Degree of Approximation of a Function Belonging To Lip $(\xi(t), p)$ And $W\left(L_r, \xi(t)\right)$ Class By (C,2) (E,1) Means of Its Fourier Series And Conjugate Fourier Series

Shobha Shukla[1], U. K. Shrivastav[2]

[1]Department of Mathematics, Dr.C.V.Raman University, Kota, Bilaspur(C.G.), India

[2]Department of Mathematics, Govt.Bilasa Girls P.G.College, Bilaspur(C.G), India

Abstract: ***This paper introduces the concept of (C,2)(E,1) product operators and establishes two new theorems on (C,2) (E,1) product summability of Fourier series and its conjugate Fourier series.***

Keywords: (C,2) operators, (E,1) operators, (C,2)(E,1) product operators, Lip $(\xi(t), p)$ class of functions, $W\left(L_r, \xi(t)\right)$ class of function, Fourier series, conjugate Fourier series, Lebesgue integral

1. Introduction

The degree of approximation of function belonging to Lipα by Cesaro means and Norlund means has been discussed by a number of researchers like Alexits[4], Sahney and Goel[3], Chandra[7], Qureshi[6], Rhoades[2].

H.K.Nigam[5] obtained the degree of approximation of the function on (C,2) (E,1) product means of its Fourier series and conjugate Fourier series. In the present we have obtained the degree of approximation of function f belonging to a lip $(\xi(t), p)$ and $W\left(L_r, \xi(t)\right)$ class of fourier series and its conjugate fourier series.

2. Definition and Notation

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with s_n for its n[th] partial sum.

Let $\left\{t_n^{(E,1)}\right\}$ denote the sequence of $(E,1)$ mean of the sequence $\{s_n\}$. If the $(E,1)$ transform of s_n is defined as

$$t_n^{(E,1)}(f;x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k(f;x) \to s \text{ as } n \to \infty. \qquad (1)$$

the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by the $(E,1)$ method.

Let $\left\{t_n^{(C,2)}\right\}$ denote the sequence of $(C,2)$ mean of the sequence $\{s_n\}$. If the $(C,2)$ transform of s_n is defined as

$$t_n^{(C,2)}(f;x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)s_k(f;x) \to s \text{ as } n \to \infty. \qquad (2)$$

the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by $(C,2)$ method.

Thus if

$$t_n^{(C,2)(E,1)}(f;x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}s_v(f;x) \to s \text{ as } n \to \infty. \qquad (3)$$

where $\{t_n^{(C,2)(E,1)}\}$ denote the sequence of $(C,2)(E,1)$ product means of the sequence s_n, the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number s by $(C,2)$ $(E,1)$ method.

Let f be a periodic function with period 2π and integrable in the Lebesgue sense. The Fourier series be given by

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} \mathrm{A_n}(x) \qquad (4)$$

with n[th] partial sums $s_n(f;x)$.

The conjugate series of Fourier series (4) is given by

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} \mathrm{B_n}(x) \qquad (5)$$

with n[th] partial sums $\tilde{s}_n(f;x)$.

A 2π - periodic function $f(x)$ is said to belong to the class $\mathrm{Lip}(\psi(t), p),\ p > 1$ if

$$|f(x+t) - f(x)| \le M(\psi(t)t^{-1/p}),\ 0 < t < \pi. \qquad (6)$$

where $\psi(t)$ is a positive increasing function and M is a positive number independent of x and t.

The degree of approximation of a function f: R→R by a trigonometric polynomial t_n of degree n under super norm $\|\ \|_\infty$ is defined by

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in \mathrm{R}\} \qquad (7)$$

A function $f \in$ Lipα if

$$|f(x+t) - f(x)| = O(|t|^\alpha),\quad 0 < \alpha \le 1 \qquad (8)$$

And $f \in \mathrm{Lip}(\alpha, p)$ if

$$\left\{\int_0^{2\pi}|f(x+t) - f(x)|^p\,dx\right\}^{\frac{1}{p}} = O(|t|^\alpha),\ 0 < \alpha \le 1,\ p \ge 1 \qquad (9)$$

Given a positive increasing function $\xi(t)$ and an integer $p \ge 1$, then $f \in$ Lip $(\xi(t), p)$ if

$$\left\{\int_0^{2\pi}|f(x+t) - f(x)|^p\,dx\right\}^{\frac{1}{p}} = O\left(\xi(t)\right) \qquad (10)$$

And $f \in W\left(L_r, \xi(t)\right)$ if

$$\left\{\int_0^{2\pi}\left|\{f(x+t)-f(x)\}sin^{\beta}x\right|^p dx\right\}^{\frac{1}{p}} = O(\xi(t)) \qquad (11)$$

We use the following notations:

$\phi(t) = f(x+t) + f(x-t) - 2f(x)$

$\psi(t) = f(x+t) - f(x-t)$

$$K_n(t) = \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left\{\frac{(n-k+1)}{2^k}\sum_{v=0}^{k}\left[\binom{k}{v}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right]\right\}$$

$$\tilde{K}_n(t) = \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left\{\frac{(n-k+1)}{2^k}\sum_{v=0}^{k}\left[\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right]\right\}$$

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\left(\frac{t}{2}\right)dt$$

3. Main Theorems

We prove the following theorems:

3.1 Theorem 1

Let f is a 2π-periodic function, Lebesgue integrable on $[-\pi,\pi]$ and $f \in \text{Lip}\,(\xi(t),p)$ then its degree of approximation by $(C,2)$ $(E,1)$ product means on its Fourier series is given by

$$||s_n - f||_r = O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\} \qquad (12)$$

where s_n is $(C,2)(E,1)$ product means of Fourier series (4).
We shall use following condition:

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\psi(t)}{t^{\frac{1}{p}}}\right)^p dt\right\}^{\frac{1}{p}} = O\left\{\psi\left(\frac{1}{n+1}\right)\right\} \qquad (13)$$

And

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\psi(t)}{t^{\frac{1}{p}+2}}\right)^q dt\right\}^{\frac{1}{q}} = O\left\{(n+1)^2\psi\left(\frac{1}{n+1}\right)\right\} \qquad (14)$$

where $\frac{1}{p}+\frac{1}{q} = 1, 1 \le r \le \infty$

3.2 Theorem 2

If a function $\tilde{f}$, conjugate to a 2π-periodic function $\tilde{f} \in \text{Lip}\,(\xi(t),p)$ then its degree of approximation by $(C,2)(E,1)$ product means on its conjugate Fourier series is given by

$$||\tilde{s}_n - \tilde{f}||_r = O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}[(n+1)^2+1]\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\}$$
(15)

where $\tilde{s}_n$ is $(C,2)(E,1)$ product means of conjugate Fourier series (5).

3.3 Theorem 3:

Let f is a 2π-periodic function, Lebesgue integrable on $[-\pi,\pi]$ and $f \in W(L_r,\xi(t))$ then its degree of approximation by $(C, 2)$ $(E, 1)$ product means on its Fourier series is given by

$$||s_n - f||_r = O\left\{(n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right\} \qquad (16)$$

where s_n is $(C,2)$ $(E,1)$ product means of Fourier series (4).
We shall use following condition:

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^p sin^{\beta p}t\,dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{n+1}\right) \qquad (17)$$

And

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^q dt\right\}^{\frac{1}{q}} = O\{(n+1)^{\delta}\} \qquad (18)$$

where $\left(\frac{\xi(t)}{t}\right)$ is non-increasing in t, and δ is an arbitrary positive number such that $s(1-\delta)-1>0$, $\frac{1}{p}+\frac{1}{q}=1$, $1 \le r \le \infty$

3.4 Theorem 4: If a function $\tilde{f}$, conjugate to a 2π-periodic function $\tilde{f} \in W(L_r,\xi(t))$ then its degree of approximation by $(C,2)$ $(E,1)$ product means on its conjugate Fourier series is given by

$$||\tilde{s}_n - \tilde{f}||_r = O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} \qquad (19)$$

where $\tilde{s}_n$ is $(C,2)$ $(E,1)$ product means of conjugate Fourier series (5).

4. Lemma

For the proof of our theorems following lemmas (see Nigam[5]) are required.

4.1 Lemma 1

For $0 \le t \le \frac{1}{n+1}$, $sinnt \le nsint$

$|K_n(t)| = O(n+1)$

4.2 Lemma 2

For $\frac{1}{n+1} \le t \le \pi$, $sin\frac{t}{2} \ge \frac{t}{\pi}$ and $sinnt \le 1$

$|K_n(t)| = O\left(\frac{1}{t}\right)$

4.3 Lemma 3

For $0 \le t \le \frac{1}{n+1}$, $sin\frac{t}{2} \ge \frac{t}{\pi}$ and $|cosnt| \le 1$

$|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)$

4.4 Lemma 4

For $0 \le a \le b \le \infty$, $0 \le t \le \pi$ and any n

$|\tilde{K}_n(t)| = O\left(\frac{1}{t}\right)$

5. Proof of Main Theorems

5.1 Proof of Theorem 1

The n[th] partial sum $s_n(f;x)$ of series (4) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{sin\frac{t}{2}}dt$$

The $(E,1)$ transform of $s_n(f;x)$ is given by

$$t_n^{(E,1)} - f(x) = \frac{1}{\pi 2^{n+1}}\int_0^{\pi}\phi(t)\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt$$

The $(C,2)(E,1)$ transform of $s_n(f;x)$ is given by

$$t_n^{(C,2)(E,1)} - f(x) =$$
$$\frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left\{\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi(t)}{sin\frac{t}{2}}\left[\sum_{v=0}^{k}\binom{k}{v}\sin\left(v+\frac{1}{2}\right)t\right]\text{dt}\right\}$$
$$= \int_0^{\pi}\phi(t)K_n(t)dt$$

$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)dt$$
$$= O(I_1) + O(I_2) \tag{20}$$

Now

$I_1 = \int_0^{\frac{1}{n+1}} \phi(t)K_n(t)dt$

$|I_1| \le \int_0^{\frac{1}{n+1}}|\phi(t)||K_n(t)|dt$

By lemma 1

$$I_1 = O(n+1)\left\{\int_0^{\frac{1}{n+1}}|\phi(t)||1|dt\right\}$$
$$= O(n+1)\left\{\int_0^{\frac{1}{n+1}}|\phi(t)|dt\right\}\left\{\int_0^{\frac{1}{n+1}}1dt\right\}$$
$$= O(n+1)\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\psi(t)}{t^{\frac{1}{p}}}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n+1}}(1)^q dt\right\}^{\frac{1}{q}}$$
$$= O(n+1)\, O\left\{\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\}$$
$$= O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\} \tag{21}$$

Next

$I_2 = \int_{\frac{1}{n+1}}^{\pi} \phi(t)K_n(t)dt$

$|I_2| \le \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||K_n(t)|dt$

By lemma 2

$$I_2 = \left\{\int_{\frac{1}{n+1}}^{\pi}|\phi(t)|O\left(\frac{1}{t}\right)dt\right\}$$
$$= \left\{\int_{\frac{1}{n+1}}^{\pi}|\phi(t)|dt\right\}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{1}{t}\right)dt\right\}$$
$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\psi(t)}{t^{\frac{1}{p}}}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{1}{t}\right)^q dt\right\}^{\frac{1}{q}}$$
$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\psi(t)}{t^{\frac{1}{p}+2}}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}(t)^q dt\right\}^{\frac{1}{q}}$$
$$= O\left\{(n+1)^2\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{q+1}}\right\}^{\frac{1}{q}}$$
$$= O\left\{(n+1)^2\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{1+\frac{1}{q}}}\right\}$$
$$= O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\} \tag{22}$$

Combining (20), (21) and (22)

$$s_n(f;x) - f(x) = O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\} + O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\}$$
$$= O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\}$$

This completes the proof of the theorem 1.

5.2 Proof of Theorem 2

The nth partial sum $\tilde{s}_n(f;x)$ of series (5) is given by

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

The $(E,1)$ transform of $\tilde{s}_n(f;x)$ is given by

$$t_n^{(E,1)} - \tilde{f}(x) = \frac{1}{\pi 2^{n+1}}\int_0^{\pi}\phi(t)\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt$$

The $(C,2)(E,1)$ transform of $\tilde{s}_n(f;x)$ is given by

$$t_n^{(C,2)(E,1)} - \tilde{f}(x) = \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left\{\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\left[\sum_{v=0}^{k}\binom{k}{v}\sin\left(v+\frac{1}{2}\right)t\right]\mathrm{d}t\right\}$$
$$= \int_0^{\pi}\phi(t)\tilde{K}_n(t)dt$$
$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)\tilde{K}_n(t)dt$$
$$= O(I_3) + O(I_4) \tag{23}$$

Now

$I_3 = \int_0^{\frac{1}{n+1}}\phi(t)\tilde{K}_n(t)dt$

$|I_3| \le \int_0^{\frac{1}{n+1}}|\phi(t)||\tilde{K}_n(t)|dt$

By lemma 3

$$I_3 = \left\{\int_0^{\frac{1}{n+1}}|\phi(t)|O\left(\frac{1}{t}\right)dt\right\}$$
$$= \left\{\int_0^{\frac{1}{n+1}}|\phi(t)|dt\right\}\left\{\int_0^{\frac{1}{n+1}}\left(\frac{1}{t}\right)dt\right\}$$
$$= \left\{\int_0^{\frac{1}{n+1}}\left(\frac{\psi(t)}{t^{\frac{1}{p}}}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n+1}}\left(\frac{1}{t}\right)^q dt\right\}^{\frac{1}{q}}$$
$$= O\left\{\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{-q+1}}\right\}^{\frac{1}{q}}$$
$$= O\left\{(n+1)^2\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{-1+\frac{1}{q}}}\right\}$$
$$= O\left\{(n+1)^3\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\} \tag{24}$$

Next

$I_4 = \int_0^{\frac{1}{n+1}}\phi(t)\tilde{K}_n(t)dt$

$|I_4| \le \int_0^{\frac{1}{n+1}}|\phi(t)||\tilde{K}_n(t)|dt$

By lemma 4

$$I_4 = \left\{\int_{\frac{1}{n+1}}^{\pi}|\phi(t)|O\left(\frac{1}{t}\right)dt\right\}$$
$$= \left\{\int_{\frac{1}{n+1}}^{\pi}|\phi(t)|dt\right\}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{1}{t}\right)dt\right\}$$
$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\psi(t)}{t^{\frac{1}{p}}}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{1}{t}\right)^q dt\right\}^{\frac{1}{q}}$$
$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\psi(t)}{t^{\frac{1}{p}+2}}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}(t)^q dt\right\}^{\frac{1}{q}}$$
$$= O\left\{(n+1)^2\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{q+1}}\right\}^{\frac{1}{q}}$$

$$= O\left\{(n+1)^2\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{1+\frac{1}{q}}}\right\}$$
$$= O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\} \qquad (25)$$

Combining (23), (24) and (25)

$$\tilde{s}_n(x) - \tilde{f}(x) = O\left\{(n+1)^3\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\} + O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\}$$
$$= O\left\{(n+1)\psi\left(\frac{1}{n+1}\right)\right\}[(n+1)^2+1]\left\{\frac{1}{(n+1)^{\frac{1}{q}}}\right\}$$

This completes the proof of the theorem 2.

5.3 Proof of Theorem 3

The n[th] partial sum $s_n(f;x)$ of series (4) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

The $(E,1)$ transform of $s_n(f;x)$ is given by

$$t_n^{(E,1)} - f(x) = \frac{1}{\pi 2^{n+1}}\int_0^\pi \phi(t)\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt$$

The $(C,2)(E,1)$ transform of $s_n(f;x)$ is given by

$$t_n^{(C,2)(E,1)} - f(x) = \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left\{\frac{(n-k+1)}{2^k}\int_0^\pi\frac{\phi(t)}{\sin\frac{t}{2}}\left[\sum_{v=0}^{k}\binom{k}{v}\sin\left(v+\frac{1}{2}\right)t\right]\mathrm{dt}\right\}$$
$$= \int_0^\pi \phi(t)K_n(t)dt$$
$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)dt$$
$$= O(I_5) + O(I_6) \qquad (26)$$

Now

$$I_5 = \int_0^{\frac{1}{n+1}}\phi(t)K_n(t)dt$$
$$|I_5| \le \int_0^{\frac{1}{n+1}}|\phi(t)||K_n(t)|dt$$
$$= \left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|\sin^\beta t}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)|K_n(t)|}{t\sin^\beta t}\right)^q dt\right\}^{\frac{1}{q}}$$

By lemma 1

$$= O\left(\frac{1}{n+1}\right)O(n+1)\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t^{1+\beta}}\right)^q dt\right\}^{\frac{1}{q}}$$
$$= O(1)\,\xi\left(\frac{1}{n+1}\right)\left\{\int_0^{\frac{1}{n+1}}\left(\frac{dt}{t^{(1+\beta)q}}\right)\right\}^{\frac{1}{q}}$$
$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-(1+\beta)q+1}}{-(1+\beta)q+1}\right\}_0^{\frac{1}{n+1}}\right]^{\frac{1}{q}}$$
$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{(1+\beta)q-1}\right\}^{\frac{1}{q}}$$
$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{(1+\beta)-\frac{1}{q}}\right\}$$
$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} \qquad (27)$$

Next

$$I_6 = \int_{\frac{1}{n+1}}^{\pi}\phi(t)K_n(t)dt$$
$$|I_6| \le \int_{\frac{1}{n+1}}^{\pi}|\phi(t)||K_n(t)|dt$$
$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|\sin^\beta t}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)|K_n(t)|}{t^{-\delta}\sin^\beta t}\right)^q dt\right\}^{\frac{1}{q}}$$

By lemma 2

$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+\beta+1}}\right)^q dt\right\}^{\frac{1}{q}}$$
$$= O\{(n+1)^\delta\}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{(-\delta+\beta+1)q}}\right)dt\right\}^{\frac{1}{q}}$$
$$= O\{(n+1)^\delta\}\,\xi\left(\frac{1}{n+1}\right)\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{dt}{t^{(-\delta+\beta+1)q}}\right)\right\}^{\frac{1}{q}}$$
$$= O\left\{(n+1)^\delta\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-q(-\delta+\beta+1)+1}}{-q(-\delta+\beta+1)+1}\right\}_{\frac{1}{n+1}}^{\pi}\right]^{\frac{1}{q}}$$
$$= O\left\{(n+1)^\delta\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{q(-\delta+\beta+1)-1}\right\}^{\frac{1}{q}}$$
$$= O\left\{(n+1)^\delta\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{(-\delta+\beta+1)-\frac{1}{q}}\right\}$$
$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} \qquad (28)$$

Combining 26), (27) and (28)

$$s_n(f;x) - f(x) = O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} + O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}$$
$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of the theorem 3.

5.4 Proof of Theorem 4

The n[th] partial sum $\tilde{s}_n(f;x)$ of series (5) is given by

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

The $(E,1)$ transform of $\tilde{s}_n(f;x)$ is given by

$$t_n^{(E,1)} - \tilde{f}(x) = \frac{1}{\pi 2^{n+1}}\int_0^\pi \phi(t)\left\{\sum_{k=0}^{n}\binom{n}{k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt$$

The $(C,2)$ $(E,1)$ transform of $\tilde{s}_n(f;x)$ is given by

$$t_n^{(C,2)(E,1)} - \tilde{f}(x) = \frac{1}{\pi(n+1)(n+2)}\sum_{k=0}^{n}\left\{\frac{(n-k+1)}{2^k}\int_0^\pi\frac{\phi(t)}{\sin\frac{t}{2}}\left[\sum_{v=0}^{k}\binom{k}{v}\sin\left(v+\frac{1}{2}\right)t\right]\mathrm{dt}\right\}$$
$$= \int_0^\pi \phi(t)\tilde{K}_n(t)dt$$
$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)\tilde{K}_n(t)dt$$
$$= O(I_7) + O(I_8) \qquad (29)$$

Now

$$I_7 = \int_0^{\frac{1}{n+1}}\phi(t)\tilde{K}_n(t)dt$$
$$|I_7| \le \int_0^{\frac{1}{n+1}}|\phi(t)||\tilde{K}_n(t)|dt$$
$$= \left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|\sin^\beta t}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)|K_n(t)|}{t\sin^\beta t}\right)^q dt\right\}^{\frac{1}{q}}$$

By lemma 3

$$= O\left(\frac{1}{n+1}\right)\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t^{2+\beta}}\right)^q dt\right\}^{\frac{1}{q}}$$
$$= O\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\left\{\int_0^{\frac{1}{n+1}}\left(\frac{dt}{t^{(2+\beta)q}}\right)\right\}^{\frac{1}{q}}$$
$$= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-(2+\beta)q+1}}{-(2+\beta)q+1}\right\}_0^{\frac{1}{n+1}}\right]^{\frac{1}{q}}$$

$$= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{(2+\beta)q-1}\right\}^{\frac{1}{q}}$$
$$= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{(2+\beta)-\frac{1}{q}}\right\}$$
$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right\} \tag{30}$$

Next

$$I_8 = \int_{\frac{1}{n+1}}^{\pi} \phi(t)\widetilde{K}_n(t)dt$$
$$|I_8| \le \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||\widetilde{K}_n(t)|dt$$
$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|\sin^{\beta}t}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)|K_n(t)|}{t^{-\delta}\sin^{\beta}t}\right)^q dt\right\}^{\frac{1}{q}}$$

By lemma 4

$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+\beta+1}}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= O\{(n+1)^{\delta}\}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{(-\delta+\beta+1)q}}\right)dt\right\}^{\frac{1}{q}}$$
$$= O\{(n+1)^{\delta}\}\,\xi\left(\frac{1}{n+1}\right)\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{dt}{t^{(-\delta+\beta+1)q}}\right)\right\}^{\frac{1}{q}}$$
$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-q(-\delta+\beta+1)+1}}{-q(-\delta+\beta+1)+1}\right\}_{\frac{1}{n+1}}^{\pi}\right]^{\frac{1}{q}}$$
$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{q(-\delta+\beta+1)-1}\right\}^{\frac{1}{q}}$$
$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{(-\delta+\beta+1)-\frac{1}{q}}\right\}$$
$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right\} \tag{31}$$

Combining (29), (30) and (31)

$$s_n(f;x) - f(x) = O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} + O\left\{(n+\right.$$
1β+1pξ1n+1
$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of the theorem 4.

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