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Some Bayesian Frailty Models

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Abstract: ***Some Bayesian estimation is carried out by many authors but none has used frailty model. In this article we used frailty model and estimate maximum of posterior distribution as the posterior mode and illustrated it by some examples of different prior distributions. Also Bayesian estimation by taking frailty normal distribution is carried out and is done not only for univariate but also for bivariate normal frailty distribution.***

Keywords: Frailty distribution, Bayesian estimation, Prior distribution, Posterior distribution, Maximum Posterior estimate, Posterior likelihood.

1. Introduction

Many authors introduced several survival models, amongst which Cox (1972) Proportional Hazard regression (PH) model is well-known. In many cases it is not possible to include all covariates, for example genetic factor comprising of all possible genes influencing survival. These unknown unobservable unmeasurable factors termed as heterogeneity or frailties are included in the model as random effect term or frailty which extends the Cox PH model. The term frailty is used first by Vaupel et al (1979) and was separately used by Clayton (1978). For failure time distribution, Hougaard (1986a, 1986b) introduced some models. The Maximum Likelihood Estimation (MLE) method and the Best Linear Unbiased Predictor (BLUP) method have been used by Handerson (1975) and McGilchrist and Aisbett (1991).

Hanagal (2011) used Weibull distribution when hazard function is linear function of frailty parameter. Parekh et al. (2015) have estimated the Survival function with the use of linear hazard function and exponential base line distribution.
The classical Bayesian approach for estimation is dealt by many authors, amongst which some of them are Akaike (1983), Le Cam (1990), Joshi (1990), Geyer and Thompson (1992) etc.

Some of the authors who have used Bayesian approach for the frailty models are Ibrahim et.al. (2001), Santos et.al. (2010) and they have used Weibull as base line distribution and Gamma, Log normal as frailty distribution. The Inverse Gaussian frailty model is used by Kheiri et.al. (2007). Sahu et.al. (1997) have used prior distribution similar to the Normal distribution with mean 0 and large variance.

Throughout this article Y represents logT where T is lifetime variable. In the special analysis we consider some of the baseline distributions and frailty prior distributions.

2. Frailty prior distribution associated with Binomial distribution

Let Y have baseline distribution as binomial distribution, $B(n, p)$. We consider following three prior frailty distribution such as Be $\left(\frac{1}{2},\frac{1}{2}\right)$, $U(0,1)$, Laplace Haldane distribution.

(i) Let frailty distribution Beta be, Be $\left(\frac{1}{2},\frac{1}{2}\right)$ with p.d.f.

$$\pi_1(p) = \frac{1}{B\left(\frac{1}{2},\frac{1}{2}\right)} p^{-\frac{1}{2}}(1-p)^{-\frac{1}{2}}, \quad 0\leq p \leq 1$$

$$\text{where } B(p,q) = \frac{\sqrt{p}\sqrt{q}}{\sqrt{p+q}}$$

(ii) Uniform distribution on interval (0,1)

$$\pi_2(P) = 1 \qquad \text{and}$$

(iii) Laplace Haldane distribution with p.d.f.

$$\pi_3(p) = \frac{1}{p(1-p)}, \qquad 0\leq p \leq 1$$

Without using any loss function a possible estimate of θ based on conditional distribution of θ given y, $\pi(\theta|y)$ is the Maximum Posterior (MP) estimate defined as the posterior mode. We note that MP estimate also maximizes $l(\theta|y)\,\pi(\theta)$, where $l(\theta|y)$ is the posterior likelihood, which bypasses the computation of the marginal distribution. In these three cases the corresponding MP estimates are then, for $n > 2$

$$\delta_1(y) = \max\left(\frac{y-\frac{1}{2}}{n-1}, 0\right)$$

$$\delta_2(y) = \frac{y}{n}$$

$$\delta_3(y) = \max\left(\frac{y-1}{n-2}, 0\right).$$

We notice that when n is large, all these three Bayesian frailty estimates are equivalent.

For $\delta_2(y) = \frac{y}{n}$, the posterior risk is

$$E^{\pi}[(\delta_2(y)-p)^2 \mid y] = E^{\pi}\left[\left(p-\frac{y}{n}\right)^2 \mid y\right]$$
$$= \left(\frac{y+\frac{1}{2}}{n+1}-\frac{y}{n}\right)^2 + \frac{\left(y+\frac{1}{2}\right)\left(n-y+\frac{1}{2}\right)}{(n+1)^2(n+2)}$$
$$= \frac{\left(y-\frac{n}{2}\right)^2}{(n+1)^2n^2} + \frac{\left(y+\frac{1}{2}\right)\left(n-y+\frac{1}{2}\right)}{(n+1)^2(n+2)}, \qquad (2.1)$$

using $\pi(p \mid y) \sim Be(y+\frac{1}{2}, n-y+\frac{1}{2})$.

Further the risk of m.l.e. being $E_p(\delta_2(y)-p)^2 = Var\left(\frac{y}{n}\right) = \frac{p(1-p)}{n}$ and $\underset{p}{\text{Sup}}\frac{p(1-p)}{n} = \frac{1}{4n}$, and also (2.1) reduces to $\frac{1}{4(n+2)}$. Since $\frac{1}{4(n+2)} < \frac{1}{4n}$, the Bayesian estimate of frailty variable is always better than the usual m.l.e.

Further considering continuous baseline distribution for Bayesian analysis as normal baseline distribution and taking its conjugate prior as normal, we obtain frailty Bayesian estimate of parameter θ in section 3. In section 4 we extended the frailty Bayesian estimate of parameter θ for bivariate priors.

3. Bayesian estimation of frailty Normal distribution

Theorem 1: Let $y \mid \theta$ have normal baseline distribution with mean θ and known variance $\sigma^2$and let the prior distribution of frailty parameter θ be normal with known mean μ and known variance τ^2. Then the Bayesian estimate, $\delta^{\pi}(y)$ of frailty parameter θ is

$$\delta^{\pi}(y) = y - \frac{\sigma^2}{\sigma^2+\tau^2}(y-\mu),$$

Proof: As $(y \mid \theta)$ has $N(\theta,\sigma^2)$ the prior distribution, $\pi(\theta)$ of frailty θ has $N(\mu,\tau^2)$. Here σ^2,τ^2,μ are known, the joint distribution, $h(y,\theta)$ of y and θ is

$$h(y,\theta) = \pi(\theta)f(y|\theta)$$
$$= \frac{1}{2\pi\sigma\tau} exp\left\{-\frac{1}{2}\left[\frac{(\theta-\mu)^2}{\tau^2}+\frac{(y-\theta)^2}{\sigma^2}\right]\right\} \qquad (3.1)$$

Defining $\rho = \tau^{-2}+\sigma^{-2}$

$$= \frac{\sigma^2+\tau^2}{\sigma^2\tau^2}$$

and completing squares of the exponent as

$$\frac{1}{2}\left[\frac{(\theta-\mu)^2}{\tau^2}+\frac{(y-\theta)^2}{\sigma^2}\right]$$
$$= \frac{1}{2}\left[\left(\frac{1}{\tau^2}+\frac{1}{\sigma^2}\right)\theta^2 - 2\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)\theta + \left(\frac{\mu^2}{\tau^2}+\frac{y^2}{\sigma^2}\right)\right]$$
$$= \frac{1}{2}\rho\left[\theta^2 - \frac{2}{\rho}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)\theta\right] + \frac{1}{2}\left(\frac{\mu^2}{\tau^2}+\frac{y^2}{\sigma^2}\right)$$
$$= \frac{1}{2}\rho\left[\theta^2 - \frac{2}{\rho}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)\theta + \frac{1}{\rho^2}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)^2 - \frac{1}{\rho^2}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)^2\right] + \frac{1}{2}\left(\frac{\mu^2}{\tau^2}+\frac{y^2}{\sigma^2}\right)$$
$$= \frac{1}{2}\rho\left[\theta - \frac{1}{\rho}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)\right]^2 - \frac{1}{2\rho}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)^2 + \frac{1}{2}\left(\frac{\mu^2}{\tau^2}+\frac{y^2}{\sigma^2}\right)$$
$$= \frac{1}{2}\rho\left[\theta - \frac{1}{\rho}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)\right]^2 + \frac{(\mu-y)^2}{2(\sigma^2+\tau^2)}$$

From (3.1), we have

$$h(y,\theta) = \frac{1}{2\pi\sigma\tau} exp\left\{-\frac{1}{2}\rho\left[\theta - \frac{1}{\rho}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)\right]^2\right\}$$
$$.exp\left\{-\frac{1}{2}\frac{(\mu-y)^2}{(\sigma^2+\tau^2)}\right\}$$

and the marginal distribution, $m(y)$ is

$$m(y) = \int_{-\infty}^{\infty} h(y,\theta)d\theta$$
$$= \frac{1}{\sqrt{2\pi\rho}\,\sigma\tau} exp\left\{-\frac{1}{2}\frac{(\mu-y)^2}{(\sigma^2+\tau^2)}\right\}$$
$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+\tau^2}} exp\left\{-\frac{1}{2}\frac{(y-\mu)^2}{(\sigma^2+\tau^2)}\right\}$$

So, $m(y) \sim N(\mu, \sigma^2+\tau^2)$

and the posterior distribution of θ given y is

$$\pi(\theta|y) = \frac{h(y,\theta)}{m(y)}$$
$$= \left(\frac{\rho}{2\pi}\right)^{\frac{1}{2}} exp\left\{-\frac{\rho}{2}\left[\theta - \frac{1}{\rho}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)\right]^2\right\}$$

Thus the marginal distribution of y is $N(\mu, \sigma^2+\tau^2)$ and posterior distribution of θ given y is

$$N\left(\mu(y), \frac{1}{\rho}\right)$$

where,

$$\mu(y) = \frac{1}{\rho}\left(\frac{\mu}{\tau^2}+\frac{y}{\sigma^2}\right)$$
$$= \frac{\sigma^2}{\sigma^2+\tau^2}\mu + \frac{\tau^2}{\sigma^2+\tau^2}y \qquad (3.2)$$
$$= y - \frac{\sigma^2}{\sigma^2+\tau^2}(y-\mu) \qquad (3.3)$$

If the loss function is squared error loss function then the posterior mean is the Bayesian estimate of frailty parameter θ.Thus the frailty parameter has Bayesian estimate which is given by (3.3) and the frailty variance is $\frac{1}{\rho} = \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}$.

Remark: We note that if $y_1, y_2, \ldots, y_n$ is a sample from $N(\theta,\sigma^2)$ and if θ has frailty prior distribution $N(\mu,\tau^2)$ then $\bar{y}$ being sufficient statistics for θ and $\bar{y} \mid \theta$ has $N(\theta, \frac{\sigma^2}{n})$ and taking $\pi(\theta|y)$as $\pi(\theta|\bar{y})$, (3.2) reduces to

$$\mu(y) = \frac{\sigma^2}{\sigma^2+n\tau^2}\mu + \frac{n\tau^2}{\sigma^2+n\tau^2}\bar{y} \qquad (3.4)$$

and

$$\rho = \frac{\sigma^2+n\tau^2}{\sigma^2\tau^2},$$

So that the frailty Bayesian predictor is given by (3.4) and the frailty variance is

$$\frac{1}{\rho}=\frac{\sigma^2\tau^2}{\sigma^2+n\tau^2}$$

We extend above result for Cox's regression model with two regressors in the next section.

4. Bivariate frailty distribution

Considering Cox's linear regression model

$$y=b_1X_1+b_2X_2+\epsilon,0\le b_1,b_2\le 1 \tag{4.1}$$

Let $(y_1,x_{11},x_{21}),\dots,(y_n,x_{1n},x_{2n})$ be a sample from (4.1) and assuming that the errors (frailty variable) ϵ_i are independently distributed as $N(0,1)$, i.e. $y_i\sim N(b_1X_{1i}+b_2X_{2i},1)$. Also the frailty noninformative prior distribution is

$$[\pi(b_1,b_2)]^2=1,\ \text{ if } 0\le b_1\le 1,\ 0\le b_2\le 1$$

The posterior frailty mean of b_i is given by $(i\ =\ 1,2)$

$$E^{\pi}(b_i\mid y_1,y_2,\dots,y_n)=\frac{\int_0^1\int_0^1 b_i\prod_{j=1}^n\phi(y_j-b_1X_{1j}-b_2X_{2j})db_1db_2}{\int_0^1\int_0^1\prod_{j=1}^n\phi(y_j-b_1X_{1j}-b_2X_{2j})\,db_1db_2}$$

where E^{π} stands for expectation when π is true and ϕ stands for the density of $N(0,1)$.

Let $(\hat{b}_1,\hat{b}_2)$ be least square estimate of (b_1,b_2). Then the posterior distribution of (b_1,b_2) is

$$\begin{pmatrix}b_1\\b_2\end{pmatrix}\sim N_2\left(\begin{pmatrix}b_1\\b_2\end{pmatrix},(X'X)^{-1}\right) \tag{4.2}$$

with

$$X=\begin{pmatrix}X_{11} & X_{21}\\ \dots & \dots\\ X_{1n} & X_{2n}\end{pmatrix}$$

Therefore the restricted frailty base estimate is given by

$$\delta_i^{\pi}(y_1,y_2,\dots,y_n)=\frac{E^{\pi}(b_i,i=1,2,0\le b_1\le 1,0\le b_2\le 1\mid y_1,y_2,\dots,y_n)}{P^{\pi}((b_1,b_2)\in[0,1]^2\mid y_1,y_2,\dots,y_n)}. \tag{4.3}$$

Let us denote

$$V=(v_{ij})=(X'X)^{-1}=\begin{pmatrix}v_{11}^2 & v_{12}\\ v_{12} & v_{22}^2\end{pmatrix}.$$

Then the conditional distribution of b_1 given b_2 is

$$f(b_1\mid b_2)=N\left(\hat{b_1}+\frac{v_{12}}{v_{22}^2}(b_2-\hat{b}_2),\ v_{11}^2-\frac{v_{12}^2}{v_{22}^2}\right)$$

And

$$P^{\pi}((b_1,b_2)\in[0,1]^2\mid y_1,y_2,\dots,y_n)$$

$$=\int_0^1\left\{\Phi\left(\frac{1-\hat{b}_1-\frac{v_{12}}{v_{22}^2}(b_2-\hat{b}_2)}{\sqrt{v_{11}^2-\frac{v_{12}^2}{v_{22}^2}}}\right)-\Phi\left(\frac{-\hat{b}_1-\frac{v_{12}}{v_{22}^2}(b_2-\hat{b}_2)}{\sqrt{v_{11}^2-\frac{v_{12}^2}{v_{22}^2}}}\right)\right\}$$

$$.\frac{1}{v_{22}}\phi\left(\frac{b_2-\hat{b}_2}{v_{22}}\right)db_2, \tag{4.4}$$

where Φ is the cumulative distribution function of standard normal variate,

$$E^{\pi}(b_i,i=1,2,0\le b_1\le 1,0\le b_2\le 1\mid y_1,y_2,\dots,y_n)$$

$$=\int_0^1\left[\hat{b}_1+\frac{v_{12}}{v_{22}^2}(b_2-\hat{b}_2)+\right.$$

$$\left.\sqrt{v_{11}^2-\frac{v_{12}^2}{v_{22}^2}}\left\{\phi\left(\frac{1-\hat{b}_1-\frac{v_{12}}{v_{22}^2}(b_2-\hat{b}_2)}{\sqrt{v_{11}^2-\frac{v_{12}^2}{v_{22}^2}}}\right)-\phi\left(\frac{-\hat{b}_1-\frac{v_{12}}{v_{22}^2}(b_2-\hat{b}_2)}{\sqrt{v_{11}^2-\frac{v_{12}^2}{v_{22}^2}}}\right)\right\}\right]$$

$$.\frac{1}{v_{22}}\phi\left(\frac{b_2-\hat{b}_2}{v_{22}}\right)db_2. \tag{4.5}$$

Using (4.4) and (4.5) in (4.3), one gets the frailty Bayesian estimates of $b_1$and b_2. (4.4) can be computed by using table of cumulative density function Φ manually but (4.5) cannot be computed analytically.

If $b_1$and $b_2$are independent, that is, $v_{12}=0$, the Bayesian frailty estimates, $\hat{b}_1$ *and* $\hat{b}_2$ are given by

$$\delta_i^{\pi}(y_1,y_2,\dots,y_n)$$

$$=E^{\pi}(b_i\mid y_1,y_2,\dots,y_n),i=1,2,$$

$$=\hat{b}_i-v_{ii}\frac{exp\left\{-\frac{(1-\hat{b}_i)^2}{2v_{ii}^2}\right\}-exp\left\{-\frac{\hat{b}_i{}^2}{2v_{ii}^2}\right\}}{\sqrt{2\pi}\left\{\Phi\left(\frac{(1-\hat{b}_i)}{v_{ii}}\right)-\Phi\left(-\frac{\hat{b}_i}{v_{ii}}\right)\right\}}$$

5. Conclusion

Even with different priors as frailty distributions, we observe that with these frailty priors the maximum posterior estimates are better than the corresponding Bayesian or classical estimates. Further frailty parameter θ of normal mean is better than the usual Bayesian estimator. Extension of normal prior of frailty for Cox model, it is observed that the frailty Bayesian estimates of primary regression coefficients $b_1$and b_2 are better than the non-Bayesian estimates of $b_1$and $b_2$when normality of baseline distribution is assumed.

References

[1] Akaike, H. (1983). Information measures and model selection. Bull. Int. statist. Inst. 50, 277-290.
[2] Clayton, D. (1978). A model for association in bivariate life tables and its applications to epidemiological studies of familial tendency in chronic disease incidence. Biometrica, 65, 141-151.
[3] Cox, D. (1972). Regression models and life tables (with discussion). Journal of the Royal Statistical Society, Series B 34, 187-200.
[4] Geyer, C.J. and Thompson, E.A. (1992). Constrained Monte Carlo maximum likelihood for dependent data (with discussion). J. Roy. Statist. Soc., Ser B 54, 657-699.
[5] Hangal, D. (2011). *Modeling survival data using frailty models*. CRC Press.
[6] Hougaard, P. (1986a). Survival models for hetrogeneous populations derived from stable distribution. Biometrika, 73, 387-396.

[7] Hougaard, P. (1986b). A class of multivariate failure time distributions. Biometrika , 73, 671-678.
[8] Ibrahim, J. G., Chen, M., and Sinha, D. (2001). *Bayesian Survival Analysis*. New York: Springer.
[9] Joshi, V.M. (1990). The censoring concept and the likelihood principle. J. Statist. Plann. Inference 26, 109-111
[10] Kheiri, S., Kimber, A., Meshkani M.R. (2007). Bayesian analysis of an inverse Gaussian correlated frailty model. Computat. Statist. Data Anal. 51, 5317-5326.
[11] Le Cam, L. (1990). Maximum likelihood: an introduction. Int. Statist. Rev. 58, 153-172.
[12] McGilchrist, C. A. (1991). Regression with frailty in survival analysis. Biometrics, 47, 461-466.
[13] Parekh, S.G., Ghosh, D. K. and Patel, S.R. (2015). On frailty models for kidney infection data with exponential baseline distribution, International Journal of Applied Mathematics & Statistical Sciences (JAMSS) 4 (5), 31-40.
[14] Sahu, K.S., Dey, D.K., Aslanidou, H., Sinha, D. (1997). A Weibull regression model with Gamma frailties for multivariate survival data. Lifetime Data Analysis 3, 123 – 137
[15] Santos dos, C. A. and Achcar, J. A. (2010). A Bayesian analysis for multivariate survival data in the presence of covariates. Journal of Statistical Theory and Applications 9, 233-253.
[16] Vaupel, J. S. (1979). The impact of heterogeneity in individual frailty on the dynamics of mortality. Demography, 16, 439-454.