On the Compound Poisson Model with Debit Interest under Absolute Ruin

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Abstract: ***This paper considers the dividend payments in a compound poisson model with debit interest under absolute ruin. Firstly, we obtain the moment generating function and moments of the discounted aggregate dividend payments, then applying these results, we get the explicit expressions of them for exponential claims. Finally, we give the numerical solution of the optimal dividend barrier and the expected discounted aggregate dividend payment which are influenced by the credit interest.***

Keywords: Compound poisson model, Absolute ruin, Moment generating function, Dividend payment

1. Introduction

Since its practical importance, the issue of absolute ruin problem has attracted growing attention in risk theory. Here we consider the dividend payments in a compound poisson model with a constant debit interest r. That is to say, the insurer could borrow an amount of money equal to the deficit at a debit interest force r when the surplus is negative.

Meanwhile, the insurer will repay the debts continuously from his premium income.

We denote the surplus of the insurer at time t with the debit interest r by U(t) which is the solution to

$$dU(t)=\begin{cases} cdt-dS(t), & U(t)\geq 0 \\ (c+rU(t))dt-dS(t), & \frac{-c}{r}\leq U(t)<0 \end{cases} \tag{1.1}$$

where:

$c>0$ represents the rate of premium $S(t)=\sum_{n=1}^{N(t)} X_n$ represents the aggregate claims in time interval $[0,t]$; $\{N(t),t\geq 0\}$ is the poisson claim number process with intensity $\lambda>0$; $\{X_n, n=1,2,\cdots\}$ independent of the claim number process,are independent and identically distributed claim size random variables with common distribution function $F(x)$ that satisfies $F(0)=0$ and has a positive mean here $\overline{F}(x)=1-F(x)$.

For the feature of dividend payments, it is assumed that dividends are paid to shareholders according to a barrier strategy with parameter $b>0$.Under the barrier strategy, the premium incomes no longer go into the surplus process but are paid out as dividends when the surplus reaches b, that is to say, when the value of the surplus hits b, dividends are paid continuously at rate c and the surplus remains at level b until the next claim occurs. Denote the aggregate dividends paid in the time $[0,t]$ by $D(t)$.So we modified surplus by

$$U_b(t)=U(t)-D(t) \tag{1.2}$$

Define $T_b=\inf\{t\geq 0: U_b(t)\leq \frac{-c}{r}\}$ as the time of absolute ruin.Let $D_{u,b}$ be the present value of all dividends payable to shareholders till time T_b calculated at constant force of interest $\delta>0$.Then

$$D_{u,b}=\int_0^{T_b} e^{-\delta t}dD(t) \tag{1.3}$$

For $\frac{-c}{r}\leq u\leq b$,we denote the expectation of $D_{u,b}$ by

$$V(u,b)=E[D_{u,b}] \tag{1.4}$$

The moment generating function of $D_{u,b}$ by

$$M(u,y;b)=E[e^{yD_{u,b}}] \tag{1.5}$$

where $M(u,y;b)$ exists for all finite y since $V(u,b)\leq c\int_0^{\infty} e^{-\delta t}dt=\frac{c}{\delta}$ The nth moment of $D_{u,b}$ by

$$V_n(u,b)=E[D_{u,b}^n] \tag{1.6}$$

Where $V_0(u,b)=1$,and obviously, when $n=1, V_1(u,b)=V(u,b)$.

The contributions and novelty of this work can be summarized as follows. Firstly, the integro-differential equations for the moment generating function $M(u,y;b)$ are derived. We get the moments of $D_{u,b}$ in section 3 and find the explicit expressions of them for exponential in section 4. Finally, the optimal choice of dividend for exponential claims is discussed.

2. Moment generating function of Du,b

In this section, we analyze the moments of $D_{u,b}$ through $M(u,y;b)$ which has different paths in $0\leq u\leq b$ and $\frac{-c}{r}\leq u<0$, then we can define

$$M(u,y;b)=\begin{cases} M_1(u,y;b), & 0\le u\le b, \\ M_2(u,y;b), & \frac{-c}{r}\le u<0, \end{cases} \quad (2.1)$$

We assume that $M_1(u,y;b)$ and $M_2(u,y;b)$ are sufficiently smooth in u and y in their respective definition domains. The results are summarized in the following theorem.

Theorem 2.1 When $0<u<b$, we have

$$c\frac{\partial M_1(u,y;b)}{\partial u}=\delta y\frac{\partial M_1(u,y;b)}{\partial y}+\lambda M_1(u,y;b)-\lambda \left[\int_0^u M_1(u-x,y;b)dF(x)+\int_0^{u+\frac{c}{r}} M_1(u-x,y;b)dF(x)\right] -\lambda\bar{F}(u+\frac{c}{r}) \quad (2.2)$$

when $\frac{-c}{r}<u<0$, we have

$$(ru+c)\frac{\partial M_2(u,y;b)}{\partial u}=\delta y\frac{\partial M_2(u,y;b)}{\partial y}+\lambda M_2(u,y;b) -\lambda\int_0^{u+\frac{c}{r}} M_2(u-x,y;b)dF(x)-\lambda\bar{F}(u+\frac{c}{r}) \quad (2.3)$$

In addition, $M_1(u,y;b)$ and $M_2(u,y;b)$ satisfy

$$\frac{\partial M_1(u,y;b)}{\partial u}=yM_1(u,y;b), \quad (2.4)$$

$$M_2(-\frac{c}{r},y;b)=1 \quad (2.5)$$

Proof:

From the strong Markov property of the surplus process,we have

$$M(u,y;b)=E[M(U_b(t),e^{-\delta t}y;b)]+o(t) \quad (2.6)$$

When $0<u<b$, considering a small time interval [0,dt] and conditioning on the time and amount of the first claim we have

$$M_1(u,y;b)=(1-\lambda dt)M_1(u+cdt,e^{-\delta dt}y;b) +\lambda dt\left[\int_0^{u+ct} M_1(u+cdt-x,e^{-\delta dt}y;b)dF(x) +\int_{u+cdt}^{u+cdt+\frac{c}{r}} M_2(u+cdt-x,e^{-\delta dt}y;b)dF(x) +\bar{F}(u+cdt+\frac{c}{r})\right]+o(dt)$$

(2.7)

From the Taylor's expansion,we can get

$$M_1(u+ct,e^{-\delta t}y;b)=M_1(u,y;b)+cdt\frac{\partial M_1(u,y;b)}{\partial u} -\delta ydt\frac{\partial M_1(u,y;b)}{\partial y}+o(t) \quad (2.8)$$

Substituting (2.8)into (2.7),dividing dt on both sides,and let $dt\downarrow 0$,we can get(2.2).

When $\frac{-c}{r}<u<0$,using the same technique above,we can obtain that

$$M_2(u,y;b)=(1-\lambda dt)M_2(u+cdt,e^{-\delta dt}y;b) +\lambda dt\left[\int_0^{u+ct+\frac{c}{r}} M_2(u+cdt-x,e^{-\delta dt}y;b)dF(x) +\bar{F}(u+cdt+\frac{c}{r})\right]+o(dt) \quad (2.9)$$

Since

$$M_2(u+ct,e^{-\delta t}y;b)=M_2(u,y;b)+(ru+c)dt\frac{\partial M_2(u,y;b)}{\partial u} -\delta ydt\frac{\partial M_2(u,y;b)}{\partial y}+o(dt)$$

So we also can get (2.3)from(2.9).

When the initial surplus is b,we can have

$$M_1(b,y;b)=(1-\lambda dt)e^{ycdt}M_1(b,e^{-\delta dt}y;b) +\lambda dt\left[\int_0^b M_1(b-x,e^{-\delta dt}y;b)dF(x) +\int_b^{b+\frac{c}{r}} M_2(b-x,e^{-\delta dt}y;b)dF(x) +\bar{F}(b+\frac{c}{r})\right]+o(dt) \quad (2.10)$$

Taylor's expansion gives

$$M_1(b,e^{-\delta t}y;b)=M_1(b,y;b)-\delta ydt\frac{\partial M_1(b,y;b)}{\partial y}+o(dt)$$

Substituting this into (2.10),and suing the same way above ,we can obtain

$$\delta y\frac{\partial M_1(b,y;b)}{\partial y}=(yc-\lambda)M_1(b,y;b) +\lambda(\left[\int_0^b M_1(b-x,y;b)dF(x) +\int_b^{b+\frac{c}{r}} M_2(b-x,y;b)dF(x)\right])-\lambda\bar{F}(b+\frac{c}{r}) \quad (2.11)$$

Letting $u\uparrow b$ in(2.2)and comparing it to (2.11),(2.4)can be obtained.

When $u=\frac{-c}{r}$,the absolute ruin occurs immediately, there is no dividend, so we have(2.5).

Remark2.1 $M_1(u,y;b)$ and $M_2(u,y;b)$ satisfy

$$M_1(0+,y;b)=M_2(0-,y;b) \quad (2.12)$$

3. Moments of Du,b

By the definitions of $M(u,y;b)$ and $V_n(u,b)$,we obtain

$$M(u,y;b)=1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_n(u,b). \quad (3.1)$$

We denote

$$V_n(u,b)=\begin{cases}V_{n1}(u,b), & 0\le u\le b,\\ V_{n2}(u,b), & \frac{-c}{r}\le u<0,\end{cases} \quad (3.2)$$

We also assume that $V_{n1}(u,b)$ and $V_{n2}(u,b)$ are sufficiently smooth in their respective definition domains.

Now we take the (3.2)into (2.2)and (2.3),compare the coefficients of y^n ,so we conclude that

$$cV_{n1}'(u,b)=(\delta n+\lambda)V_{n1}(u,b) -\lambda[\int_0^u V_{n1}(u-x,b)dF(x) +\int_u^{u+\frac{c}{r}}V_{n2}(u-x,b)dF(x)] \quad (3.3)$$

for $0<u<b$,and for $\frac{-c}{r}<u<0$,we have

$$(ru+c)V_{n2}'(u,b)=(\delta n+\lambda)V_{n2}(u,b) -\lambda[\int_0^{u+\frac{c}{r}}V_{n2}(u-x,b)dF(x)] \quad (3.4)$$

When $u=b$,taking(3.2) into(2.4),we conclude that

$$V_{n1}'(u,b)\big|_{u=b}=nV_{n-1,1}(b,b) \quad (3.5)$$

Because $V_{0,1}(b,b)=1$,we can have $V_{1,1}'(b,b)=0$.

Similarity,from (2.5)and (2.12)we can obtain

$$V_{n2}(-\frac{c}{r},b)=0,n=1,2,\ldots \quad (3.6)$$

$$V_{n1}(0+,b)=V_{n2}(0-,b) \quad (3.7)$$

Further,we can have

$$V_{n1}'(0+,b)=V_{n2}'(0-,b) \quad (3.8)$$

4. Expression of $M(u,y;b)$ and $V_n(u,b)$ under Exponential Claims

In this section, we assume that $F(x)=1-e^{-\frac{x}{\mu}}$ which have the mean μ. Substituting it into (3.3)and (3.4),we can obtain that, when $0<u<b$

$$cV_{n1}'(u,b)=(\lambda+n\delta)V_{n1}(u,b)-\frac{\lambda}{\mu}e^{-\frac{u}{\mu}}[\int_0^u V_{n1}(x,b)e^{\frac{x}{\mu}}dx +\int_{-\frac{c}{r}}^{0}V_{n2}(x,b)e^{\frac{x}{\mu}}dx] \quad (4.1)$$

and when $-\frac{c}{r}<u<0$

$$(ru+c)V_{n2}'(u,b)=(\lambda+n\delta)V_{n2}(u,b) -\frac{\lambda}{\mu}e^{-\frac{u}{\mu}}\int_{-\frac{c}{r}}^{u}V_{n2}(x,b)e^{\frac{x}{\mu}}dx \quad (4.2)$$

Further more,we can take the derivative and rearrange them yields

$$cV_{n1}''(u,b)+(\frac{c}{\mu}-(\lambda+n\delta))V_{n1}'(u,b)-\frac{n\delta}{\mu}V_{n1}(u,b)=0 \quad (4.3)$$

for $0<u<b$ .With

$$x_{1/2}=\frac{\lambda+n\delta-\frac{c}{\mu}\pm\sqrt{[\frac{c}{\mu}-(\lambda+n\delta)]^2+\frac{4cn\delta}{\mu}}}{2c} \quad (4.4)$$

the solution is

$$V_{n1}(u,b)=a_{n1}h_{n1}(u)+a_{n2}h_{n2}(u) \quad (4.5)$$

where a_{n1}, a_{n2} are arbitrary constants, $h_{n1}(u)=e^{x_1u}$, $h_{n2}(u)=e^{x_2u}$.

And when $-\frac{c}{r}<u<0$

$$(ru+c)V_{n2}''(u,b)+(\frac{ru+c}{\mu}+r-(\lambda+n\delta))V_{n2}'(u,b) -\frac{n\delta}{\mu}V_{n2}(u,b)=0 \quad (4.6)$$

Using the transforms $V_{n2}(u,b)=g_{(n)}(y)$ and $y=-\frac{ru+c}{r\mu}$,we can obtain that

$$yg_{(n)}''(y)+(1-\frac{\lambda+n\delta}{r}-y)g_{(n)}'(y)+\frac{n\delta}{r}g_{(n)}(y)=0 \quad (4.7)$$

which is a confluent hypergeometric equation.

By Abramowitz[2],we can have

$$g_{(n)}(y)=a_{n3}e^yU(1-\frac{\lambda}{r},1-\frac{\lambda+n\delta}{r};-y) +a_{n4}y^{\frac{\lambda+n\delta}{r}}e^yM(1+\frac{n\delta}{r},1+\frac{\lambda+n\delta}{r};-y) \quad (4.8)$$

where a_{n3}, a_{n4} are arbitrary constants, $M(a,b;z)$ and $U(a,b;z)$ are confluent hypergeometric functions.

Thus

$$V_{n2}(u,b)=a_{n3}h_{n3}(u)+a_{n4}h_{n4}(u) \quad (4.9)$$

Where

$$h_{n3}(u)=(\frac{ru+c}{r\mu})^{\frac{\lambda+n\delta}{r}}e^{-\frac{ru+c}{r\mu}}M(1+\frac{n\delta}{r},1+\frac{\lambda+n\delta}{r};\frac{ru+c}{r\mu})$$

$$h_{n4}(u)=e^{-\frac{ru+c}{r\mu}}U(1-\frac{\lambda}{r},1-\frac{\lambda+n\delta}{r};\frac{ru+c}{r\mu}).$$

Letting $u\downarrow -\frac{c}{r}$ on both (4.9) and using the property of the confluent hypergeometric functions ,we can conclude $a_{n4}=0$.

So we have

$$V_{n2}(u,b)=a_{n3}h_{n3}(u) \qquad (4.10)$$

when $-\frac{c}{r}<u<0$

Remark 4.1 For $0<u<b$

$$M_1(u,y;b)=1+\sum_{n=1}^{\infty}\frac{y^n}{n!}(a_{n1}h_{n1}(u)+a_{n2}h_{n2}(u)) \qquad (4.11)$$

and for $-\frac{c}{r}<u<0$

$$M_2(u,y;b)=1+\sum_{n=1}^{\infty}\frac{y^n}{n!}a_{n3}h_{n3}(u) \qquad (4.12)$$

5. Optimal Choice of dividend barrier for Exponential Claims

In this section, we assume the barrier is a constant and we need to find the optimal level for the barrier, that is to say, we should find the $b*\geq 0$ which maximize $V(u,b)$ .we had $V_n(u,b)=V(u,b)$,when $n=1$.

When $n=1$,from (3.5)-(3.8),(4.5)and (4.7),we have the following equations:

$$\begin{cases} a_{11}h_{11}'(b)+a_{12}h_{12}'(b)=1, \\ a_{11}h_{11}(0)+a_{12}h_{12}(0)=a_{13}h_{13}(0), \\ a_{11}h_{11}'(0)+a_{12}h_{12}'(0)=a_{13}h_{13}'(0). \end{cases} \qquad (5.1)$$

Solving the equations, letting

$$d'(b)=h_{12}'(b)(h_{11}(0)h_{13}'(0)-h_{11}'(0)h_{13}(0))$$

$$-h_{11}'(b)(h_{12}(0)h_{13}'(0)-h_{12}'(0)h_{13}(0))$$

we obtain

$$a_{11}=-\frac{h_{12}(0)h_{13}'(0)-h_{12}'(0)h_{13}(0)}{d'(b)}$$

$$a_{12}=\frac{h_{11}(0)h_{13}'(0)-h_{11}'(0)h_{13}(0)}{d'(b)} \qquad (5.2)$$

$$a_{13}=\frac{h_{11}(0)h_{12}'(0)-h_{11}'(0)h_{12}(0)}{h_{12}'(b)d'(b)}$$

Substituting the solutions into (4.5)and (4.7),we obtain that when $0<u<b$

$$V(u,b)=\frac{d(u)}{d'(b)} \qquad (5.3)$$

and when $\frac{-c}{r}<u<0$

$$V(u,b)=\frac{h_{13}(u)(h_{11}(0)h_{12}'(0)-h_{11}'(0)h_{12}(0))}{h_{13}(0)d'(b)} \qquad (5.4)$$

For getting the $b*>0$ which makes the $V(u,b)$ maximize i.e.we need to make the denominator minimize thus $b*>0$ is solution to $d''(b)=0$,then we have $b*$ satisfies

$$d''(b*)=0 \qquad (5.5)$$

We can conclude that

$$b*=\frac{ln\frac{[h_{13}'(0)-x_2(\frac{c}{r\mu})^{(\frac{\lambda+\delta}{r})}e^{-\frac{c}{r\mu}}M(1+\frac{\delta}{r},1+\frac{\lambda+\delta}{r};\frac{c}{r\mu})]}{[h_{13}'(0)-x_1(\frac{c}{r\mu})^{(\frac{\lambda+\delta}{r})}e^{-\frac{c}{r\mu}}M(1+\frac{\delta}{r},1+\frac{\lambda+\delta}{r};\frac{c}{r\mu})]}+2ln\frac{x_1}{x_2}}{x_1-x_2} \qquad (5.6)$$

and the optimal dividend is

$$V(u,b*)=\frac{e^{x_2b*}(h_{13}'(0)-x_1h_{13}(0))-e^{x_1b*}(h_{13}'(0)-x_2h_{13}(0))}{x_2e^{x_2b*}(h_{13}'(0)-x_1h_{13}(0))-x_1e^{x_1b*}(h_{13}'(0)-x_2h_{13}(0))}. \qquad (5.7)$$

Remark5.1 From (5.5)and(5.3),we have $V_{11}''(b*,b*)=0$, substituting it into (4.3),we can obtain

$$\frac{c}{\mu}-(\lambda+\delta)-\frac{\delta}{\mu}V(b*,b*)=0 \qquad (5.8)$$

then we can have

$$V(b*,b*)=\frac{c-\mu(\lambda+\delta)}{\delta} \qquad (5.9)$$

which is a constant .

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