

# Chebyshev Best Approximation of $(p, q)$ -Type and Lower $(p, q)$ -Type of Entire Functions of Several Complex Variables

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**Abstract:** This paper deals with the characterization of generalized  $(p, q)$ -type and generalized lower  $(p, q)$ -type with respect to proximate order of an entire function  $f : C^2 \rightarrow C$  in terms of the Chebyshev best approximation to  $f$  on compact set  $E \subset C^2$  by polynomials. In this paper we want to establish formulae for lower  $(p, q)$ -type of entire functions of two complex variables with index pair  $(p, q)$ .

**Keywords:** Chebyshev approximation, polynomials, generalized, bounded, proximate order

## 1. Introduction

Let  $E$  be a bounded closed set in the space  $C^2$  of two complex variables  $z = (z_1, z_2)$  with the norm

$$\|f\|_E = \sup \{ |f(z)| : z \in E \}$$

For a function  $f$  defined and bounded on  $E$ .

Let  $P_\nu$  denote the set of all polynomials in  $z$  of degree  $< \nu$ . Set

$$E_\nu(f, E) = \inf \{ \|f - p\|_E : p \in P_\nu \}.$$

The following theorem for single complex variable was proved by Winiarski [1].

**Theorem 1:** A function  $f$  defined and bounded on a closed set  $E$  with a positive transfinite diameter  $d$ , can be continued to an entire function  $f$  of order  $\rho$  ( $0 < \rho < \infty$ ) and of type  $\sigma$  ( $0 < \sigma < \infty$ ) if and only if

$$(1.1) \limsup_{\nu \rightarrow \infty} \nu^{1/\rho} (E_\nu(f, E))^{1/\nu} = d(e\sigma\rho)^{1/\rho}.$$

It has been noted that in two or more than two complex variables, the type and lower type of  $f(z)$  can not be characterized by means of the measure of the Chebyshev best approximation to  $f$  on  $E$  by polynomials of degree  $\leq \nu$  with respect to all variable. Due to this fact Kumar D. [2] considered the measures  $E_k^*(f, E)$ ,  $k = (k_1, k_2)$  of the Chebyshev best approximation to  $f$  in  $E = E^{(1)} \times E^{(2)}$  by polynomials of degree  $\leq k_j$  with respect to the  $j^{\text{th}}$  variable,  $j = 1, 2$ , where  $E_j$  is bounded closed set with a positive transfinite diameter  $d_j = d(E^j)$  in the complex  $z_j$  plane. He extended the above theorem

for two complex variables and to estimate the slow and fast growth of entire functions. This theorem also have been extended to  $(p, q)$ -scale introduced by Juneja et al. ([3],[4]). But these results are inadequate for comparing the growth of those entire functions which are of same  $(p, q)$ -order but of infinite  $(p, q)$ -types. To refine this scale we shall obtain  $(p, q)$ -type and lower  $(p, q)$ -type with respect to proximate order of index pair  $(p, q)$ , for integers  $p$  and  $q$  such that  $1 \leq q \leq p$ . Analogous results for generalized lower  $(p, q)$ -type also have been studied.

Let  $D$  be a complex Banach space with norm  $\|\cdot\|$ . Let  $f : C^2 \rightarrow D$  be an entire function and  $P_k = P_k(C^2, D)$ ,  $k = (k_1, k_2)$  be the set of all polynomials  $p : C^2 \rightarrow D$  of degree  $\leq k_j$ , with respect to  $j^{\text{th}}$  variable, respectively,  $j = 1, 2$ .

Let  $E$  be a compact set in  $C^2$  and let  $f : E \rightarrow D$  be a function defined and bounded on  $E$ . Set

$$E_k^*(f, E) = \inf \{ \|f - p\|_E : p \in P_k \}.$$

Let  $E = E^{(1)} \times E^{(2)}$ , when  $E^{(j)}$  ( $j = 1, 2$ ) is a compact set in  $C$  containing infinitely many different points. Let  $n_j^{k_j} = (n_{j0}, \dots, n_{jk_j})$ ,  $j = 1, 2$ , be a system of  $k_j + 1$  extremal points of  $E_j$  (see [2]). The polynomial

$$L_k(z) = \sum_{u_1, u_2=0}^{k_1, k_2} (n_{1u_1}, n_{2u_2}) L^{(u_1)}(z_1) L^{(u_2)}(z_2)$$

is the Lagrange interpolation polynomial for  $f$  with nodes  $n_1^{(k_1)} \times n_2^{(k_2)}$  of degree  $\leq k_j$  with respect to the  $j^{\text{th}}$  variable. Here

$$L^{(u_j)}(z_j) = L^{(u_j)}(z_j - E_j) = \frac{(z_j - n_{j^0}) \cdots}{(n_{j^0} - n_{j^0}) \cdots} \left| \frac{(z_j n_j k_j)}{(n_j u_j - n_j k_j)} \right|$$

where  $|u_j$  means that the factor  $u_j$  is omitted. The inequality (1.2)

$$E_k^*(f, E) \leq \|f - L_k\|_E \left( 1 + \prod_{j=1}^2 (k_j + 1) \right) E_k^*(f, E)$$

can be proved in a similar manner as Lemma 1.1 of [2].

**Definition 1:** An entire function defined on  $C^2$  is said to be  $(p, q)$ -order  $\rho(p, q)$  and if  $(b < \rho(p, q) < \infty)$   $(p, q)$ -type  $\tau(p, q)$  and lower  $(p, q)$ -type  $t(p, q)$  if

$$\rho(p, q) = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} \rho(r, f)}{\log^{[q]} r}$$

$$\tau(p, q) = \limsup_{r \rightarrow \infty} \frac{\log^{[p-1]} S(r, f)}{(\log^{[q-1]} r)^{\rho(p, q)}}, \quad 0 < t(p, q) \leq \tau(p, q) \leq \infty$$

$$S(r, f) = \sup_{|z|=r} \|f(z)\| \quad \forall r \in R^+, \quad \log^{[m]} x = \exp^{[m]} x < \infty$$

with  $\log^{[0]} x = \exp^{[0]} x = x, b = 1$  if  $p = q$  and  $b = 0$  if  $p > q$ .

**Definition 2:** The generalized  $(p, q)$ -type  $\tau^*$  and generalized lower  $(p, q)$ -type  $t^*$  of  $f(z)$  are defined as

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p-1]} S(r, f)}{(\log^{[q-1]} r)^{\rho(r)}} = \tau^*(p, q) \equiv \tau^*$$

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p-1]} S(r, f)}{(\log^{[q-1]} r)^{\rho(r)}} = t^*(p, q) \equiv t^*$$

The function  $\rho(r)$  is said to be proximate order and lower proximate order of the given function  $f$  if  $\tau^*$  and  $t^*$  are nonzero and finite respectively.

Now we shall prove the main results.

**Theorem 2:** Let  $f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$  be an entire function of index-pair  $(p, q)$ . If  $0 < U < \infty$ , the function  $f(z_1, z_2)$  is of  $(p, q)$ -order

$$\rho(p, q) = (\rho_1(p, q), \rho_2(p, q)) > (b, b) \quad \text{and}$$

$$\text{generalized } (p, q)\text{-type } \tau^*(p, q)$$

$= (\tau_1^*(p, q), \tau_2^*(p, q)) (0, 0)$ , with respect to proximate order  $\rho(r)$ . Then

$$\tau^*(p, q) = M' U$$

where

$$U \equiv U(p, q) = \limsup_{\min(k_j) \rightarrow \infty} \left[ \frac{\varphi(\log^{[p-2]}(k))}{\log^{[q-1]} E_k^*(f, E)^{-1/k}} \right]^{\rho(p, q) - A}$$

$$M' = \begin{cases} \frac{(\rho(2, 2) - 1)^{\rho(2, 2) - 1}}{\rho(2, 2)^{\rho(2, 2)}} & \text{if } (p, q) = (2, 2) \\ 1/e\rho(2, 1)d^{\rho(2, 1)} & \text{if } (p, q) = (2, 1) \\ 1 & \text{otherwise.} \end{cases}$$

$$A = \begin{cases} 1 & \text{for } (p, q) = (2, 2) \\ 0 & \text{otherwise;} \end{cases}$$

$$b = \begin{cases} 1 & \text{for } (p, q) = (2, 2) \\ 0 & \text{otherwise;} \end{cases}$$

$k = (k_1, k_2)$  and  $d_j = d(E^j) > 0$  ( $j = 1, 2$ ) are the transfinite diameters of  $E^j$ .

**Proof.** From [2], eq. 1.10], we have

$$\|f - L_k\|_E \leq \lambda \frac{M(r, f)}{r^k} (d e^\epsilon)^k, \quad (2.1)$$

$$\text{for } r > r^{(1)} = (r_1^{(1)}, r_2^{(1)}), k > k^{(1)} = (k_1^{(1)}, k_2^{(1)}),$$

where  $\lambda = \lambda_1, \lambda_2, \epsilon = (\epsilon_1, \epsilon_2), e^\epsilon = (e^{\epsilon_1}, e^{\epsilon_2})$ .

Let  $k(p, q) = (k_1(p, q), k_2(p, q)) > \tau^*(p, q)$ . By definite of generalized  $(p, q)$ -type of  $f(z_1, z_2)$  there exists an  $r^{(2)} > r^{(1)}$  such that

$$\frac{\log^{[p-1]} S(r, f)}{(\log^{[q-1]} r)^{\rho(r)}} \leq K(p, q) \text{ for } r > r^{(2)},$$

or

$$S(r, f) \leq \exp^{[p-1]} [K(p, q) (\log^{[q-1]} r)^{\rho(r)}]. \quad (2.2)$$

For  $(p, q) = (2, 1)$ , using (2.2) with (2.1) we get

$$\|f - L_k\|_E \leq \lambda (d e^\epsilon)^k \left[ \frac{e^{k(2, 1)} r^{\rho(r)}}{r^k} \right]. \quad (2.3)$$

Let  $k^{(2)} > k^{(1)}$  such that  $\frac{k_j}{r_j \rho_j(2, 1)} > r_j^{(\rho(r))}$  for

$$j = 1, 2, k > k^{(2)}.$$

Choosing

$$r = \left[ \varphi \left( \frac{k_1}{K_2(2, 1) \rho_2(2, 1)} \right), \varphi \left( \frac{k_2}{K_2(2, 1) \rho_2(2, 1)} \right) \right]$$

in (2.3) we get

$$\|f - L_k\|_E < \lambda (d e^\epsilon)^k \frac{(e^\phi k(2, 1) \rho(2, 1))^{k/\rho(2, 1)}}{[\varphi(k)]^k}$$

$$\leq \lambda d^k \left( \frac{(e\tau^*(2,1)\rho(2,1))^{k/\rho(2,1)}}{[\varphi(k)]^k} \right) (e^{-\delta/k})^k \text{ for } k > k^{(2)}, \delta = (\delta_1, \delta_2)$$

which gives

$$[\varphi(k) (\|f - L_k\|_E)^{1/k}]^{\rho(2,1)} \leq e\rho(2,1)\tau^*(2,1)d^{\rho(2,1)}\lambda^{\rho(2,1)/k} (e^{-\delta/k})^{\rho(2,1)}$$

or

$$\left[ \frac{\varphi(k)}{\log\|f - L_k\|_E^{-1/k}} \right]^{\rho(2,1)} \leq \frac{\tau^*(2,1)}{M'} \text{ as } k \rightarrow \infty. \quad (2.4)$$

Choosing

$$r = \left[ \exp\varphi\left(\frac{k_1}{K_1(2,2)\rho_2(2,2)}\right), \exp\varphi\left(\frac{k_2}{K_2(2,2)\rho_2(2,2)}\right) \right]$$

in (2.5), we get

For  $(p, q) = (2, 2)$ , in view of (2.2), we have

$$\|f - L_k\|_E \leq \lambda (de^\epsilon)^k \exp[k(2,2)(\log r)^{\rho(r)}] \left(\frac{1}{r^k}\right). \quad (2.5) \quad \|f - L_k\|_E \leq \frac{\lambda (de^\epsilon)^k \left\{ \exp\left[\frac{k}{\rho(2,2)} \cdot \frac{\varphi(k)}{(K(2,2)\rho(2,2))^{1/(\rho(2,2)-1)}}\right] \right\}}{\left[ \exp\left[\frac{\varphi(k)}{(K(2,2)\rho(2,2))^{1/(\rho(2,2)-1)}}\right] \right]^k}$$

Let  $k^{(2)} > k^{(1)}$  such that  $r^{\rho(r)} = \frac{k_j}{k(2,2)\rho_j(2,2)} > r_j^{\rho(r)}$ ,  $(j = 1, 2)$ ,  $k > k^{(2)}$ .

$$\log\|f - L_k\|_E \leq \log\lambda + \frac{k\varphi(k)}{K(2,2)\rho(2,2)^{\rho/\rho-1}} - \frac{k\varphi(k)}{[K(2,2)\rho(2,2)]^{1/(\rho(2,2)-1)}}.$$

$$\left[ \log\|f - L_k\|_E^{-1/k} \right]^{\rho(2,2)-1} \geq \frac{[\varphi(k)]^{\rho(2,2)-1}}{K(2,2)} \left( \frac{(\rho(2,2)-1)^{\rho(2,2)-1}}{\rho(2,2)^{\rho(2,2)}} \right) [1 - O(1)]^{\rho(2,2)-1}$$

or

$$\frac{K(2,2)}{M'} \geq \limsup_{\min(k_j) \rightarrow \infty} \left[ \frac{\varphi(k)}{\log\|f - L_k\|_E^{-1/k}} \right]^{\rho(2,2)-1}. \quad (2.6)$$

Now we consider the case when  $(p, q) \neq (2, 1)$  and  $(2, 2)$  i.e.  $3 \leq q \leq p < \infty$ , let  $k^{(2)} > k^{(1)}$  such that

$$\exp^{[q-1]} \left[ \frac{\log^{[p-2]}(k_j / K(p, q)\rho_j(p, q))}{K(p, q)} \right] > r_j^{\rho(r)}$$

for  $k > k^{(2)}$ ,  $j = 1, 2$ .

Choosing

$$r = \left[ \exp^{[q-1]} \left[ \varphi\left(\frac{\log^{[p-2]}(k_1 / K_1(p, q)\rho_1(p, q))}{K_1(p, q)}\right) \right], \exp^{[q-1]} \left[ \varphi\left(\frac{\log^{[p-2]}(k_2 / K_2(p, q)\rho_2(p, q))}{K_2(p, q)}\right) \right] \right]$$

in (2.1) and (2.2), we obtain

$$\log\|f - L_k\|_E^{-1/k} \geq \exp^{[q-2]} \left[ \varphi\left(\frac{\log^{[p-2]}(k / K(p, q)\rho(p, q))}{K(p, q)}\right) \right] [1 - O(1)]$$

for sufficiently large values of  $k_j$ 's or

$$K(p, q) \geq \left[ \frac{\varphi(\log^{[p-2]}(k / K(p, q)\rho(p, q)))}{\log^{[q-1]} \|f - L_k\|_E^{-1/k}} \right]^{\rho(p, q)} + o(1).$$

Proceeding to limits, we get

$$K(p, q) \geq \limsup_{\min(k_j) \rightarrow \infty} \left[ \frac{\varphi(\log^{[p-2]}(k / K(p, q)\rho(p, q)))}{\log^{[q-1]} \|f - L_k\|_E^{-1/k}} \right]^{\rho(p, q)}. \quad (2.7)$$

Since (2.4), (2.6) and (2.7) are valid for every  $K(p, q) = (K_1(p, q), K_2(p, q)) > \tau^*(p, q)$ , it follows that

$$\limsup_{\min(k_j) \rightarrow \infty} \left[ \frac{\varphi(\log^{[p-2]} k)}{\log^{[q-1]} \|f - L_k\|_E^{-1/k}} \right]^{\rho(p, q)-A} \leq \frac{\tau^*(p, q)}{M'}. \quad (2.8)$$

To prove the reverse inequality, let  $\tilde{\nu} = (\nu, \nu) \in \mathbb{R}^2, \nu = 0; 1 \dots$  and in view of Lemma 1.1 of [2] to the function  $f$  in the series

$$f(z) = L_{\tilde{0}}(z) + \sum_{\nu=0}^{\infty} (L_{\tilde{\nu+1}}(z) - L_{\tilde{\nu}}(z)) \quad z \in C^2.$$

$$\text{We get } \|f(z)\| \leq \|L_{\tilde{0}}(z)\| + \sum_{\nu=0}^{\infty} \|L_{\tilde{\nu+1}}(z) - L_{\tilde{\nu}}(z)\|,$$

We have the property of extremal function  $\varphi(z, E)$  [5]

$$\|p(z)\| \leq \|p\|_E \varphi^{\nu}(z), \quad z \in C^2. \quad (2.9)$$

Applying (2.9) in above for every variable separately, we get

$$\|f(z)\| \leq a_0 + 2 \sum_{\nu=0}^{\infty} \|f - L_{\tilde{\nu}}\|_E (r/d)^{\tilde{\nu}} \text{ for } z \in E_{r_j}^{(j)}. \quad (2.10)$$

Consider the function

$$g(z) = \sum_{\nu=0}^{\infty} \|f - L_{\tilde{\nu}}\|_E z^{\tilde{\nu}}.$$

Since  $\lim_{\tilde{\nu} \rightarrow \infty} \|f - L_{\tilde{\nu}}\|_E^{1/\tilde{\nu}} = 0$  in view of Lemma 1.1 of [2] it

follows that  $g(z)$  is entire function. From (2.10) we obtain

$$S(r, f) \leq a_0 + 2g(r/d). \quad (2.11)$$

Now applying Theorem , for each variable separately, we get

$$\frac{\tau^*(p, q)}{M'} \leq \limsup_{\min(\tilde{\nu}_j) \rightarrow \infty} \left[ \frac{\varphi(\log^{[p-2]} \tilde{\nu})}{\log^{[q-1]} \|f - L_k\|_E^{-1/\tilde{\nu}}} \right]^{\rho(p, q)-A}. \quad (2.12)$$

Using inequality (2.4) with (2.8) and (2.12) together proves the theorem.

## References

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