Study of Different Models for Ranking Error in Ranked Set Sampling

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Abstract: ***Ranked set sampling was first introduced by McIntyre(1952) for the purpose of estimating mean pasture and forage yields. McIntyre's intension was to maintain the unbiasedness of simple random sampling while introducing the extra information provided by visual inspection. Ranked Set Sampling may not be as efficient as it should be if ranking is not perfect. This Paper discuses some possible models for ranking errors in ranked set sampling.***

Keywords: Ranked set sampling, error functions, judgmental ranking, judgmental order statistic, true order statistic

1. Introduction

McIntyre (1952) proposed a method of unbiased selective sampling using ranked set. This sampling procedure is estimation of average yield from large plots of arable crops without a concerned more about data collection rather than sample selection. It is based on simple random sample, though of a larger size than required, and hence inherits some properties of simple random sampling. In the original form ranked set sampling assumes that ranking of sampling units within a set is based on judgment regarding the characteristic of interest. It was a later development that extended this and also consider the possibility of ranking on the basis of an easy to observe concomitant variable. These two ways of ranking sampling units give rise to two different types of ranking errors. The first is due to judgmental error while the second is due to imperfect association between the concomitant variable and variable of interest. This paper is restricted only to the first type, namely ranking errors arising out of judgmental errors. Such an errors is caused by similarity among sampling units that are to be ranked. The closer these sampling units each other the higher is the probability of error in ranking. This will be made clear in the next section. Let us first define the notation and set up of ranked set sampling.

2. The Method of Ranked Set Sampling

The procedure of selection of a ranked set sample of size m begin with selection of m samples of size m each, equivalently, a sample of size m^2 may be selected and them randomly partitioned in to m samples of equal size. In either case, m is a design parameter and is to be specified by the investigator. The samples of size m are called sets. Next, the sampling units in every set are ranked by visual inspection or by method that does not require measurement on the variable of interest. For he purpose of qualification, the unit with smallest rank is selected from the first set the unit having second smallest rank is selected from the second set, and so on until the unit having the largest rank is selected from the m^{th} set. A ranked set sample, in this way, obtains m measurements from an initial sample of size m^2. Since units in every set are to be judgmentally ranked, m is restricted to small. If the desires sample size is large in view of facilitating inference, the above procedure, now known as a cycle, is repeated r times, resulting in $n = mr$ quantified units out of m^2r units constitute a ranked set sample. The procedure described above is often called the MTW sampling in recognition of McIntyre's (1952) original formulation and the subsequent mathematical contributions of Takahasi and Wakimoto (1968).

Let us consider an example to illustrate the method of ranked set sampling. Let he set size be $m = 4$ and let $r = 3$ be the number of replications or cycles. Then the sampling scheme can be presented as follows.

Cycle	Rank			
	1	2	3	4
1	⊙	●	●	●
	●	⊙	●	●
	●	●	⊙	●
	●	●	●	⊙
2	⊙	●	●	●
	●	⊙	●	●
	●	●	⊙	●
	●	●	●	⊙
3	⊙	●	●	●
	●	⊙	●	●
	●	●	⊙	●
	●	●	●	⊙
4	⊙	●	●	●
	●	⊙	●	●
	●	●	⊙	●
	●	●	●	⊙

Here every row denotes a judgment- ordered sample and the circled units indicate units that are to be quantified. In are $M^2r = 48$ Units have been drawn initially, but only $mr = 12$ units circled in the diagram, are quantified to obtain a ranked set sample.

Mathematically, let the m sets of size m each be denoted by $X_{11}, X_{12}, \dots, X_{1m}, X_{21}, X_{22}, \dots, X_{2m}, ..$ $, X_{m1}, X_{m2}, \dots, X_{mm}$. Note that all the m^2 random variables are independent and identically distributed random variables having common distribution function $F(x)$. Let $X_{i(1)}$,

$X_{i(2)}, \dots, X_{i(m)}$ denote the order statistics corresponding to $X_{i1}, X_{i2}, \dots, X_{im}$ for i $= 1, 2, \cdots, $m. Then the ranked set sample is given by $(X_{1(1)}, X_{2(2)}, \dots, X_{i(i)}, \dots, X_{i(m)})$, considering only one cycle. The following diagram may be useful for visualizing this description.

Set	Initial Sample			
1	X_{11},	X_{12},	...,	X_{1m}
2	X_{21},	X_{22},	...,	X_{2m}
3	X_{31},	X_{32},	...,	X_{3m}
⋮	⋮	⋮		⋮
m	X_{m1},	X_{m2},	...,	X_{mm}

Rank-Ordered Sample

Set	Rank-Ordered Sample			
1	$X_{1(1)}$,	$X_{1(2)}$,	...,	$X_{1(m)}$
2	$X_{2(1)}$,	$X_{2(2)}$,	...,	$X_{2(m)}$
3	$X_{3(1)}$,	$X_{3(2)}$,	...,	$X_{3(m)}$
⋮	⋮	⋮		⋮
m	$X_{m(1)}$,	$X_{m(2)}$,	...,	$X_{m(m)}$

The final ranked set sample

Set	Rank-Ordered Sample			
1	$X_{1(1)}$,	*,	...,	*
2	*,	$X_{2(2)}$,	...,	*
3	*,	*,	...,	*
⋮	⋮	⋮		⋮
m	*,	*,	...,	$X_{m(m)}$

In case of a single cycle, the mean of a ranked set sample is

$$\bar{X}_{RSS} = \frac{1}{m} \sum_{i=1}^{m} X_{i(i)}$$
$$\bar{X}_{RSS} = \frac{1}{m} \sum_{i=1}^{m} X_{(i:m)}$$

where $X_{(i:m)}$ is the i^{th} order statistic in a sample 0 size m.

Without going in details about the statistical properties of the ranked set sample mean, the next section discusses ranking errors arising due to error in the judgment and their effect on the efficiency of ranked set sampling in comparison with simple random sampling.

3. Judgmental Error in the Ranking

The most common difficulty in carrying out the protocol of ranking. The result of ranking errors is that units are assigned ranks that are different from their true ranks according the variable of interest. This, in turn, leads to a measurement difference between the units that is quantified and the unit that ought to have been quantified. This effect was formally studied for the first time by Dell and Clutter(1972). In this section, the quantified units from the i^{th} set dose not represent the i^{th} order statistic from that set, but rather the i^{th} "judgmental order statistic". If we denote the "judgment order statistic" and the "true order statistic" by $X_{[i:m]}$ and $X_{(i:m)}$ respectively, then Dell and Clutter(1972) use the model

$$X_{[i:m]} = X_{(i:m)} + \epsilon_i$$

where $X_{(i:m)}$ and ϵ_i are independent and $\varepsilon_i \sim N(0, \sigma_\epsilon^2)$ in order to study the impact of ranking errors.

Dell and Clutter (1972) have pointed out that this model may not be realistic in some situations. First, errors in judgment are likely to be influenced by set size m. Second, the assumption of independence between the observation and the error may not be appropriate. An error may be influenced by the values of the units, or more reasonably, by values of other units within the set. This drawback is hard to avoid because such a situation may be peculiar to a given situation.

Here we take a different approach, where the emphasis is on the fact that ranking errors results in quantifying a unit that ought not have been quantified. It is important to note that ranking error depends o the proximity of the incorrectly ranked unit to the truly ranked unit. It is therefore proposed that ranking error will be modeled as a function of the difference between the ranks of two units. The closer ranks of two units are, more likely it is to rank one in place of the order. In other words, the units with true rank j will be assigned rank i with probability that is a function of $(i - j)$. If $R_1(i,j)$ is the probability of assigning rank i to the unit that has the true rank j, then the first model states that

$$R_1(i,j) = \frac{c_1(i,j,m)}{1+|i-j|}, \quad i,j = 1,2,\dots,m$$

Where $c_1(i,j,m)$ is the normalizing constant, which may be a function of $i, j, and\ m$.
The following table shows the numerical values of ranking error function for the set size $m = 4$.

Table: Ranking Error Function $c_1(i,j,m)$. $R_1(i,j)$

True Rank	Judgment Rank			
	1	2	3	4
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
2	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{3}$
3	$\frac{1}{3}$	$\frac{1}{2}$	1	$\frac{1}{2}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1

Table: Ranking Error Function $R_1(i,j,4)$

True Rank	Judgment Rank			
	1	2	3	4
1	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$
2	$\frac{3}{14}$	$\frac{3}{7}$	$\frac{3}{14}$	$\frac{1}{7}$
3	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{3}{7}$	$\frac{3}{14}$
4	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{6}{25}$	$\frac{12}{25}$

As a result of ranking error, the ranked set sample contains values that should not have been included, while it does not contain values that should have there. More precisely, the observed (judgment rank) values can be expressed as follows.

$$P[X_{[1:4]} = X_{(1:4)}] = \frac{12}{25}$$
$$P[X_{[1:4]} = X_{(2:4)}] = \frac{6}{25}$$
$$P[X_{[1:4]} = X_{(3:4)}] = \frac{4}{25}$$
$$P[X_{[1:4]} = X_{(4:4)}] = \frac{3}{25}$$

This gives, on average, the following values to be obtained on the unit that has judgment rank one.

$$E[X_{[1:4]}] = \frac{12}{25}X_{(1:4)} + \frac{6}{25}X_{(2:4)} + \frac{4}{25}X_{(3:4)} + \frac{3}{25}X_{(4:4)}$$

Similarly for other judgment ranks. Finally, the ranked set sample mean is the average of the four values in the ranked set sample. Its expected value is then given in terms of true ranks by the following expression.

$$\frac{1}{4}[X_{[1:4]} + X_{[2:4]} + X_{[3:4]} + X_{[4:4]}]$$
$$= \frac{1}{4}\{[\frac{12}{25}X_{(1:4)} + \frac{6}{25}X_{(2:4)} + \frac{4}{25}X_{(3:4)} + \frac{3}{25}X_{(4:4)}]$$
$$+[\frac{3}{14}X_{(1:4)} + \frac{3}{7}X_{(2:4)} + \frac{3}{14}X_{(3:4)} + \frac{1}{7}X_{(4:4)}] + [\frac{1}{7}X_{(1:4)} +$$
314 X2:4+ 37 X3:4+314X4:4+ [325X1:4+ 425 X2:4+
$$\frac{6}{25}X_{(3:4)} + \frac{12}{25}X_{(4:4)}]\}$$

$$= \frac{1}{4}\left(\frac{12}{25} + \frac{3}{14} + \frac{1}{7} + \frac{3}{25}\right)X_{(1:4)}$$
$$+ \frac{1}{4}\left(\frac{6}{25} + \frac{3}{7} + \frac{3}{14} + \frac{4}{25}\right)X_{(2:4)}$$
$$+ \frac{1}{4}\left(\frac{4}{25} + \frac{3}{14} + \frac{3}{7} + \frac{6}{25}\right)X_{(3:4)}$$
$$+ \frac{1}{4}\left(\frac{3}{25} + \frac{1}{7} + \frac{3}{14} + \frac{12}{25}\right)X_{(4:4)}$$

$$= \frac{335}{350}X_{(1:4)} + \frac{365}{350}X_{(2:4)} + \frac{365}{350}X_{(3:4)} + \frac{335}{350}X_{(4:4)}$$
$$= \bar{X}_{RSS} - \frac{3}{70}[X_{(1:4)} - X_{(2:4)} - X_{(3:4)} - X_{(4:4)}]$$
$$= \bar{X}_{RSS} - \frac{3}{70}[(X_{(1:4)} - X_{(2:4)}) - (X_{(3:4)} - X_{(4:4)})]$$

This expression indicates the role of underlying population distribution. In particular, if the population distribution is symmetric, the ranked set sample mean has the same expected value even under ranking error function R_1. In other words, ranking errors described by R_1 have no effect on the unbiased property of the ranked set sample mean as long as the parent distribution is symmetric.

Next, Let $R_2(i,j,m)$ denote the probability that is assigned to judgment rank i has the true rank j, where

$$R_2(i,j,m) = \frac{c_2(i,j,m)}{1+|i-j|^2}, \quad i,j = 1,2,\dots,m$$

Where $c_2(i,j,m)$ is the normalizing constant. The following table shows the numerical values of $R_2(i,j,m)$ for the set size $m = 4$.

Table: Ranking Error Function $c_2(i,j,m)$. $R_2(i,j,m)$

True Rank	Judgment Rank			
	1	2	3	4
1	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$
2	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{5}$
3	$\frac{1}{5}$	$\frac{1}{2}$	1	$\frac{1}{5}$
4	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1

Table: Ranking Error Function $R_2(i,j,m)$

True Rank	Judgment Rank			
	1	2	3	4
1	$\frac{5}{9}$	$\frac{5}{18}$	$\frac{1}{9}$	$\frac{1}{18}$
2	$\frac{5}{22}$	$\frac{5}{11}$	$\frac{5}{22}$	$\frac{1}{11}$
3	$\frac{1}{11}$	$\frac{5}{22}$	$\frac{5}{11}$	$\frac{5}{22}$
4	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{5}{18}$	$\frac{5}{9}$

Under the model R_2, the mathematical expressions relating the judgment ranks with the true ranks are as follows.

$$E[X_{[1:4]}] = \frac{5}{9}X_{(1:4)} + \frac{5}{18}X_{(2:4)} + \frac{1}{9}X_{(3:4)} + \frac{1}{18}X_{(4:4)}$$
$$E[X_{[2:4]}] = \frac{5}{22}X_{(1:4)} + \frac{5}{11}X_{(2:4)} + \frac{5}{22}X_{(3:4)} + \frac{1}{11}X_{(4:4)}$$
$$E[X_{[3:4]}] = \frac{1}{11}X_{(1:4)} + \frac{5}{22}X_{(2:4)} + \frac{5}{11}X_{(3:4)} + \frac{5}{22}X_{(4:4)}$$
$$E[X_{[4:4]}] = \frac{1}{18}X_{(1:4)} + \frac{1}{9}X_{(2:4)} + \frac{5}{18}X_{(3:4)} + \frac{5}{9}X_{(4:4)}$$

This relations leads to the following expressions or what would constitute this rank set sample.

$$\frac{1}{4}[X_{[1:4]} + X_{[2:4]} + X_{[3:4]} + X_{[4:4]}]$$
$$= \frac{1}{4}\left(\frac{5}{9} + \frac{5}{22} + \frac{1}{11} + \frac{1}{18}\right)X_{(1:4)}$$
$$+ \frac{1}{4}\left(\frac{5}{18} + \frac{5}{11} + \frac{5}{22} + \frac{1}{9}\right)X_{(2:4)}$$
$$+ \frac{1}{4}\left(\frac{1}{9} + \frac{5}{22} + \frac{5}{11} + \frac{5}{18}\right)X_{(3:4)}$$
$$+ \frac{1}{4}\left(\frac{1}{18} + \frac{1}{11} + \frac{5}{22} + \frac{5}{9}\right)X_{(4:4)}$$

$$= \frac{184}{198}X_{(1:4)} + \frac{212}{198}X_{(2:4)} + \frac{212}{198}X_{(3:4)} + \frac{184}{198}X_{(4:4)}$$
$$= \bar{X}_{RSS} - \frac{53}{198}[X_{(1:4)} - X_{(2:4)} - X_{(3:4)} - X_{(4:4)}]$$
$$= \bar{X}_{RSS} - \frac{53}{198}[(X_{(1:4)} - X_{(2:4)}) - (X_{(3:4)} - X_{(4:4)})]$$

Again a symmetric distribution of the variable of interest X ensures that the ranked set sample mean is unbiased for the population mean under the ranking error described by ranking error function R_2. A more general and parametric form of the ranking error function is given by

$$R_p(i,j,m) = \frac{c_p(i,j,m)}{1+|i-j|^p}, \quad i,j = 1,2,\dots,m$$

Note that the limiting case as $p \to 0$ represents randomly ranking units in a set while other limiting case as $p \to \infty$

represent perfect ranking The value of parameter p can be at a value that is deemed to be reasonable in a given situation. In every case, the ranked set sample mean continues to be unbiased even when ranking is not perfect.

4. Ranking Error Function $D_p(i,j,m)$

Another way to consider the situation where ranking may not be perfect is where values of the variable of interest may be too close to one another for ranking. Hence this model uses the difference between values of different units rather than their ranks. In other words if we denote the difference between $E(X_{(i:m)})$ and $E(X_{(j:m)})$ by $D(i,j,m)$, then the ranking error function
$D_p(i,j,m)$ is defined by

$$D_p(i,j,m) = \frac{C(i,j,m)}{1+[D(i,j,m)]^p},$$
$p \geq 0;\ i,j = 1,2,\dots,m$

This is same as

$$D_p(i,j,m) = \frac{C(i,j,m)}{1+[E(X_{(i:m)}) - E(X_{(j:m)})]^p},$$
$p \geq 0;\ i,j = 1,2,\dots,m$

It may be noted that the difference $D_p(i,i+1,m)$ is called the spacing between the i^{th} and $(i+1)^{th}$ order statistics. The spacing involves the distribution of X more closely than the difference between the ranks.

More generally, let $p_{i,j}$ denote the probability that the unit that has true rank i is assigned judgment rank j. The

$p_{i,j} = P[X_{[j:m]} = X_{[i:m]}], i = j = 1,2,\dots,m.$ In case of perfect ranking we have $p_{i,i} = 1$ and $p_{i,j} = 0\ for\ j \neq i$

On the other hand, when the ranking is done completely randomly, we have $p_{i,j} = \frac{1}{m}\ for\ all\ i = j = 1,2,\dots,m.$ In general, however, there may be no specific structure for the $P_{i,j}$. What can be said with certainly is that $\sum_{i=1}^{m} p_{i,j} = 1$ because the j^{th} judgment order statistic will be $X_{(i:m)}$ for some i = 1, 2, …, m. In general model for imperfect ranking, thus, the $p_{i,j}'s$ are unknown parameters. Further, they cannot be estimated only on the basis of ranked set sample because the true order statistic will not be known. The only way to some hold on the ranking error is to assume it to be a function of what is observed. Using this argument, Bohn and Wolfe(1994) took $p_{i,j}$ to be inversely proportional to $E(X_{(j:m)} - X_{(i:m)})$, the expected spacing between $X_{(j:m)}$ and $X_{(i:m)}$. In mathematical terms, they assumed

$$p_{i,j} = \alpha_j / |E(X_{(j:m)} - X_{(i:m)})|$$

and $p_{i,j} = c_j$. This specification of the $p_{i,j}$ and a doubly stochastic restriction on the matrix P of the $p_{i,j}'s$ gives a completely specified model, but the values of $p_{i,j}$ depends on underlying distribution through the expected spacings. The relationship between judgment rank and true ranks can be stated in terms of their distribution function as follows.

$$F_{[j:m]}(x) = \sum_{i=1}^{m} p_{i,j}\, F_{(i:m)}(x) \qquad x \in \mathbb{R}$$

The relationship can be used to study properties of different statistics, whether estimators or test statistics.

The expected spacing model is also intuitive because it appears reasonable to say that two order statistics are more likely to be mistaken for each other when the expected difference between them is smaller than when it is larger. It does not imply that no errors occur when the expected spacing is large. It only states that it is more likely than when the expected spacing is small.

Presnell and Bohn (1999) have criticized this model because it depends on the assumption of independence, which is questionable. It is on the other hand argued that this model can be considered to be reasonable approximation, if not exact, for most of the distribution that are likely to involved. Another criticism of this model is that it uses the expected spacings but ignore the variances of this spacings. Finally the model is extend beyond set size 5.

5. Conclusions

The discussion in this paper leads to some important points regarding ranked set sampling. First and foremost, ranked set sampling may not be able to achieve perfect ranking. Second, imperfect ranking cannot be verified from observed data. Third, the only way of handling the possible ranking errors is through mathematical model. Finally, the model to be chosen from different proposed models may be chosen using the principle of pasimony. It is therefore recommended that the models of section 3 be used as the preference. It may be noted that these models from the family of models and hence the interest may be in selecting an appropriate member of this family. It is an open problem to develop method for this selection. This problem is however beyond the scope of this paper and hence not attempted in this paper.

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