# Characterization of Pathos VICT Graph of a Tree

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Abstract: In this paper, the concept of pathos Vict graph of a tree is introduced. Its study is concentrated only on trees. A characterization of those graphs whose pathos Vict graph of a tree are planar, outerplanar, maximal outerplanar, minimally nonouterplanar and crossing number one were obtained. Also a necessary and sufficient conditions for  $PV_n(T)$  to be eulerian is established.

Keywords: PVn(T)

## 1. Introduction

The concept of pathos of a graph G was introduced by Harary[2] as a collection of minimum number of edge disjoint open paths whose union is G. The path number of a graph G is the number of paths in pathos.

Stanton [6] and Harary [4] have calculated the path number for certain classes of graphs like trees and complete graphs. The path number of a tree T is equal to K, where 2K is the number of odd degree vertices of T. Also the endvertices of each path of any pathos of a tree are odd vertices is given by Gudagudi [1]. All undefined terminologies will conform with that in Harary [3]. All graphs considered here are finite, undirected and without loops or multiple edges. The pathos vict graph of a tree T, denoted as  $PV_n(G)$  is defined as the graph whose vertex set is the union of the set of vertices, set of cutvertices and set of paths of pathos of T, in which two vertices are adjacent if and only if corresponding vertices of T are adjacent and the vertices lies on the path P<sub>i</sub> of pathos and the vertices are adjacent to the cutvertices.

Since the system of pathos for a tree is not unique, the corresponding pathos vict graph is also not unique. In Fig-1, a tree T and its different pathos Vict graphs  $PV_n(T)$  are shown.

The edge degree of an edge uv of a tree T is the sum of the degrees of u and v. The pathos length is the number of edges which lie on a particular path  $P_i$  of pathos in T. A pendant pathos is a path  $P_i$  of pathos having unit length which corresponds to a pendant edge in T. A pathos vertex is a vertex in  $PV_n(T)$  corresponding to the path  $P_i$  of pathos in T.

The following results are required to prove our further results.



Figure 1

The following results are required to prove our further results.

**Theorem A[5]:** If G is a nontrivial connected (p, q) graph, Ci be the number of cutvertices in G and li be the number of edges incident with the cutvertices in G. Then vict graph  $V_n$  (G) has p+ i=0nC<sub>i</sub> vertices and [i=0n(l<sub>i</sub>+C<sub>i</sub>)]+q edges.

**Theorem B[5]:** Let G be (p, q) graph. Then vict graph  $V_n(G)$  is outerplanar if and only if G is nonseparable outerplanar and G is either a path or a cycle.

**Theorem C[3]:** A graph G is outerplanar if and only if it has no subgraph homeomorphic to  $K_4$  or  $K_{2,3}$ .

**Theorem D[3]:** Every maximal outerplanar graph G with p vertices has (2p-3) edges.

**Theorem E[3]:** If G is any planar (p, q) graph with  $p \ge 3$ , then  $q \le 3p-6$ . Furthermore, if G has no triangles then  $q \le 2p-4$ .

## 2. Pathos Vict Graphs

We start with few preliminary results.

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**Remark 1:** No vertex of  $PV_n(T)$  is a cutvertex.

**Remark 2:** The edge degree of an edge uv in a tree is odd if the degree of one vertex is even and other odd.

**Remark 3:** The edge degree of every edge in a tree is even if and only if every vertex is of odd degree.

**Remark 4:** For any tree T,  $V_n(T)$  is a subgraph of  $PV_n(T)$ .

**Remark 5:** The degree of the pathos vertex in  $PV_n(T)$  is equals to the pathos length of the corresponding path  $P_i+1$  of paths in T.

**Remark 6:** Every pendantpathos in a tree T corresponds to a pendant edge in  $PV_n(T)$  which adds one vertex to  $PV_n(T)$ .

**Remark 7:** If  $T=K_{1, n}$  where n is even. Then degree of each pathos vertex  $p_i$  is 3.

**Remark 8:** If  $T=K_{1, n}$  where n is odd. Then degree of each pathos vertex  $p_i$  is 3 except one pathos vertex, which is of degree two.

In the following theorem we obtained the number of vertices and edges in a pathos vict graph.

**Theorem 1:** If a graph G is a (p, q) graph where Ci be the number of cutvertices in G, li be the number of edges incident with the cutvertices in G, k be the number of paths in G and Pi be the path length of a pathos, the pathos vict graph  $PV_n(T)$  has  $p+k+i=1n(C_i)$  vertices and  $[i=1n(C_i+l_i)+q]+[i=1n(P_i+1)]$  edges.

Proof: By the definition, the number of vertices in PVn(T) is p+k+i=1n(Ci). By Theorem A, the number of edges in Vn(G) is [i=1n(Ci+li)+q].

The number of edges in PVn(T) is the sum of edges in Vn(G) and the number of vertices which lie on the paths Pi of pathos of G, by the Remark 5, which is  $i=1n(P_i+1)$ . Hence the number of edges in PVn(T) is  $[i=1n(C_i+I_i)+q]+[i=1n(P_i+1)]$ .

## 3. Planar pathos vict graphs

A criterion for pathos vict graph to be planar is presented in the next theorem.

**Theorem 2:** The pathos vict graph  $PV_n(T)$  of a tree T is planar if and only if T does not contain two adjacent cutverices, which are adjacent to at least two paths, in which path of pathos does not contain these adjacent cutvertices and their neighbours.

**Proof:** Suppose  $PV_n(T)$  is planar. Then  $V_n(T)$  is planar. Now assume there exist two adjacent cutvertices u and w which are adjacent with at least two paths  $P_n$ ,  $P_m$  at u and  $P_s$ ,  $P_t$  at w with path length n, m, s,  $t \ge 1$ . Suppose there exist a path of pathos  $P_1$  which contains  $x \in N(u)$ ,  $y \in N(w)$ , u, w and paths either  $P_n$ ,  $P_t$  or  $P_s$ ,  $P_m$ . Now assume  $P_1$  contains  $P_m$ ,  $P_s$ , u, w, x and y. Then the remaining paths of pathos  $P_A$  contains  $P_n$  and path of pathos  $P_B$  contains  $P_t$ .

Since u, w, x, y,  $p_l$ ,  $p_A$  and  $p_B$  are the vertices of  $PV_n(T)$ and u', w' be the vertices of  $PV_n(T)$  corresponding to u and w. Then in  $PV_n(T)$ , the edges joining uu', ux, uw, ww', wy,  $p_lu$ ,  $p_lw$ ,  $p_lx$ ,  $p_ly$  any two edges intersecting in any plane embedding of  $PV_n(T)$ , a contradiction.

Conversely, suppose T Satisfies the conditions of the theorem, on plane embedding of  $PV_n(T)$ , the vertex  $p_1$  is not adjacent to either x or y. Suppose it is not adjacent to x. Then the both the cutvertices u', w' are adjacent to x and y without any intersection in any plane embedding.

Hence  $PV_n(T)$  is planar.

We now present a characterization of tree whose pathos vict graphs are outerplanar.

Theorem 3. The pathos vict graph  $PV_n(T)$  of a tree T is outerplanar if and only if  $G \cong K_2$ .

Proof. Suppose  $PV_n(T)$  is outerplanar. Assume T has a vertex v of degree 2. The edges incident to v and the cutvertex v forms an induced subgraph homeomorphic to  $K_4$ . Hence  $PV_n(T)$  is nonouterplanar, a contradiction.

Conversely, suppose T is a path  $P_t$  of length t $\leq 1$ . By Theorem B,  $V_n(T)$  is nonseperable outerplanar.

For t=0, the result is obvious.

For t=1, the vertices joining to  $V_n(T)$  from the corresponding pathos vertex gives  $PV_n(T)$  which is a triangle. By Theorem C,  $PV_n(T)$  is outerplanar.

We now present a characterization of tree whose pathos vict graphs are maximal outerplanar.

**Theorem 4:** The pathos vict graph  $PV_n(T)$  of a tree T is maximal outerplanar if and only if T is a  $K_2$ .

**Proof:** Suppose  $PV_n(T)$  is maximal outerplanar. Then  $PV_n(T)$  is connected. Hence T is connected. If  $PV_n(T)$  is  $K_3$ , then obviously T is  $K_2$ .

Let T be any connected tree with  $p\leq 2$  vertices, q edges,  $C_i$  cutvertices,  $l_i$  be the number of edges incident with the cutvertices in T, k be the number of paths in G and  $P_i$  be the path length of a pathos.

 $[i=1n(l_i+C_i)+q]+[i=1n(P_i+1)]=2[p+k+i=1n(C_i)]-3.$ 

Thus for  $T=K_2$  we have  $l_i=0$ ,  $C_i=0$ , p=2, k=1,  $P_i=1$  and q=1.

[i=1n(0+0)+1]+i=1n(1+1) = 2[2+1+i=1n0]-3

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1+1+1=2[2+1]-3 3=2[3]-3 3=6-3 3=3

By Theorem D, it follows that G is a K<sub>2</sub> Necessity is thus proved.

For sufficiency, suppose T is a path,  $P_n(n{\leq}2).$  Then we consider two cases.

**Case1:** Assume T is  $K_1$ . Then  $PV_n(T) = K_1$ . Hence it is maximal outerplanar.

**Case2:** Assume T is  $K_{2}$ . Then  $PV_n(T) = K_{3}$ , Which is maximal outerplanar.

For any plane graph G the inner vertex number i(G) of G is the minimum number of vertices not belonging the boundary of the exterior region in any embedding of G in the plane. We call the inner vertex number i(G) as Kulli number.

A graph G is said to be minimally nonouterplanar if Kulli number is one or i(G)=1.

In the next theorem we establish the result in which  $\mathrm{PV}_n(T)$  has a Kulli number one.

**Theorem 5:** For any tree T,  $PV_n(T)$  has a Kulli number one if and only if  $\Delta(T) \le 2$  for every vertex v of T and T has a unique vertex of degree two.

**Proof:** Suppose  $PV_n(T)$  has a Kulli number. Assume  $\Delta$  (T)>2. Let v be a vertex with deg(v)=3, let  $v_1$ ,  $v_2$ ,  $v_3$  are adjacent to v. In  $V_n$  (T) has a subgraph homeomorphic to  $K_{2, 3}$ . Since it has two pathos vertices. Then any one vertex lies in the interior region of embedded  $PV_n(T)$ . Hence Kulli number of  $PV_n(T)$  is more than one, a contradiction. Hence  $\Delta$  (T)=2.

Assume that there exist at least two vertices of degree 2 in T. Then  $V_n(T)$  has at least two blocks as  $P_3+K_1$ . Since T has exactly one pathos, let v be a pathos vertex which is adjacent to all the vertices of  $P_3+K_1$ .  $P_3+K_1$ . On embedding  $PV_n(T)$  in any plane it has at least two Kulli number, a contradiction.Hence T has exactly one vertex of degree two.

Conversely, suppose every vertex of T has degree  $\leq 2$  and has a unique vertex of degree 2. Then  $V_n(T)$  has exactly one block as  $K_4$ -x. A pathos vertex v is adjacent to every vertex of tree T.

Which gives  $W_5$  as a subgraph. Hence  $PV_n(T)$  has a Kulli number one.

In the next theorem we prove that  $PV_n(T)$  is maximal minimally nonouterplanar.

**Theorem6:** For any tree T,  $PV_n(T)$  has maximal Kulli number one if and only if  $\Delta(T)\leq 2$  has unique vertex of degree 2.

**Proof:** Suppose  $PV_n(T)$  of tree T has maximal Kulli number one. We consider the following cases.

**Case1:** If  $\Delta$  (T) <2. Then by Theorem 4,  $PV_n(T)$  is maximal outerplanar, a contradiction.

**Case2:** If  $\Delta(T)>3$ . Then by Theorem 5,  $PV_n(T)$  has greater than Kulli number one, a contradiction.

**Case3:** If T has at least two vertices of degree 2. Then  $PV_n(T)$  has greater than Kulli number one, a contradiction.

**Case4:** If T has a unique vertex v of degree 2. By Theorem 5,  $PV_n(T)$  has Kulli number one. Now we Show that  $PV_n(T)$  has maximal Kulli number one. Since  $T=K_{1, 2}$ , then  $V_n(T)=K_4$ -x and  $PV_n(K_{1, 2})=W_5$ .

Which has maximal Kulli number one.

The next theorem characterizes  $PV_n(T)$  in terms of crossing number one.

**Theorem 7:** The pathos vict graph  $PV_n(T)$  of a tree T has a crossing number one if and only if for any tree T with  $\Delta$  (T) $\geq$ 3 has exactly two adjacent vertices  $v_1$  and  $v_2$  with  $\Delta$  (T) and remaining vertices are of degree either one or two. Also the path of pathos contains  $v_1$ ,  $v_2$ ,  $N(v_1)$  and  $N(v_2)$ .

**Proof:** Suppose pathos vict graph of a tree T has crossing number one. Then  $V_n(T)$  is planar. Now we assume  $\Delta(T)\geq 2$ . Then we consider the following cases.

**Case1:** Assume  $\Delta(T)=2$ . Then T=Pn. By Theorem1 and Theorem E,  $[i=1n(Ci+Ii)+q]+ [i=1n(Pi+1)] \le 3n-6$ . Hence Cr[PVn(T)]=0, a contradiction.

**Case2:** Assume  $\Delta(T) \ge 3$  and T has exactly two adjacent vertices  $v_1$  and  $v_2$  with  $\Delta(T)$  and remaining vertices are of degree either one or two. Also the path of pathos contain  $v_1$ ,  $v_2$ ,  $N(v_1)$  and  $N(v_2)$ .Then we consider following subcases of case2.

**Subcase 2.1:** Suppose T has three cutvertices  $v_1$ ,  $v_2$  and  $v_3$  of degree  $\Delta(T)$ . Further assume that  $v_1$  is adjacent to  $v_2$  and  $v_2$  is adjacent to  $v_3$  and path of pathos contains these three vertices. Then in  $PV_n(T)$ , { $v_1, v_2, N(v_1), v_3 v_1^I, v_2^I$ } forms a subgraph homeomorphic to  $K_{3, 3}$  where  $v_1^I$ ,  $v_2^I$  are corresponding vertices of  $v_1$ ,  $v_2$ . Also { $v_3, v_2, v_1, v_2^I, v_3^I$ ,  $N(v_3)$ } forms another subgraph homeomorphic to  $K_{3, 3}$  where  $v_2^I$ ,  $v_3^I$  are cut vertices and  $v_2^I$ ,  $v_3^I \in V[PV_n(T)]$ . Hence  $C_r [PV_n(T)] > 1$ , a contradiction.

**Subcase 2.2:** Suppose path of pathos does not contain either  $N(v_1)$  or  $N(v_2)$ . Now assume path of pathos contain  $N(v_1)$ . Then there exist  $v_1^{I}$  and  $v_2^{I} \in PV_n(T)$  where  $v_1^{I}$ ,  $v_2^{I}$  corresponds to  $v_1$ ,  $v_2 \in T$ . In  $PV_n(T)$ ,  $v_1^{I}$  is adjacent to  $v_1$  and  $N(v_1)$ . And  $v_2^{I}$  is adjacent to  $v_2$  and  $N(v_2)$  and pathos vertex  $p_1$  which is adjacent to  $v_1$ ,  $v_2$  and  $N(v_1)$ . Thus in planar embedding of  $PV_n(T)$  in any plane,  $PV_n(T)$  is planar. Thus  $C_r[PV_n(T)]=0$ , a contradiction. On the other hand if path of pathos contains  $N(v_2)$ , we have  $C_r[PV_n(T)]=0$  again, a contradiction.

Conversely, suppose T holds the condition of the Theorem. Let  $v_1$  and  $v_2$  are two adjacent cut vertices of degree  $\Delta(T)\geq 2$ , such that  $v_1^{\ I}$  and  $v_2^{\ I} \in PV_n(T)$ , corresponding to  $v_1$ ,  $v_2 \in T$ . Then there exist a pathos vertex  $p_1$  which lies on  $v_1$ ,  $v_2$ ,  $N(v_1)$  and  $N(v_2)$ .

In  $PV_n(T)$ ,  $v_1^{I}$  is adjacent to  $v_1$  and  $N(v_1)$ , also  $v_2^{I}$  is adjacent to  $v_2$  and  $N(v_2)$ ,  $p_1$  is adjacent to  $v_1$ ,  $v_2$ ,  $N(v_1)$  and  $N(v_2)$ . This adjacency produces an induced subgraph homeomorphic to  $K_{3,3}$ .

Hence  $C_r[PV_n(T)]=1$ .

The noneulerian property of  $PV_n(T)$  is giving by the following theorem.

**Theorem 8:** For any nontrivial tree T with  $p \ge 3$  vertices  $PV_n(T)$  is noneulerian.

**Proof:** Suppose T is a tree with p<3 vertices. Then T is either  $K_1$  or  $K_2$ . Assume if  $T=K_1$ ,  $PV_n(T) = K_1$  and if  $T=K_2$ ,  $PV_n(T) = K_3$ . Then  $PV_n(T)$  is eulerian.

Suppose T is a tree with  $p \ge 3$  vertices. Let  $A = \{ v_1, v_2, \dots, v_n \}$  be the set of vertices such that  $deg(v_i) = odd$ ,  $\forall V_i \in A$  and  $B = \{ v_1, v_2, \dots, v_m \}$  be the set of vertices in which each

 $v_i \in B$ , deg( $v_i$ )=even. In  $V_n(T)$ , there exist subsets  $A_1 \subset A$ ,

 $\begin{array}{l} A_2 \subset A \text{ and } B_1 \subset B, \ B_2 \subset B \text{ such that if } deg(v_i) \ \forall \ v_i \in A_1 \text{ is} \\ \text{odd, then } deg(v_i) \ \forall \ V_i \in A_2 \text{ is even. Similarly for } B_1 \text{ and} \\ B_2. \ In \ PV_n(T), \ deg(v_i) \ \forall \ V_i \in A_1, \text{ is even and } deg(v_i) \ \forall \ v_i \in A_2 \text{ is odd. Similarly it is true for the subsets } B_1 \text{ and } B_2. \\ \text{Hence there exists at least one subset containing the} \\ \text{vertices of odd degree. Hence } PV_n(T) \text{ is noneulerian.} \end{array}$ 

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