International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2015): 6.391

Characterization of Pathos VICT Graph of a Tree

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Abstract: ***In this paper, the concept of pathos Vict graph of a tree is introduced. Its study is concentrated only on trees. A characterization of those graphs whose pathos Vict graph of a tree are planar, outerplanar, maximal outerplanar, minimally nonouterplanar, maximal minimally nonouterplanar and crossing number one were obtained. Also a necessary and sufficient conditions for $PV_n(T)$ to be eulerian is established.***

Keywords: PVn(T)

1. Introduction

The concept of pathos of a graph G was introduced by Harary[2] as a collection of minimum number of edge disjoint open paths whose union is G. The path number of a graph G is the number of paths in pathos.

Stanton [6] and Harary [4] have calculated the path number for certain classes of graphs like trees and complete graphs. The path number of a tree T is equal to K, where 2K is the number of odd degree vertices of T. Also the endvertices of each path of any pathos of a tree are odd vertices is given by Gudagudi [1]. All undefined terminologies will conform with that in Harary [3]. All graphs considered here are finite, undirected and without loops or multiple edges. The pathos vict graph of a tree T, denoted as $PV_n(G)$ is defined as the graph whose vertex set is the union of the set of vertices, set of cutvertices and set of paths of pathos of T, in which two vertices are adjacent if and only if corresponding vertices of T are adjacent and the vertices lies on the path P_i of pathos and the vertices are adjacent to the cutvertices.

Since the system of pathos for a tree is not unique, the corresponding pathos vict graph is also not unique. In Fig-1, a tree T and its different pathos Vict graphs $PV_n(T)$ are shown.

The edge degree of an edge uv of a tree T is the sum of the degrees of u and v. The pathos length is the number of edges which lie on a particular path P_i of pathos in T. A pendant pathos is a path P_i of pathos having unit length which corresponds to a pendant edge in T. A pathos vertex is a vertex in $PV_n(T)$ corresponding to the path P_i of pathos in T.

The following results are required to prove our further results.

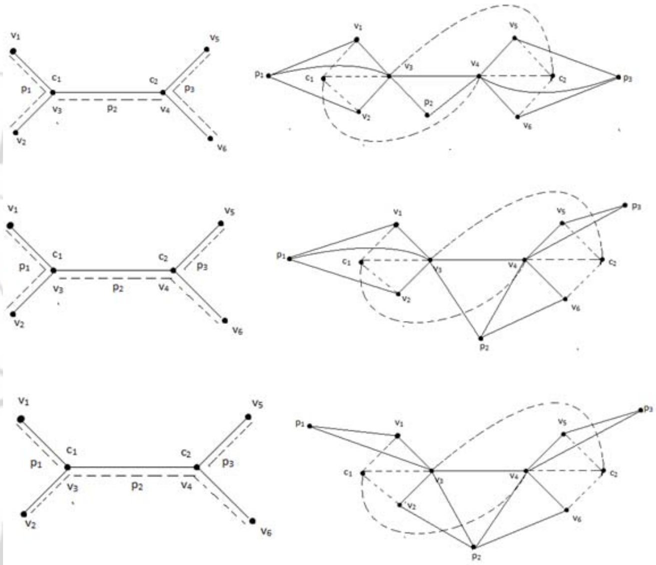

Figure 1

The following results are required to prove our further results.

Theorem A[5]: If G is a nontrivial connected (p, q) graph, Ci be the number of cutvertices in G and li be the number of edges incident with the cutvertices in G. Then vict graph $V_n(G)$ has $p+ i=0nC_i$ vertices and $[i=0n(l_i+C_i)]+q$ edges.

Theorem B[5]: Let G be (p, q) graph. Then vict graph $V_n(G)$ is outerplanar if and only if G is nonseparable outerplanar and G is either a path or a cycle.

Theorem C[3]: A graph G is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$.

Theorem D[3]: Every maximal outerplanar graph G with p vertices has (2p-3) edges.

Theorem E[3]: If G is any planar (p, q) graph with $p \geq 3$, then $q \leq 3p-6$. Furthermore, if G has no triangles then $q \leq 2p-4$.

2. Pathos Vict Graphs

We start with few preliminary results.

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Remark 1: No vertex of $PV_n(T)$ is a cutvertex.

Remark 2: The edge degree of an edge uv in a tree is odd if the degree of one vertex is even and other odd.

Remark 3: The edge degree of every edge in a tree is even if and only if every vertex is of odd degree.

Remark 4: For any tree T, $V_n(T)$ is a subgraph of $PV_n(T)$.

Remark 5: The degree of the pathos vertex in $PV_n(T)$ is equals to the pathos length of the corresponding path P_i+1 of paths in T.

Remark 6: Every pendantpathos in a tree T corresponds to a pendant edge in $PV_n(T)$ which adds one vertex to $PV_n(T)$.

Remark 7: If $T=K_{1,n}$ where n is even. Then degree of each pathos vertex p_i is 3.

Remark 8: If $T=K_{1,n}$ where n is odd. Then degree of each pathos vertex p_i is 3 except one pathos vertex, which is of degree two.

In the following theorem we obtained the number of vertices and edges in a pathos vict graph.

Theorem 1: If a graph G is a (p, q) graph where Ci be the number of cutvertices in G, li be the number of edges incident with the cutvertices in G, k be the number of paths in G and Pi be the path length of a pathos, the pathos vict graph $PV_n(T)$ has $p+k+i=1n(C_i)$ vertices and $[i=1n(C_i+l_i)+q]+[i=1n(P_i+1)]$ edges.

Proof: By the definition, the number of vertices in PVn(T) is p+k+i=1n(Ci). By Theorem A, the number of edges in Vn(G) is [i=1n(Ci+li)+q].

The number of edges in PVn(T) is the sum of edges in Vn(G) and the number of vertices which lie on the paths Pi of pathos of G, by the Remark 5, which is $i=1n(P_i+1)$. Hence the number of edges in PVn(T) is $[i=1n(C_i+l_i)+q]+[i=1n(P_i+1)]$.

3. Planar pathos vict graphs

A criterion for pathos vict graph to be planar is presented in the next theorem.

Theorem 2: The pathos vict graph $PV_n(T)$ of a tree T is planar if and only if T does not contain two adjacent cutverices, which are adjacent to at least two paths, in which path of pathos does not contain these adjacent cutvertices and their neighbours.

Proof: Suppose $PV_n(T)$ is planar. Then $V_n(T)$ is planar. Now assume there exist two adjacent cutvertices u and w which are adjacent with at least two paths P_n, P_m at u and P_s, P_t at w with path length n, m, s, t ≥1. Suppose there exist a path of pathos P_l which contains x ∈ N(u), y ∈ N(w), u, w and paths either P_n, P_t or P_s, P_m. Now assume P_l contains P_m, P_s, u, w, x and y. Then the remaining paths of pathos P_A contains P_n and path of pathos P_B contains P_t.

Since u, w, x, y, p_l, p_A and p_B are the vertices of $PV_n(T)$ and u', w' be the vertices of $PV_n(T)$ corresponding to u and w. Then in $PV_n(T)$, the edges joining uu', ux, uw, ww', wy, p_lu, p_lw, p_lx, p_ly any two edges intersecting in any plane embedding of $PV_n(T)$, a contradiction.

Conversely, suppose T Satisfies the conditions of the theorem, on plane embedding of $PV_n(T)$, the vertex p_l is not adjacent to either x or y. Suppose it is not adjacent to x. Then the both the cutvertices u', w' are adjacent to x and y without any intersection in any plane embedding.

Hence $PV_n(T)$ is planar.

We now present a characterization of tree whose pathos vict graphs are outerplanar.

Theorem 3. The pathos vict graph $PV_n(T)$ of a tree T is outerplanar if and only if $G \cong K_2$.

Proof. Suppose $PV_n(T)$ is outerplanar. Assume T has a vertex v of degree 2. The edges incident to v and the cutvertex v forms an induced subgraph homeomorphic to K_4. Hence $PV_n(T)$ is nonouterplanar, a contradiction.

Conversely, suppose T is a path P_t of length t≤1. By Theorem B, $V_n(T)$ is nonseperable outerplanar.

For t=0, the result is obvious.

For t=1, the vertices joining to $V_n(T)$ from the corresponding pathos vertex gives $PV_n(T)$ which is a triangle. By Theorem C, $PV_n(T)$ is outerplanar.

We now present a characterization of tree whose pathos vict graphs are maximal outerplanar.

Theorem 4: The pathos vict graph $PV_n(T)$ of a tree T is maximal outerplanar if and only if T is a K_2.

Proof: Suppose $PV_n(T)$ is maximal outerplanar. Then $PV_n(T)$ is connected. Hence T is connected. If $PV_n(T)$ is K_3, then obviously T is K_2.

Let T be any connected tree with p≤2 vertices, q edges, C_i cutvertices, l_i be the number of edges incident with the cutvertices in T, k be the number of paths in G and P_i be the path length of a pathos.

Then $PV_n(T)$ has $[p+k+i=1n(C_i)]$ vertices and $[i=1n(l_i+C_i)+q]+[i=1n(P_i+1)$ edges. Since $PV_n(T)$ is maximal outerplanar, by Theorem D, it has $2[p+k+i=1n(C_i)]$ -3 edges. Hence,

$$[i=1n(l_i+C_i)+q]+[i=1n(P_i+1)]=2[p+k+i=1n(C_i)]-3.$$

Thus for $T=K_2$ we have $l_i=0$, $C_i=0$, p=2, k=1, $P_i=1$ and q=1.

$$[i=1n(0+0)+1]+i=1n(1+1)=2[2+1+i=1n0]-3$$

1+1+1=2[2+1]-3
3=2[3]-3
3=6-3
3=3

By Theorem D, it follows that G is a K_2
Necessity is thus proved.

For sufficiency, suppose T is a path, $P_n(n\leq 2)$. Then we consider two cases.

Case1: Assume T is K_1. Then $PV_n(T)= K_1$. Hence it is maximal outerplanar.

Case2: Assume T is K_2.Then $PV_n(T)= K_3$, Which is maximal outerplanar.

For any plane graph G the inner vertex number i(G) of G is the minimum number of vertices not belonging the boundary of the exterior region in any embedding of G in the plane. We call the inner vertex number i(G) as Kulli number.

A graph G is said to be minimally nonouterplanar if Kulli number is one or i(G)=1.
In the next theorem we establish the result in which $PV_n(T)$ has a Kulli number one.

Theorem 5: For any tree T, $PV_n(T)$ has a Kulli number one if and only if Δ (T)≤2 for every vertex v of T and T has a unique vertex of degree two.

Proof: Suppose $PV_n(T)$ has a Kulli number. Assume Δ (T)>2. Let v be a vertex with deg(v)=3, let v_1, v_2, v_3 are adjacent to v. In V_n (T) has a subgraph homeomorphic to $K_{2,3}$. Since it has two pathos vertices.Then any one vertex lies in the interior region of embedded $PV_n(T)$. Hence Kulli number of $PV_n(T)$ is more than one, a contradiction. Hence Δ (T)=2.

Assume that there exist at least two vertices of degree 2 in T. Then $V_n(T)$ has at least two blocks as P_3+K_1. Since T has exactly one pathos, let v be a pathos vertex which is adjacent to all the vertices of P_3+K_1. P_3+K_1. On embedding $PV_n(T)$ in any plane it has at least two Kulli number, a contradiction.Hence T has exactly one vertex of degree two.

Conversely, suppose every vertex of T has degree≤2 and has a unique vertex of degree 2. Then $V_n(T)$ has exactly one block as K_4-x. A pathos vertex v is adjacent to every vertex of tree T.

Which gives W_5 as a subgraph. Hence $PV_n(T)$ has a Kulli number one.

In the next theorem we prove that $PV_n(T)$ is maximal minimally nonouterplanar.

Theorem6: For any tree T, $PV_n(T)$ has maximal Kulli number one if and only if Δ (T)≤2 has unique vertex of degree 2.

Proof: Suppose $PV_n(T)$ of tree T has maximal Kulli number one. We consider the following cases.

Case1: If Δ (T) <2. Then by Theorem 4, $PV_n(T)$ is maximal outerplanar, a contradiction.

Case2: If Δ(T)>3. Then by Theorem 5, $PV_n(T)$ has greater than Kulli number one, a contradiction.

Case3: If T has at least two vertices of degree 2. Then $PV_n(T)$ has greater than Kulli number one, a contradiction.

Case4: If T has a unique vertex v of degree 2. By Theorem 5, $PV_n(T)$ has Kulli number one. Now we Show that $PV_n(T)$ has maximal Kulli number one. Since $T=K_{1,2}$, then $V_n(T)= K_4$-x and $PV_n(K_{1,2})=W_5$.

Which has maximal Kulli number one.

The next theorem characterizes $PV_n(T)$ in terms of crossing number one.

Theorem 7: The pathos vict graph $PV_n(T)$ of a tree T has a crossing number one if and only if for any tree T with Δ (T)≥3 has exactly two adjacent vertices v_1 and v_2 with Δ (T) and remaining vertices are of degree either one or two. Also the path of pathos contains $v_1, v_2, N(v_1)$ and $N(v_2)$.

Proof: Suppose pathos vict graph of a tree T has crossing number one. Then $V_n(T)$ is planar. Now we assume Δ(T)≥2. Then we consider the following cases.

Case1: Assume Δ(T)=2. Then T=Pn. By Theorem1 and Theorem E, [i=1n(Ci+Ii)+q]+ [i=1n(Pi+1)]≤3n-6. Hence Cr[PVn(T)]=0, a contradiction.

Case2: Assume Δ(T)≥3 and T has exactly two adjacent vertices v_1 and v_2 with Δ(T) and remaining vertices are of degree either one or two. Also the path of pathos contain v_1, v_2, $N(v_1)$ and $N(v_2)$.Then we consider following subcases of case2.

Subcase 2.1: Suppose T has three cutvertices v_1, v_2 and v_3 of degree Δ(T). Further assume that v_1 is adjacent to v_2 and v_2 is adjacent to v_3 and path of pathos contains these three vertices. Then in $PV_n(T)$, { $v_1, v_2, N(v_1), v_3\ v^I_1, v^I_2$} forms a subgraph homeomorphic to $K_{3,3}$ where v^I_1, v^I_2 are corresponding vertices of v_1, v_2. Also {v_3, v_2, v_1, v^I_2, v^I_3, $N(v_3)$} forms another subgraph homeomorphic to $K_{3,3}$ where v_2^I, v_3^I are cut vertices and v_2^I, $v_3^I \in V[PV_n(T)]$. Hence $C_r\ [PV_n(T)]>1$, a contradiction.

Subcase 2.2: Suppose path of pathos does not contain either $N(v_1)$ or $N(v_2)$. Now assume path of pathos contain $N(v_1)$. Then there exist v_1^I and $v_2^I \in PV_n(T)$ where v_1^I, v_2^I corresponds to v_1, $v_2 \in T$. In $PV_n(T)$, v_1^I is adjacent to v_1 and $N(v_1)$. And v_2^I is adjacent to v_2 and $N(v_2)$ and pathos vertex p_1 which is adjacent to v_1, v_2 and $N(v_1)$. Thus in planar embedding of $PV_n(T)$ in any plane, $PV_n(T)$ is planar. Thus $C_r[PV_n(T)]=0$, a contradiction. On the other hand if path of pathos contains $N(v_2)$, we have $C_r[PV_n(T)]=0$ again, a contradiction.

Paper ID: NOV164311 http://dx.doi.org/10.21275/v5i6.NOV164311

Conversely, suppose T holds the condition of the Theorem. Let v_1 and v_2 are two adjacent cut vertices of degree $\Delta(T)\geq 2$, such that v_1^I and $v_2^I \in PV_n(T)$, corresponding to $v_1, v_2 \in T$. Then there exist a pathos vertex p_1 which lies on v_1, v_2, $N(v_1)$and $N(v_2)$.

In $PV_n(T)$, v_1^I is adjacent to v_1 and $N(v_1)$, also v_2^I is adjacent to v_2 and $N(v_2)$, p_1 is adjacent to v_1, v_2, $N(v_1)$ and $N(v_2)$. This adjacency produces an induced subgraph homeomorphic to $K_{3,3}$.

Hence $C_r[PV_n(T)]=1$.

The noneulerian property of $PV_n(T)$ is giving by the following theorem.

Theorem 8: For any nontrivial tree T with $p\geq 3$ vertices $PV_n(T)$ is noneulerian.

Proof: Suppose T is a tree with $p<3$ vertices. Then T is either K_1 or K_2. Assume if $T=K_1$, $PV_n(T) = K_1$ and if $T=K_2$, $PV_n(T) = K_3$. Then $PV_n(T)$ is eulerian.

Suppose T is a tree with $p\geq 3$ verices. Let $A=\{ v_1, v_2, \text{--------}, v_n\}$ be the set of vertices such that $\deg(v_i)$=odd, $\forall\ V_i \in A$ and $B=\{ v_1, v_2, \text{------}, v_m\}$ be the set of vertices in which each $v_j \in B$, $\deg(v_j)$=even. In $V_n(T)$, there exist subsets $A_1 \subset A$, $A_2 \subset A$ and $B_1 \subset B$, $B_2 \subset B$ such that if $\deg(v_i)\ \forall\ v_i \in A_1$ is odd, then $\deg(v_i)\ \forall\ V_i \in A_2$ is even. Similarly for B_1 and B_2. In $PV_n(T)$, $\deg(v_i)\ \forall\ V_i \in A_1$, is even and $\deg(v_i)\ \forall\ v_i \in A_2$ is odd. Similarly it is true for the subsets B_1 and B_2. Hence there exists at least one subset containing the vertices of odd degree. Hence $PV_n(T)$ is noneulerian.

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