$, $0 < \alpha \leq 1$ (8)

And $f \in \text{Lip}(\alpha, p)$ if $\left\{ \int_0^{2\pi} |f(x+t) - f(x)|^p dx \right\}^{\frac{1}{p}} = O(|t|^\alpha)$, $0 < \alpha \leq 1, p \geq 1$ (9)

Given a positive increasing function $\xi(t)$ and an integer $p \geq 1$, then $f \in \text{Lip}(\xi(t), p)$ if $\left\{ \int_0^{2\pi} |f(x+t) - f(x)|^p dx \right\}^{\frac{1}{p}} = O(\xi(t))$ (10) And $f \in W(L_r, \xi(t))$ if

$$\left\{ \int_0^{2\pi} |f(x+t) - f(x)| \sin^\beta x |dx| \right\}^{\frac{1}{p}} = O(\xi(t)) \quad (11)$$

We use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$\psi(t) = f(x+t) - f(x-t)$$

$$K_n(t) = \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^n \frac{(n-k+1)}{2^k} \sum_{v=0}^k \left[\binom{k}{v} \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right]$$

$$\tilde{K}_n(t) = \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^n \frac{(n-k+1)}{2^k} \sum_{v=0}^k \left[\binom{k}{v} \frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right]$$

$$\tilde{f}(x) = -\frac{1}{2\pi} \int_0^\pi \psi(t) \cot\left(\frac{t}{2}\right) dt$$

3. Main Theorems

We prove the following theorems:

3.1 Theorem 1

Let f is a 2π -periodic function, Lebesgue integrable on $[-\pi, \pi]$ and $f \in \text{Lip}(\xi(t), p)$ then its degree of approximation by $(C,2)(E,1)$ product means on its Fourier series is given by

$$\|s_n - f\|_r = O\left\{ (n+1)\psi\left(\frac{1}{n+1}\right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\} \quad (12)$$

where s_n is $(C,2)(E,1)$ product means of Fourier series (4).

We shall use following condition:

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\psi(t)}{t^{\frac{1}{p}}} \right)^p dt \right\}^{\frac{1}{p}} = O\left\{ \psi\left(\frac{1}{n+1}\right) \right\} \quad (13)$$

And

$$\left\{ \int_{\frac{1}{n+1}}^\pi \left(\frac{\psi(t)}{t^{\frac{1}{p}+2}} \right)^q dt \right\}^{\frac{1}{q}} = O\left\{ (n+1)^2 \psi\left(\frac{1}{n+1}\right) \right\} \quad (14)$$

where $\frac{1}{p} + \frac{1}{q} = 1, 1 \leq r \leq \infty$

3.2 Theorem 2

If a function \tilde{f} , conjugate to a 2π -periodic function $\tilde{f} \in \text{Lip}(\xi(t), p)$ then its degree of approximation by $(C,2)(E,1)$ product means on its conjugate Fourier series is given by

$$\|\tilde{s}_n - \tilde{f}\|_r = O\left\{ (n+1)\psi\left(\frac{1}{n+1}\right) \right\} [(n+1)^2 + 1] \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\} \quad (15)$$

where \tilde{s}_n is $(C,2)(E,1)$ product means of conjugate Fourier series (5).

3.3 Theorem 3:

Let f is a 2π -periodic function, Lebesgue integrable on $[-\pi, \pi]$ and $f \in W(L_r, \xi(t))$ then its degree of approximation by $(C, 2)(E, 1)$ product means on its Fourier series is given by

$$\|s_n - f\|_r = O\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\} \quad (16)$$

where s_n is $(C,2)(E,1)$ product means of Fourier series (4).

We shall use following condition:

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t|\psi(t)|}{\xi(t)} \right)^p \sin^{\beta p} t dt \right\}^{\frac{1}{p}} = O\left(\frac{1}{n+1}\right) \quad (17)$$

And

$$\left\{ \int_{\frac{1}{n+1}}^\pi \left(\frac{t^{-\delta} |\psi(t)|}{\xi(t)} \right)^q dt \right\}^{\frac{1}{q}} = O\{(n+1)^\delta\} \quad (18)$$

where $\left(\frac{\xi(t)}{t}\right)$ is non-increasing in t , and δ is an arbitrary

positive number such that $s(1-\delta) - 1 > 0, \frac{1}{p} + \frac{1}{q} = 1,$

$1 \leq r \leq \infty$

3.4 Theorem 4: If a function \tilde{f} , conjugate to a 2π -periodic function $\tilde{f} \in W(L_r, \xi(t))$ then its degree of approximation by $(C,2)(E,1)$ product means on its conjugate Fourier series is given by

$$\|\tilde{s}_n - \tilde{f}\|_r = O\left\{ (n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right) \right\} \quad (19)$$

where \tilde{s}_n is $(C,2)(E,1)$ product means of conjugate Fourier series (5).

4. Lemma

For the proof of our theorems following lemmas (see Nigam[5]) are required.

4.1 Lemma 1

For $0 \leq t \leq \frac{1}{n+1}, \sin nt \leq nsint$

$$|K_n(t)| = O(n+1)$$

4.2 Lemma 2

For $\frac{1}{n+1} \leq t \leq \pi, \sin \frac{t}{2} \geq \frac{t}{\pi}$ and $\sin nt \leq 1$

$$|K_n(t)| = O\left(\frac{1}{t}\right)$$

4.3 Lemma 3

For $0 \leq t \leq \frac{1}{n+1}, \sin \frac{t}{2} \geq \frac{t}{\pi}$ and $|\cos nt| \leq 1$

$$|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)$$

4.4 Lemma 4

For $0 \leq a \leq b \leq \infty, 0 \leq t \leq \pi$ and any n

$$|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)$$

5. Proof of Main Theorems

5.1 Proof of Theorem 1

The n^{th} partial sum $s_n(f; x)$ of series (4) is given by

$$s_n(f; x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt$$

The $(E,1)$ transform of $s_n(f; x)$ is given by

$$t_n^{(E,1)} - f(x) = \frac{1}{\pi 2^{n+1}} \int_0^\pi \phi(t) \left\{ \sum_{k=0}^n \binom{n}{k} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\} dt$$

The $(C,2)(E,1)$ transform of $s_n(f; x)$ is given by

$$\begin{aligned} t_n^{(C,2)(E,1)} - f(x) &= \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^n \frac{(n-k+1)}{2^k} \int_0^\pi \frac{\phi(t)}{\sin\frac{t}{2}} \left[\sum_{v=0}^k \binom{k}{v} \sin\left(v+\frac{1}{2}\right)t \right] dt \\ &= \int_0^\pi \phi(t) K_n(t) dt \end{aligned}$$

$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right] \phi(t) K_n(t) dt$$

$$= O(I_1) + O(I_2) \tag{20}$$

Now

$$I_1 = \int_0^{\frac{1}{n+1}} \phi(t) K_n(t) dt$$

$$|I_1| \leq \int_0^{\frac{1}{n+1}} |\phi(t)| |K_n(t)| dt$$

By lemma 1

$$I_1 = O(n+1) \left\{ \int_0^{\frac{1}{n+1}} |\phi(t)| |1| dt \right\}$$

$$= O(n+1) \left\{ \int_0^{\frac{1}{n+1}} |\phi(t)| dt \right\} \left\{ \int_0^{\frac{1}{n+1}} 1 dt \right\}$$

$$= O(n+1) \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\psi(t)}{t^{\frac{1}{p}}} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{1}{n+1}} (1)^q dt \right\}^{\frac{1}{q}}$$

$$= O(n+1) O \left\{ \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\}$$

$$= O \left\{ (n+1) \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\} \tag{21}$$

Next

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} \phi(t) K_n(t) dt$$

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| |K_n(t)| dt$$

By lemma 2

$$I_2 = \left\{ \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| O \left(\frac{1}{t} \right) dt \right\}$$

$$= \left\{ \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| dt \right\} \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t} \right) dt \right\}$$

$$= \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\psi(t)}{t^{\frac{1}{p}}} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t} \right)^q dt \right\}^{\frac{1}{q}}$$

$$= \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\psi(t)}{t^{\frac{1}{p}+2}} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n+1}}^{\pi} (t)^q dt \right\}^{\frac{1}{q}}$$

$$= O \left\{ (n+1)^2 \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{q+1}} \right\}^{\frac{1}{q}}$$

$$= O \left\{ (n+1)^2 \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{1+\frac{1}{q}}} \right\}$$

$$= O \left\{ (n+1) \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\} \tag{22}$$

Combining (20), (21) and (22)

$$s_n(f; x) - f(x) = O \left\{ (n+1) \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\} +$$

$$O \left\{ (n+1) \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\}$$

$$= O \left\{ (n+1) \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\}$$

This completes the proof of the theorem 1.

5.2 Proof of Theorem 2

The n^{th} partial sum $\tilde{s}_n(f; x)$ of series (5) is given by

$$\tilde{s}_n(f; x) - \tilde{f}(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin\left(\frac{n+\frac{1}{2}}{2}\right)t}{\sin\frac{t}{2}} dt$$

The $(E, 1)$ transform of $\tilde{s}_n(f; x)$ is given by

$$t_n^{(E,1)} - \tilde{f}(x) = \frac{1}{\pi 2^{n+1}} \int_0^{\pi} \phi(t) \left\{ \sum_{k=0}^n \binom{n}{k} \frac{\sin\left(\frac{k+\frac{1}{2}}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right\} dt$$

The $(C, 2)(E, 1)$ transform of $\tilde{s}_n(f; x)$ is given by

$$t_n^{(C,2)(E,1)} - \tilde{f}(x) =$$

$$\frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^n \left\{ \frac{(n-k+1)}{2k} \int_0^{\pi} \frac{\phi(t)}{\sin\frac{t}{2}} \left[\sum_{v=0}^k \binom{k}{v} \sin\left(v + \frac{1}{2}\right)t \right] dt \right\}$$

$$= \int_0^{\pi} \phi(t) \tilde{K}_n(t) dt$$

$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right] \phi(t) \tilde{K}_n(t) dt$$

$$= O(I_3) + O(I_4) \tag{23}$$

Now

$$I_3 = \int_0^{\frac{1}{n+1}} \phi(t) \tilde{K}_n(t) dt$$

$$|I_3| \leq \int_0^{\frac{1}{n+1}} |\phi(t)| |\tilde{K}_n(t)| dt$$

By lemma 3

$$I_3 = \left\{ \int_0^{\frac{1}{n+1}} |\phi(t)| O \left(\frac{1}{t} \right) dt \right\}$$

$$= \left\{ \int_0^{\frac{1}{n+1}} |\phi(t)| dt \right\} \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{1}{t} \right) dt \right\}$$

$$= \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\psi(t)}{t^{\frac{1}{p}}} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{1}{t} \right)^q dt \right\}^{\frac{1}{q}}$$

$$= O \left\{ \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{-q+1}} \right\}^{\frac{1}{q}}$$

$$= O \left\{ (n+1)^2 \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{-1+\frac{1}{q}}} \right\}$$

$$= O \left\{ (n+1)^3 \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\} \tag{24}$$

Next

$$I_4 = \int_{\frac{1}{n+1}}^{\pi} \phi(t) \tilde{K}_n(t) dt$$

$$|I_4| \leq \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| |\tilde{K}_n(t)| dt$$

By lemma 4

$$I_4 = \left\{ \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| O \left(\frac{1}{t} \right) dt \right\}$$

$$= \left\{ \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| dt \right\} \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t} \right) dt \right\}$$

$$= \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\psi(t)}{t^{\frac{1}{p}}} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t} \right)^q dt \right\}^{\frac{1}{q}}$$

$$= \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{\psi(t)}{t^{\frac{1}{p}+2}} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n+1}}^{\pi} (t)^q dt \right\}^{\frac{1}{q}}$$

$$= O \left\{ (n+1)^2 \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{q+1}} \right\}^{\frac{1}{q}}$$

$$\begin{aligned}
 &= O \left\{ (n+1)^2 \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{1+\frac{1}{q}}} \right\} \\
 &= O \left\{ (n+1) \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\} \quad (25)
 \end{aligned}$$

Combining (23), (24) and (25)

$$\begin{aligned}
 \tilde{s}_n(x) - \tilde{f}(x) &= O \left\{ (n+1)^3 \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\} + \\
 &O \left\{ (n+1) \psi \left(\frac{1}{n+1} \right) \right\} \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\} \\
 &= O \left\{ (n+1) \psi \left(\frac{1}{n+1} \right) \right\} [(n+1)^2 + 1] \left\{ \frac{1}{(n+1)^{\frac{1}{q}}} \right\}
 \end{aligned}$$

This completes the proof of the theorem 2.

5.3 Proof of Theorem 3

The n^{th} partial sum $s_n(f; x)$ of series (4) is given by

$$\begin{aligned}
 s_n(f; x) - f(x) &= \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin\left(\frac{n+\frac{1}{2}}{2}t\right)}{\sin\frac{t}{2}} dt \\
 \text{The } (E,1) \text{ transform of } s_n(f; x) \text{ is given by} \\
 t_n^{(E,1)} - f(x) &= \frac{1}{\pi 2^{n+1}} \int_0^\pi \phi(t) \left\{ \sum_{k=0}^n \binom{n}{k} \frac{\sin\left(\frac{k+\frac{1}{2}}{2}t\right)}{\sin\left(\frac{t}{2}\right)} \right\} dt \\
 \text{The } (C,2)(E,1) \text{ transform of } s_n(f; x) \text{ is given by} \\
 t_n^{(C,2)(E,1)} - f(x) &= \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^n \left\{ \frac{(n-k+1)}{2^k} \int_0^\pi \frac{\phi(t)}{\sin\frac{t}{2}} \left[\sum_{v=0}^k \binom{k}{v} \sin\left(v + \frac{1}{2}\right)t \right] dt \right\} \\
 &= \int_0^\pi \phi(t) K_n(t) dt \\
 &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^\pi \right] \phi(t) K_n(t) dt \\
 &= O(I_5) + O(I_6) \quad (26)
 \end{aligned}$$

Now

$$\begin{aligned}
 I_5 &= \int_0^{\frac{1}{n+1}} \phi(t) K_n(t) dt \\
 |I_5| &\leq \int_0^{\frac{1}{n+1}} |\phi(t)| |K_n(t)| dt \\
 &= \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)| \sin^\beta t}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\xi(t)|K_n(t)|}{t \sin^\beta t} \right)^q dt \right\}^{\frac{1}{q}} \\
 \text{By lemma 1} \\
 &= O \left(\frac{1}{n+1} \right) O(n+1) \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\xi(t)}{t^{1+\beta}} \right)^q dt \right\}^{\frac{1}{q}} \\
 &= O(1) \xi \left(\frac{1}{n+1} \right) \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{dt}{t^{(1+\beta)q}} \right) \right\}^{\frac{1}{q}} \\
 &= O \left\{ \xi \left(\frac{1}{n+1} \right) \right\} \left[\frac{t^{-(1+\beta)q+1}}{-(1+\beta)q+1} \right]_0^{\frac{1}{n+1}} \\
 &= O \left\{ \xi \left(\frac{1}{n+1} \right) \right\} \{ (n+1)^{(1+\beta)q-1} \}^{\frac{1}{q}} \\
 &= O \left\{ \xi \left(\frac{1}{n+1} \right) \right\} \{ (n+1)^{(1+\beta)-\frac{1}{q}} \} \\
 &= O \left\{ (n+1)^{\beta+\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) \right\} \quad (27)
 \end{aligned}$$

Next

$$\begin{aligned}
 I_6 &= \int_{\frac{1}{n+1}}^\pi \phi(t) K_n(t) dt \\
 |I_6| &\leq \int_{\frac{1}{n+1}}^\pi |\phi(t)| |K_n(t)| dt \\
 &= \left\{ \int_{\frac{1}{n+1}}^\pi \left(\frac{t^{-\delta} |\phi(t)| \sin^\beta t}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n+1}}^\pi \left(\frac{\xi(t)|K_n(t)|}{t^{-\delta} \sin^\beta t} \right)^q dt \right\}^{\frac{1}{q}}
 \end{aligned}$$

By lemma 2

$$\begin{aligned}
 &= \left\{ \int_{\frac{1}{n+1}}^\pi \left(\frac{t^{-\delta} |\phi(t)|}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n+1}}^\pi \left(\frac{\xi(t)}{t^{-\delta+\beta+1}} \right)^q dt \right\}^{\frac{1}{q}} \\
 &= O \left\{ (n+1)^\delta \right\} \left\{ \int_{\frac{1}{n+1}}^\pi \left(\frac{\xi(t)}{t^{(-\delta+\beta+1)q}} \right) dt \right\}^{\frac{1}{q}} \\
 &= O \left\{ (n+1)^\delta \right\} \xi \left(\frac{1}{n+1} \right) \left\{ \int_{\frac{1}{n+1}}^\pi \left(\frac{dt}{t^{(-\delta+\beta+1)q}} \right) \right\}^{\frac{1}{q}} \\
 &= O \left\{ (n+1)^\delta \xi \left(\frac{1}{n+1} \right) \right\} \left[\frac{t^{-q(-\delta+\beta+1)+1}}{-q(-\delta+\beta+1)+1} \right]_{\frac{1}{n+1}}^\pi \\
 &= O \left\{ (n+1)^\delta \xi \left(\frac{1}{n+1} \right) \right\} \{ (n+1)^{q(-\delta+\beta+1)-1} \}^{\frac{1}{q}} \\
 &= O \left\{ (n+1)^\delta \xi \left(\frac{1}{n+1} \right) \right\} \{ (n+1)^{(-\delta+\beta+1)-\frac{1}{q}} \} \\
 &= O \left\{ (n+1)^{\beta+\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) \right\} \quad (28)
 \end{aligned}$$

Combining (26), (27) and (28)

$$\begin{aligned}
 s_n(f; x) - f(x) &= O \left\{ (n+1)^{\beta+\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) \right\} + O \left\{ (n+1)^{\beta+\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) \right\} \\
 &= O \left\{ (n+1)^{\beta+\frac{1}{p}} \xi \left(\frac{1}{n+1} \right) \right\}
 \end{aligned}$$

This completes the proof of the theorem 3.

5.4 Proof of Theorem 4

The n^{th} partial sum $\tilde{s}_n(f; x)$ of series (5) is given by

$$\begin{aligned}
 \tilde{s}_n(f; x) - \tilde{f}(x) &= \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin\left(\frac{n+\frac{1}{2}}{2}t\right)}{\sin\frac{t}{2}} dt \\
 \text{The } (E,1) \text{ transform of } \tilde{s}_n(f; x) \text{ is given by} \\
 t_n^{(E,1)} - \tilde{f}(x) &= \frac{1}{\pi 2^{n+1}} \int_0^\pi \phi(t) \left\{ \sum_{k=0}^n \binom{n}{k} \frac{\sin\left(\frac{k+\frac{1}{2}}{2}t\right)}{\sin\left(\frac{t}{2}\right)} \right\} dt \\
 \text{The } (C,2)(E,1) \text{ transform of } \tilde{s}_n(f; x) \text{ is given by} \\
 t_n^{(C,2)(E,1)} - \tilde{f}(x) &= \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^n \left\{ \frac{(n-k+1)}{2^k} \int_0^\pi \frac{\phi(t)}{\sin\frac{t}{2}} \left[\sum_{v=0}^k \binom{k}{v} \sin\left(v + \frac{1}{2}\right)t \right] dt \right\} \\
 &= \int_0^\pi \phi(t) \tilde{K}_n(t) dt \\
 &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^\pi \right] \phi(t) \tilde{K}_n(t) dt \\
 &= O(I_7) + O(I_8) \quad (29)
 \end{aligned}$$

Now

$$\begin{aligned}
 I_7 &= \int_0^{\frac{1}{n+1}} \phi(t) \tilde{K}_n(t) dt \\
 |I_7| &\leq \int_0^{\frac{1}{n+1}} |\phi(t)| |\tilde{K}_n(t)| dt \\
 &= \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)| \sin^\beta t}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\xi(t)|\tilde{K}_n(t)|}{t \sin^\beta t} \right)^q dt \right\}^{\frac{1}{q}} \\
 \text{By lemma 3} \\
 &= O \left(\frac{1}{n+1} \right) \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{\xi(t)}{t^{2+\beta}} \right)^q dt \right\}^{\frac{1}{q}} \\
 &= O \left(\frac{1}{n+1} \right) \xi \left(\frac{1}{n+1} \right) \left\{ \int_0^{\frac{1}{n+1}} \left(\frac{dt}{t^{(2+\beta)q}} \right) \right\}^{\frac{1}{q}} \\
 &= O \left\{ \left(\frac{1}{n+1} \right) \xi \left(\frac{1}{n+1} \right) \right\} \left[\frac{t^{-(2+\beta)q+1}}{-(2+\beta)q+1} \right]_0^{\frac{1}{n+1}}
 \end{aligned}$$

$$\begin{aligned}
 &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\{(n+1)^{(2+\beta)q-1}\}^{\frac{1}{q}} \\
 &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\{(n+1)^{(2+\beta)-\frac{1}{q}}\} \\
 &= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} \tag{30}
 \end{aligned}$$

[8] S. Lal and J. Kushwaha, "Degree of approximation of lipschitz function by product summability method", Int.Math.Forum, 4, 2009, No.43, 2101-2107.

Next

$$\begin{aligned}
 I_8 &= \int_{\frac{1}{n+1}}^{\pi} \phi(t)\tilde{K}_n(t)dt \\
 |I_8| &\leq \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||\tilde{K}_n(t)|dt \\
 &= \left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|\sin^{\beta}t}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} \left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)|K_n(t)|}{t^{-\delta}\sin^{\beta}t}\right)^q dt\right\}^{\frac{1}{q}}
 \end{aligned}$$

By lemma 4

$$\begin{aligned}
 &= \left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} \left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)}{t^{-\delta+\beta+1}}\right)^q dt\right\}^{\frac{1}{q}} \\
 &= O\{(n+1)^{\delta}\}\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)}{t^{(-\delta+\beta+1)q}}\right) dt\right\}^{\frac{1}{q}} \\
 &= O\{(n+1)^{\delta}\}\xi\left(\frac{1}{n+1}\right)\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{dt}{t^{(-\delta+\beta+1)q}}\right)\right\}^{\frac{1}{q}} \\
 &= O\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\}\left[\frac{t^{-q(-\delta+\beta+1)+1}}{-q(-\delta+\beta+1)+1}\right]_{\frac{1}{n+1}}^{\pi}\right]^{\frac{1}{q}} \\
 &= O\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\}\{(n+1)^{q(-\delta+\beta+1)-1}\}^{\frac{1}{q}} \\
 &= O\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\}\{(n+1)^{(-\delta+\beta+1)-\frac{1}{q}}\} \\
 &= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} \tag{31}
 \end{aligned}$$

Combining (29), (30) and (31)

$$\begin{aligned}
 s_n(f; x) - f(x) &= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} + O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} \\
 &= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}
 \end{aligned}$$

This completes the proof of the theorem 4.

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