3-Equitable Labeling in Context of the Barycentric Subdivision of Cycle Related Graph

I. I. Jadav, G. V. Ghodasara

1Research Scholar, R K University, Rajkot–360020, Gujarat–India
2H & H. B. Kotak Institute of Science, Rajkot–360001, Gujarat-India

Abstract: A function from the vertex set of a graph G to the set \{0, 1, 2\} is called 3-equitable labeling if the induced edge labels are produced by the absolute difference of labels of end vertices such that the absolute difference of number of vertices of G labeled with 0, 1 and 2 differ by most 1 and similarly the absolute difference of number of edges of G labeled with 0, 1 and 2 differ by most 1[1]. In this paper, we discuss 3-equitable labeling in the context of barycentric subdivision of different cycle related graphs.

Keywords: 3-equitable labeling, barycentric subdivision

AMS Subject classification number (2010): 05C78

1. Introduction

We consider simple, finite, undirected graph. If the vertices of the graph are assigned values subject to certain conditions, then it is known as a graph labeling [5]. A survey on graph labeling is given by Gallian[3].

We follow Gross and Yellen [5] for the graph theoretical terminology and notations. The vertex set and the edge set of the graph are assigned values subject to certain conditions, then it is known as a graph labeling [5].

A mapping f from V(G) to \{0,1,2\} is called ternary vertex labeling of G[1]. A ternary vertex labeling of a graph G is called barycentric labeling if the induced edge labeling function \(f^*\) from E(G) to the set \{0,1,2\} is defined as \(f^*(e = uv) = |f(u) - f(v)|\) such that the absolute difference of number of vertices of G with label 0, 1 and 2 differ by most 1 and similarly absolute difference of number of edges of G with label 0, 1 and 2 differ by most 1.

A graph which admits 3-equitable labeling is called a 3-equitable graph [1].

The barycentric subdivision of crown \(C_n \otimes K_1\) is 3-equitable.

Proof: Let \(S(C_n \otimes K_1)\) denote the barycentric subdivision of crown \(C_n \otimes K_1\). Let \(u_1, u_2, \ldots, u_n\) be pendent vertices of \(C_n \otimes K_1\) and \(v_1, v_2, \ldots, v_2n\) are the vertices corresponding to \(C_n\), where \(v_j\) is the vertex added due to the barycentric subdivision of edge \(v_jv_{j+1}\), \(j = 2, 4, 6, \ldots, 2n - 2\) and \(v_{2n}\) is the vertex added due to the barycentric subdivision of edge \(v_{2n}v_1\). Here, \(w_i\) is the vertex added due to the barycentric subdivision of edge \(u_1v_{2i-1}\), \(i = 1, 2, 3, \ldots, n\). Note that \(|V(S(C_n \otimes K_1))| = 4n\) and \(|E(S(C_n \otimes K_1))| = 4n\). We define labeling function \(f: V(S(C_n \otimes K_1)) \to \{0,1,2\}\) as follows

\[
\begin{align*}
C_n \otimes K_1 & \quad \text{Crown obtained from the corona of } C_n \text{ with } K_1. \\
C_n \otimes 2K_1 & \quad \text{Double crown obtained from the corona of } C_n \text{ with } 2K_1. \\
AC_n & \quad \text{Armed crown obtained from } C_n \oplus P_n. \\
C'_n & \quad \text{Obtained by duplicating a vertex by an edge in a cycle } C_n, \text{where } \{v_1, v_2, \ldots, v_{2n}\} \text{ are vertices added to obtain } C'_n \text{ corresponding to the vertices } v_1, v_2, \ldots, v_n \text{ in } C_n.
\end{align*}
\]

Notation 1.1: Standard graphs used in this paper [2]: Let \(e = uw\) be an edge of a graph \(G\) and \(w\) is not a vertex of \(G\). Then edge is said to be subdivided when it is replaced by edges \(e' = uw\) and \(e'' = vw\)[5]. If every edge of graph \(G\) is subdivided, then the resulting graph is called barycentric subdivision of the graph \(G\). It is denoted by \(S(G)\). Vaidya et al.[6] proved that cycle with twin chords is cordial as well as 3-equitable. In [7] Vaidya et al. proved that the barycentric subdivision of cycle with one chord, cycle with twin chords and cycle with triangle are cordial. G. V. Ghodasra and I. I. Jadav[4] proved that barycentric subdivision of cycle with one chord, cycle with twin chords, cycle with triangle, shell graph and wheel graph are 3-equitable. In this paper, we prove that the barycentric subdivision of crown, double crown, armed crown and some standard graph are 3-equitable graphs.

2. Main Results

Theorem 1: The barycentric subdivision of crown \(C_n \otimes K_1\) is 3-equitable.
Hence $S(C_n \odot K_1)$ is 3-equitable, for all $n \in \mathbb{N}, n \geq 3$. Above defined labeling pattern satisfies the conditions of 3-equitable labeling as shown in Table 1.

Let $n = 6a + b$, where $b \in \mathbb{N}$ and $a \in \mathbb{N}\cup\{0\}$.

<table>
<thead>
<tr>
<th>b, $i$</th>
<th>Vertex Conditions</th>
<th>Edge Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0,3$</td>
<td>$v_f(0) = v_f(1) = v_f(2)$</td>
<td>$\sigma_f(0) = \sigma_f(1) = \sigma_f(2)$</td>
</tr>
<tr>
<td>$1,4$</td>
<td>$v_f(0) + 1 = v_f(1) + 1 = v_f(2)$</td>
<td>$\sigma_f(0) = \sigma_f(1) = \sigma_f(2) + 1$</td>
</tr>
<tr>
<td>$2,5$</td>
<td>$v_f(0) = v_f(1) + 1 = v_f(2)$</td>
<td>$\sigma_f(0) + 1 = \sigma_f(1) = \sigma_f(2)$</td>
</tr>
</tbody>
</table>

Example 1. 3-equitable labeling of the graph obtained by barycentric subdivision of crown $C_5 \odot K_1$ is shown in Figure 1. It is the case related to $n \equiv 5(\text{mod } 6)$.

![Figure 1](image1.png)

**Theorem 2.** The barycentric subdivision of double crown $C_n \odot 2K_1$ is 3-equitable.

**Proof.** Let $S(C_n \odot 2K_1)$ denote the barycentric subdivision of double crown $C_n \odot 2K_1$. Let $u_1, u_2, \ldots, u_{2n}$ be the pendant vertices of $C_n \odot 2K_1$ and $v_1, v_2, \ldots, v_{2n}$ are vertices corresponding to $C_n$, where $v_j$ is the vertex added due to barycentric subdivision of edge $v_{j-1}v_j$, $j = 2, 4, 6, \ldots, 2n - 2$ and $v_{2n}$ is the vertex added due to the barycentric subdivision of edge $v_{2n-1}v_1$. Here, $w_i$ is the vertex added due to the barycentric subdivision of edge $u_iv_i$ for odd $i$ and $u_iv_{i-1}$ for even $i$ ($i = 2, 4, 6, \ldots, 2n$). Note that $|V(S(C_n \odot 2K_1))| = 6n$ and $|E(S(C_n \odot 2K_1))| = 6n$. We define labeling function $f: V(S(C_n \odot 2K_1)) \to \{0, 1, 2\}$ as follows.

![Figure 2](image2.png)

**Theorem 3:** The barycentric subdivision of armed crown $AC_n$ is 3-equitable.

**Proof:** Let $S(AC_n)$ denote the barycentric subdivision of armed crown $AC_n$. Let $u_i$ denote the pendant vertices of $AC_n$ and $v_1, v_2, \ldots, v_{2n}$ are the vertices corresponding to $C_n$, where $v_j$ is the vertex added due to the barycentric subdivision of edge $v_{j-1}v_j$, $j = 2, 4, 6, \ldots, 2n - 2$ and $v_{2n}$ is the vertex added due to the barycentric subdivision of edge $v_{2n-1}v_1$. Here, $w_i$ is the vertices of degree $wo$ and $w'_i$ is the vertex added due to barycentric subdivision of edge.
$w_i, v_{2i-1}, u'_i$ is the vertex added due to barycentric subdivision of edge $w_i u_i$, $1 \leq i \leq n$.

Note that $|V(S(A\mathcal{C}_n^0))| = 6n$ and $|E(S(A\mathcal{C}_n^0))| = 6n$.

We define labeling function $f: V(S(A\mathcal{C}_n^0)) = 6n \rightarrow \{0, 1, 2\}$ as follows.

Case 1: $n$ is even

$$f(v_i) = \begin{cases} 2; & i = 0, 1 (\text{mod } 4) \\ 0; & i = 2, 3, (\text{mod } 4), 1 \leq i \leq 2n. \end{cases}$$

$$f(w_i) = 1; \text{if } 1 \leq i \leq n.$$  

$$f(u'_i) = 1; \text{if } 1 \leq i \leq n.$$  

$$f(u_i) = 2; \text{if } 1 \leq i \leq n.$$  

Case 2: $n$ is odd.

$$f(v_i) = \begin{cases} 2; & i = 0, 1 (\text{mod } 4) \\ 0; & i = 2, 3, (\text{mod } 4), 1 \leq i \leq 2n. \end{cases}$$

$$f(w_i) = 1; \text{if } 1 \leq i \leq n.$$  

$$f(u'_i) = 1; \text{if } 1 \leq i \leq n.$$  

$$f(u_i) = 2; \text{if } 1 \leq i \leq n.$$  

From the above labeling pattern, we get $v'_f(0) = v_f(1) = v_f(2)$ and $e_f(0) = e_f(1) = e_f(2)$. Hence $S(A\mathcal{C}_n^0)$ is 3-equitable graph.

Example 3. 3-equitable labeling of the graph obtained by barycentric subdivision of armed crown $A\mathcal{C}_3$ is shown in Figure 3. It is the case related to $n \equiv 3 (\text{mod } 6)$.

![Figure 3](image-url)

**Theorem 4.** The barycentric subdivision of $\mathcal{C}_n'$ is 3-equitable.

**Proof.** Let denote the barycentric subdivision of $\mathcal{C}_n'$ by $u_1, u_2, \ldots, u_{2n}$ be the outer vertices of $\mathcal{C}_n'$ and $v_1, v_2, \ldots, v_{2n}$ are the vertices corresponding to $\mathcal{C}_n$, where $v_j$ is the vertex added due to the barycentric subdivision of edge $v_{j-1}v_j, j = 2, 4, 6, \ldots, 2n - 2$ and $v_{2n}$ is the vertex added due to the barycentric subdivision of edge $v_{2n-1}v_1$ and $u'_i$ is the vertex added due to barycentric subdivision of edge $u_i u_{i+1}, i = 1, 2, 3, \ldots, n$. Here $u'_i$ is the vertex added due to barycentric subdivision of edge $u_i v_i$ for odd $i$ and edge $u_i v_{i-1}$ for even $i, i = 1, 2, 3, \ldots, 2n$.

Note that $|V(S(\mathcal{C}_n'))| = 8n$ and $|E(S(\mathcal{C}_n'))| = 8n$.

We define labeling function $f: V(S(\mathcal{C}_n')) = 6n \rightarrow \{0, 1, 2\}$ as follows.

Case 1: $n \equiv 0, 3 \ (\text{mod } 6)$

$$f(v_i) = 1; \text{if } 1 \leq i \leq 2n.$$  

$$f(u'_i) = 1; \text{if } 1 \leq i \leq 2n.$$  

$$f(u_i) = 2; \text{if } 1 \leq i \leq 2n.$$  

Case 2: $n \equiv 1, 2, 4, 5 \ (\text{mod } 6)$

$$f(u_i) = 0; \text{if } 1 \leq i \leq 2n.$$  

$$f(w_i) = \begin{cases} 0; & i = 1, 4 (\text{mod } 6) \\ 1; & i = 2, 5 (\text{mod } 6) \\ 2; & i = 0, 3 (\text{mod } 6), 1 \leq i \leq n. \end{cases}$$

Above defined labeling pattern satisfies the conditions of 3-equitable labeling as shown in Table 4.

Let $n = 6a + b$, where $b \in \mathbb{N}$ and $a \in \mathbb{N} \cup \{0\}$.

**Table 2: Table for Theorem 4**

<table>
<thead>
<tr>
<th>$b$</th>
<th>Vertex Conditions</th>
<th>Edge Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 3</td>
<td>$v_f(0) = v_f(1) = v_f(2)$</td>
<td>$e_f(0) = e_f(1) = e_f(2)$</td>
</tr>
<tr>
<td>2, 5</td>
<td>$v_f(0) + 1 = v_f(1) = v_f(2)$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) + 1$</td>
</tr>
<tr>
<td>1, 4</td>
<td>$v_f(0) + 1 = v_f(1) + 1 = v_f(2)$</td>
<td>$e_f(0) + 1 = e_f(1) + 1 = e_f(2)$</td>
</tr>
</tbody>
</table>
Hence the barycentric subdivision of $C_n'$ is 3-equitable.

**Example 3.3:** equitable labeling of the graph obtained by barycentric subdivision of $C_n'$ is shown in Figure 4.

![Figure 4](image)

**References**