Optimal Dividend and Financing Control Problem in The Risk Model with Non-Cheap Proportional Reinsurance

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Abstract: ***In this paper, we consider the optimal dividend and financing control problem in the risk model with non-cheap proportional reinsurance and two kinds of transaction costs. The company control its reserves by paying dividends, issuing equity and taking reinsurance. In our model, the objective is to find the strategy which maximizes the expected present value of the dividends payout minus the equity issuance until the time of ruin. We solve the optimal control problem and identify the optimal strategy by constructing two categories of suboptimal control problems, one is the classical model without equity issuance,the other never goes bankrupt by equity issuance.***

Keywords: dividend, equity issuance, non-cheap proportional reinsurance, HJB equation, optimal strategy

1. Introduction

Optimal dividend strategy, as one major public concern to assess the stability of companies that take on risks, has become an increasingly popular topic in actuarial research. Its origin can be traced as early as the work of De Finetti [1], which introduced a discrete-time model for optimal dividend. De Finetti showed that the optimal strategy is a barrier strategy and determined the optimal level of the barrier by maximizing the expected discounted dividends paid to shareholders. Since then the optimal dividend strategy has been studied extensively. Some recent works include Højgaard and Taksar [2], Gerber and Shiu [3], Cadenillas et al.[4], Avanzi et al.[5]and so on.

Meanwhile, in the real financial market, equity issuance is an important approach for the insurance company to earn profit and reduce risk. Both equity issuance and dividend are important issues in modeling insurance risk. Sethi and Taksar [6] recently considered the model for the company that controls its risk exposure by issuing new equity and paying dividends to shareholders and discussed the problem using singular and impulse controls. Løkka and Zervos [7] studied the combined optimal dividend and equity issuance problem by taking into account the possibility of bankruptcy. The aforementioned authors mainly focused on developing optimal dividend and equity issuance strategies. They did not consider reinsurance.

Reinsurance plays a significant role in both theory and practice of insurance risk modeling. It is a means by which a direct insurance company can transfer the risks from their liabilities to a second insurance carrier. The three popular types of reinsurance strategies are stop-loss reinsurance, proportional reinsurance and excess-of-loss reinsurance. The academia and practitioners have paid more attention to the proportional reinsurance and excess-of-loss reinsurance. Some works on the excess-of-loss reinsurance are Asmussen et al.[8], Choulli et al.[9], Centeno [10] and so on. Some literature on the proportional reinsurance includes He and Liang [11], [12] and so on. He and Liang [11], [12]incorporated the proportional reinsurance strategy in the combined dividend and equity issuance problem using both singular and mixed singular-impulse controls.

Motivated by these works, we consider the optimal dividend and financing control problem of an insurance company. We assume that the company can control its reserves by paying dividends, issuing equity and taking non-cheap proportional reinsurance. Moreover, there exists a minimal reserve requirement. And some costs will be incurred: reinsurance company will need more premium for the risk ceded by the insurer; fixed cost is generated by advisory and consulting fees when payingdividends; proportional transaction costs are generated by the tax. In this paper, we consider the dividends payout and the equity issuance as the reflecting and absorbing boundaries of the reserve process, respectively. Firstly, we study the solutions of two models: one is diffusion control model without equity issuance, the other stands for the model with equity issuance to meet the minimal reserve requirement, so it never goes bankrupt. Our objective is to maximize the expected present value of the dividends payout minus the equity issuance until the time of ruin. Then we prove that the value functions and the optimal strategies are the solutions of the two control problems. We provide a rigorous and detailed mathematical analysis for the combined effect of the optimal dividend, equity issuance and non-cheap proportional reinsurance strategies.

The rest of the paper is organized as follows. In Section 2, we introduce the control model of an insurance company with non-cheap proportional reinsurance. In Section 3, we present two lemmas for proving the main results of this paper. In Section 4, we construct solutions of two categories of suboptimal models. In Section 5, we verify the value function and the optimal strategy with the corresponding solution in either category of suboptimal models, tightly connecting with the relationships among the parameters.

The Model

Let $(\Omega, \mathrm{F}, \{\mathrm{F}_t\}_{t\geq 0}, P)$ be a filtered probability space and B_t is a standard Brownian motion on this probability space, where F_t represents the information available at time t and any decision made up to time t is based on this information. To pay dividends to the share holders, the insurance company has to determine the times and amounts of dividend payments. A dividend stream is defined by $L := \{(\tau_i, \xi_i) \mid i = 1,2,\ldots\}$,where τ_i, ξ_i are the time and amount of the i th dividend payment, respectively. We assume that $\{\tau_i, \mid i = 1,2,\ldots\}$ is a sequence of increasing stopping times and $\{\xi_i, \mid i = 1,2,\ldots\}$ is a sequence of non-negative,i.i.d random variables.Let L_t denote the total amount of dividends paid until time t. Then we can define $L_t := \sum_{i=1}^{\infty} I_{\{\tau_i \leq t\}} \xi_i$, where I_E is the indicator function of the event E. We suppose the liquid reserves of the insurance company must satisfy some minimal reserve requirement. In this case, we assume that the company needs to keep its reserves above m, $m > 0$ is the minimal reserve requirement. The company is considered bankrupt as soon as the reserves fall below m. However, to avoid bankruptcy, the company should issue some equity. We denote G_t as the total amount raised by issuing equity from time 0 to t. We assume that the process $\{G_t, t \geq 0\}$ is $\{\mathrm{F}_t\}_{t\geq 0} - adapted$, increasing, right-continuous with left limits and $G(0) = 0$. By the reinsurance strategy $a_t \in [0,1]$ (the proportional retention level), the amount of dividend L_t and equity issuance G_t, the liquid reserve of the insurance company evolves according to the stochastic equation,

$$R_t = x + \int_0^t \left[a_s\lambda - (\lambda - \mu)\right] d_s + \int_0^t a_s dB_s - \sum_{n=1}^{\infty} I_{\{\tau_i \leq t\}} \xi_i + G_t, \quad (2.1)$$

where x is the initial reserve, $\mu > 0$ is the relative safety loading of the insurer and λ is the relative safety loading of the reinsurer.

The proportional reinsurance is cheap when $\lambda = \mu$. We consider the case of $\lambda > \mu$ which is called non-cheap proportional reinsurance.

We will define an admissible strategy as follows.

A strategy $\pi = \{a_t, t \geq 0; \tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots; \mathrm{G}\}$ is said to be admissible if

(i) $\{a_t\}$ is an $\{\mathrm{F}_t\}_{t\geq 0} - adapted$ process and $P(a_t \in [0,1]) = 1$ for any $t \geq 0$.

(ii)For each $i = 1,2,\ldots$, $\{\tau_i \leq t\} \in F_t$ and $\xi_i \in \mathrm{F}_{\tau_i}$.

(iii) $\{L_t\}$ is $\{\mathrm{F}_t\}_{t\geq 0} -$ progressively measurable, increasing and cádlág, $\{G_t, t \geq 0\}$ is $\{\mathrm{F}_t\}_{t\geq 0} - adapted$, increasing, cádlág and $\xi_i \leq R_{\tau_i^-} - m$.

(iv) $P(\lim_{i\to\infty} \tau_i \leq t) = 0, \ \forall t \geq 0$.

(v) $\Delta L_t \cdot \Delta G_t = 0$, here $\Delta L_t = L_t - L_{t^-}, \Delta G_t = G_t - G_{t^-}$.

We write $\Pi(x)$ for the space of these admissible policies. For each $\pi \in \Pi(x)$, we write $R^\pi := \{R_t^\pi \mid t \geq 0\}$ for the surplus progress of the company associated with π. The surplus progress is written as follows,

$$R_t^\pi = x + \int_0^t \left[a_s^\pi \lambda - (\lambda - \mu)\right] d_s + \int_0^t a_s^\pi dB_s - \sum_{n=1}^{\infty} I_{\{\tau_i^\pi \leq t\}} \xi_i^\pi + G_t^\pi. \quad (2.2)$$

The ruin time correspond to π is defined as: $\tau^\pi := \inf\{t \geq 0 : R_t^\pi < m\}$ and τ^π is an $\mathrm{F}_t -$ stopping time. For each dividend payment, we have to pay a fixed set-up cost $K \in (0, \infty)$, which is independent of the amount of the payment. Let $\beta_1 < 1$ be a positive number. Then $1 - \beta_1$ is the tax rate if the dividend is taxed. Consequently, the amount of the money that the shareholder receives is $-K + \beta_1 \xi_i^\pi$ if the amount ξ_i^π of the liquid assets is distribute. In the meanwhile, the shareholders must pay out $\beta_2 g (\beta_2 > 1)$ to meet the amount of g as new equity of the company. β_2 is the proportional transaction cost generated by the issuance of equity.

So our optimal control problem is to maximize the expected present value of the dividends payout minus the equity issuance before bankruptcy, i.e. we need to find $\pi \in \Pi$ maximizing the following performance function as

$$V(x, \pi) = E\left[\sum_{i=1}^{\infty} e^{-\delta \tau_i^\pi} (-K + \beta_1 \xi_i^\pi) I_{\{\tau_i^\pi \leq \tau^\pi\}} - \int_0^{\tau^\pi} e^{-\delta s} \beta_2 dG_s^\pi\right]. \quad (2.3)$$

The optimal value function is defined as

$$V(x) = \sup_{\pi \in \Pi} V(x, \pi). \quad (2.4)$$

In addition, the minimal reserve requirement asks for $V(x) = 0$, for $\forall x < m$. To solve the optimization problem, our must determine the value function $V(x)$ and the optimal strategy π^* satisfies $V(x) = V(x, \pi^*)$.

Next, we will divide $\lambda > \mu$ into two parts: $\mu < \lambda < 2\mu$ and $\lambda \geq 2\mu$.

(I) We discuss the case of $\mu < \lambda < 2\mu$. It includes two situations: (i) $x_1 > d > x_0$ and (ii) $0 \leq d < x_0$, where

$$x_0 = G\left(\frac{2\mu - \lambda}{2\delta}\right), \quad G(u) = \int_0^u \frac{\delta y + \lambda - \mu}{(\delta + \frac{\lambda^2}{2\sigma^2})y + \lambda - \mu} dy, \ u \geq 0,$$

d is a nonnegative constant.

First, we consider the situation (i):

((i) will be discussed in Section 3 and Theorem 4.1 and Theorem 4.2. (ii) and (II): $\lambda \geq 2\mu$ will be discussed after that.)

3. Two Primary Lemmas

In this section we give two main lemmas before proving the Theorems in Section 4.

Lemma 3.1.
There exists a unique $x_{1_*} = x_{1_*}(\delta,\mu,\sigma,\beta_1,K) > m$ *satisfying the following equation in* x_1,

$$\frac{\beta_1 b_2}{b_1(b_2-b_1)} e^{b_1(m-x_1)} + \frac{\beta_1 b_1}{b_2(b_1-b_2)} e^{b_2(m-x_1)} = 0, \tag{3.1}$$

where $b_1 = \frac{-\mu - \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}$, $b_2 = \frac{-\mu + \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}$.

Proof. Denote the left-hand side of (3.1)by $k(x_1)$.
Differentiating $k(x_1)$ with respect t x_1 ,we have

$$k'(x_1) = \frac{-\beta_1 b_2}{b_2 - b_1} e^{b_1(m-x_1)} + \frac{\beta_1 b_1}{b_2 - b_1} e^{b_2(m-x_1)} < 0.$$

Then $k(x_1)$ is a strictly decreasing function of x_1. $k(x_1)$ reaches its maximum at m on $[m,+\infty)$.We deduce
$k(x_1) \to -\infty$, as $x_1 \to \infty$ and

$$k(m) = \frac{\beta_1 b_2}{b_1(b_2-b_1)} + \frac{\beta_1 b_1}{b_2(b_1-b_2)} > 0,$$

thus (3.1) has a unique solution x_{1_*} and $x_{1_*} > m$.

Lemma 3.2
There exists a unique $x_{1_{**}} = x_{1_{**}}(\delta,\mu,\sigma,\beta_1,\beta_2) > m$ satisfying the following equation in x_1,

$$\frac{\beta_1 b_2}{b_2 - b_1} e^{b_1(m-x_1)} + \frac{\beta_1 b_1}{b_1 - b_2} e^{b_2(m-x_1)} = \beta_2, \tag{3.2}$$

where b_1, b_2 are the same as in Lemma 3.1.

Proof. Denote the left-hand side of (3.2) by $h(x_1)$.
Differentiating $h(x_1)$ with respect to x_1, we have

$$h'(x_1) = \frac{-\beta_1 b_2 b_1}{b_2 - b_1} e^{b_1(m-x_1)} - \frac{\beta_1 b_1 b_2}{b_1 - b_2} e^{b_2(m-x_1)} > 0.$$

Then $h(x_1)$ is a strictly increasing function of x_1. $h(x_1)$ reaches its minimum at m on $[m,+\infty)$. We deduce
$h(x_1) \to +\infty$, as $x_1 \to \infty$, and
$h(m) = \frac{\beta_1 b_2}{b_2 - b_1} + \frac{\beta_1 b_1}{b_1 - b_2} = \beta_1 < \beta_2$, thus (3.2) has a unique solution $x_{1_{**}}$ and $x_{1_{**}} > m$.

4. Two categories of suboptimal solutions

In this section, we consider two categories of suboptimal control problems.

Let $\pi_A = \{a_A, L_A, 0\} \in \Pi$ be the control process for the company in which equity issuance is not permitted.We define the associated optimal value function as

$$V_A(x) = \sup_{\pi_A \in \Pi} V(x,\pi_A), \text{ for } x \geq m.$$

Let $\pi_B = \{a_B, L_B, G_B\} \in \Pi$ be the control process for the company with equity issuance procedures. In this case, the insurance company will never go bankrupt. The associated optimal value function is $V_B(x) = \sup_{\pi_B \in \Pi} V(x,\pi_B)$, $x \geq m$.

According to (2.4), it follows that $V(x) \geq \max\{V_A(x), V_B(x)\}$ for $x \geq m$.

The two suboptimal solutions will play a key role in constructing the optimal policy π^*. Thus we will first study the solutions to the two suboptimal control problems.

4.1 The solution to the problem without equity issuance

In this subsection, our objective is to maximize the expected discounted dividends payout.

Theorem 4.1. We assume $x_{1_*} \leq x_{1_{**}}$, where x_{1_*} and $x_{1_{**}}$ are defined in Lemmas 3.1. and 3.2. Then the function f defined by

$$f(x) = \begin{cases} f_1(x) = C_1(x_{1_*}) e^{b_1(x+d-x_0)} + C_2(x_{1_*}) e^{b_2(x+d-x_0)}, \\ \qquad m \leq x < x_{1_*}, \\ f_2(x) = \beta_1(x - x_{1_*}) + f_1(x_{1_*}), \\ \qquad x \geq x_{1_*}, \end{cases} \tag{4.1.1}$$

satisfies the following HJB equation and the boundary conditions for $x \geq m$,

$$\max\left\{\max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2 a^2 f''(x) + [a\lambda - (\lambda - \mu)]f'(x) - \delta f(x)\right\}, \beta_1 - f'(x), \Im f(x) - f(x)\right\} = 0, \tag{4.1.2}$$

$$f(m) = 0. \tag{4.1.3}$$

Moreover, for $x \geq m$,

$$f'(x) \leq \beta_2, \tag{4.1.4}$$

where b_1, b_2 are the same as in Lemma 3.1, $C_1(x_{1_*}), C_2(x_{1_*})$ are defined by

$$C_1(x_{1_*}) = \frac{\beta_1 b_2}{e^{b_1(x_{1_*} - x_0 + d)} b_1(b_2 - b_1)}, \quad C_2(x_{1_*}) = \frac{\beta_1 b_1}{e^{b_2(x_{1_*} - x_0 + d)} b_2(b_1 - b_2)};$$

$$\Im f(x) = \sup_{m \leq y \leq x}\{-K + \beta_1(x+y) + f(y)\}.$$

Proof. By the standard theory of optimal control, we use the same method as in Wendell and Fleming [13] and Højgaard and Taksar [2] to get a function f satisfying the following HJB equations,

$$\begin{cases} \max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2 a^2 f''(x) + [a\lambda - (\lambda - \mu)]f'(x) - \delta f(x)\right\} = 0, \\ \qquad m \leq x < x_1, \\ f'(x) = \beta_1, \qquad x \geq x_1, \\ f''(x) = 0, \qquad x \geq x_1. \end{cases} \tag{4.1.5}$$

Then differentiating w.r.t a for the first equation of (4.1.5), we can find $a(x) = \frac{-\lambda f'(x)}{\sigma^2 f''(x)}$.

Since $a(x)$ belongs to $[0,1]$, putting the expression $a(x)$

into the first equation of (4.1.5), we get

$$-\frac{\lambda^2 f'(x)}{2\sigma^2 f''(x)} - c\frac{f(x)}{f'(x)} + (\mu - \lambda) = 0 \cdot$$

Denoting $p(x) = \frac{f(x)}{f'(x)}$, we get that

$$p'(x) = \frac{(c + \frac{\lambda^2}{2\sigma^2})p(x) + \lambda - \mu}{cp(x) + \lambda - \mu}, x \geq m.$$

By a simple calculation, we find that there exists a nonnegative constant d such that

$$p(x) = \frac{f(x)}{f'(x)} = G^{-1}(x+d), \quad x \geq m,$$

where $G^{-1}(\cdot)$ denote the inverse function of G.

We have $a(x) = \frac{2}{\lambda}\left[cG^{-1}(x+d) + \lambda - \mu\right]$.

Since $a(x) \in [0,1]$, we have $x + d \leq x_0 = G\left(\frac{2\mu - \lambda}{2c}\right)$. The above expression requires $d < x_0$ and $\lambda \leq 2\mu$. Therefore, we need to consider two cases: $\lambda < 2\mu$ and $\lambda \geq 2\mu$.

First we suppose $x_1 > d > x_0$ under the case of $\mu < \lambda < 2\mu$. Since $a(x)$ is an increasing function, we know $a(x) = 1$ on $[m, x_1)$, which implies that the first equation of (4.1.5) becomes

$$\frac{1}{2}\sigma^2 a^2 f''(x) + [a\lambda - (\lambda - \mu)]f'(x) - \delta f(x) = 0, \quad x \in [m, x_1) \qquad (4.1.7)$$

Therefore

$$f_1(x) = C_1(x_1)e^{b_1(x+d-x_0)} + C_2(x_1)e^{b_2(x+d-x_0)}, \quad x \in [m, x_1);$$

$$f_2(x) = \beta_1(x - x_1) + f_1(x_1), \quad x \geq x_1.$$

Due to the continuity of the function $f'(x)$ and $f''(x)$ at point x_1, we can derive that

$$f_1'(x) = C_1(x_1)b_1 e^{b_1(x+d-x_0)} + C_2(x_1)b_2 e^{b_2(x+d-x_0)} = \beta_1, \qquad \text{i.e.}$$

$$f_1''(x) = C_1(x_1)b_1^2 e^{b_1(x+d-x_0)} + C_2(x_1)b_2^2 e^{b_2(x+d-x_0)} = 0,$$

$$C_1(x_1) = \frac{\beta_1 b_2}{e^{b_1(x_1-x_0+d)}b_1(b_2 - b_1)}, \quad C_2(x_1) = \frac{\beta_1 b_1}{e^{b_2(x_1-x_0+d)}b_2(b_1 - b_2)}.$$

From $f(m) = 0$, we have

$$f_1(m) = C_1(x_1)e^{b_1(m+d-x_0)} + C_2(x_1)e^{b_2(m+d-x_0)} = 0,$$

which implies that x_1 is a solution of (3.1). Using Lemma 3.1, we have $x_1 = x_{1_*}$. Similarly, if $x_1 = x_{1_*}$, then $f(m) = 0$. So $f(m) = 0 \Leftrightarrow x_1 = x_{1_*}$.

We will prove that f satisfies (4.1.2)-(4.1.4). Noticing that $\beta_1 < 1, \beta_2 > 1$, it suffices to prove the following:

$$\begin{cases} \max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2 a^2 f''(x) + [a\lambda - (\lambda - \mu)]f'(x) - \delta f(x)\right\} \leq 0, \\ f'(x) \geq \beta_1, \ f'(x) \leq \beta_2, \ m \leq x < x_{1_*}; \\ \max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2 a^2 f''(x) + [a\lambda - (\lambda - \mu)]f'(x) - \delta f(x)\right\} \leq 0, \\ x \geq x_{1_*}. \end{cases} \qquad (4.1.8)$$

The proof is as follows: for $x \geq x_{1_*}$,

$f_2(x) = \beta_1(x - x_{1_*}) + f_1(x_{1_*})$, we have $\mu\beta_1 - \delta f_1(x_{1_*}) = 0$.

Therefore

$$\max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2 a^2 f_2''(x) + [a\lambda - (\lambda - \mu)]f_2'(x) - \delta f_2(x)\right\}$$

$$= \max_{a\in[0,1]}\left\{[a\lambda - (\lambda - \mu)]f_2'(x) - \delta f_2(x)\right\}$$

$$= \mu\beta_1 - \delta f_1(x_{1_*}) - \delta\beta_1(x - x_{1_*}) \leq 0$$

Using the same way as in Højgaard and Taksar [2], it is easy to prove that

$$\max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2 a^2 f_1''(x) + [a\lambda - (\lambda - \mu)]f_1'(x) - \delta f_1(x)\right\} \leq 0$$

holds for $m \leq x < x_{1_*}$.

Since $f_1'(x) = \frac{\beta_1 b_1 b_2}{b_2 - b_1}\left[e^{b_1(m-x_{1_*})} - e^{b_2(m-x_{1_*})}\right] \leq 0$,

then $f'(x)$ is a decreasing function on $[m, x_{1_*}]$. Moreover, $f'(x_{1_*}) = \beta_1$ and $f'(x) \geq \beta_1$ are obvious.

The problem remaining is to prove that the solution f satisfies $\Im f(x) \leq f(x)$: $f(y) + \beta_1(x - y) - K - f(x) \Leftrightarrow$ $\int_y^x(\beta_1 - f'(\partial))d\partial - K \leq \int_m^x(\beta_1 - f'(\partial))d\partial - K \leq 0$. The proof of (4.1.4) is as follows.

Since $f'(m) = \frac{\beta_1 b_2}{b_2 - b_1}e^{b_1(m-x_{1_*})} + \frac{\beta_1 b_1}{b_1 - b_2}e^{b_2(m-x_{1_*})} = h(x_{1_*}), \ x_{1_*} \leq x_{1_{**}}$ and $h(x)$ is a strictly increasing function, we have $f'(m) = h(x_{1_*}) \leq h(x_{1_{**}}) = \beta_2$ by Lemma 3.2.

4.2. The solution to the problem with equity issuance

In this subsection, our aim is to maximize the expected discounted dividends payout minus the expected discounted equity issuance over all reinsurance, dividends payout and equity issuance strategies. This kind of insurance companies will never go bankrupt.

Theorem 4.2. Assume that $x_{1_*} \geq x_{1_{**}}$, where x_{1_*} and $x_{1_{**}}$ are defined in Lemmas 3.1 and 3.2. Then the function g defined by

$$g(x) = \begin{cases} g_1(x) = C_1(x_{1_{**}})e^{b_1(x+d-x_0)} + C_2(x_{1_{**}})e^{b_2(x+d-x_0)}, \\ \qquad m \leq x < x_{1_{**}}, \\ g_2(x) = \beta_1(x - x_{1_{**}}) + g_1(x_{1_{**}}), \\ \qquad x \geq x_{1_{**}}, \end{cases} \qquad (4.2.1)$$

satisfies the following HJB equation and the boundary conditions:

$$\max\left\{\max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2 a^2 g''(x) + [a\lambda - (\lambda - \mu)]g'(x) - \delta g(x)\right\}, \beta_1 - f'(x), \right.$$

$$\left. \Im g(x) - g(x), \ g'(x) - \beta_2\right\} = 0, \qquad (4.2.2)$$

$$g'(m) \geq 0 \qquad (4.2.3)$$

where b_1, b_2 are the same as in Lemma 3.1, $C_1(x_{1_*}), C_1(x_{1_{**}})$ are defined as same as in Theorem 4.1 by replacing x_{1_*} with $x_{1_{**}}$.

Proof. Considering the time value of money leads us to the conclusion that it is optimal to postpone the new equity issuance as long as possible. If we issue equity at the reserve n prior to m, $g'(n) = \beta_2$ and $g'(x)$ is a decreasing function, so $g''(n)$ must be 0 to meet the requirement $g'(x) \leq \beta_2$. But it is not compatible with $a \in [0,1]$. Thus we know that it is optimal to issue equity only when the reserves become m.

By the same argument as in Theorem 4.1, we know the function g should be characterized by

$$g'(m)=\beta_2 \tag{4.2.4}$$

$$\begin{cases} \max\limits_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2a^2g''(x)+[a\lambda-(\lambda-\mu)]g'(x)-\delta g(x)\right\}=0, \\ \qquad m\le x<x_1, \\ g'(x)=\beta_1, \qquad x\ge x_1, \\ g''(x)=0, \qquad x\ge x_1. \end{cases} \tag{4.2.5}$$

Doing the same procedures as in proof of Theorem 4.1, we can prove the function $g(x)$ of (4.2.4) and (4.2.5) has the same form as $f(x)$, and x_1 satisfies the following equation

$$\frac{\beta_1 b_2}{b_2-b_1}e^{b_1(m-x_1)}+\frac{\beta_1 b_1}{b_1-b_2}e^{b_2(m-x_1)}=\beta_2.$$

By Lemma 3.2, we have $x_1=x_{1_{**}}$ and $x_{1_{**}}>m.$

We will prove that the solution g satisfies the conditions mentioned in Theorem 4.2. It suffices to prove the following:

$$\begin{cases} \max\limits_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2a^2g''(x)+[a\lambda-(\lambda-\mu)]g'(x)-\delta g(x)\right\}\le 0, \\ g'(x)\ge\beta_1,\ g'(x)\le\beta_2,\ m\le x<x_{1_{**}}; \\ \max\limits_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2a^2g''(x)+[a\lambda-(\lambda-\mu)]g'(x)-\delta g(x)\right\}\le 0, \\ x\ge x_{1_{**}}, \end{cases} \tag{4.2.6}$$

and $\Im g(x)\le g(x)$.

Using the similar procedures as in Section 4.1, we can prove the above affirms.

We will verify $g(m)\ge 0$, i.e.

$$\begin{aligned} g(m)&=\frac{\beta_1 b_2}{b_1(b_2-b_1)}e^{b_1(m-x_{1_{**}})}+\frac{\beta_1 b_1}{b_2(b_1-b_2)}e^{b_2(m-x_{1_{**}})} \\ &=k(x_{1_{**}})\ge k(x_{1_*}) \\ &=\frac{\beta_1 b_2}{b_1(b_2-b_1)}e^{b_1(m-x_{1_*})}+\frac{\beta_1 b_1}{b_2(b_1-b_2)}e^{b_2(m-x_{1_*})}. \end{aligned}$$

(ii) $0\le d<x_0$.

By the same argument as in (i), we can get the Lemmas and Theorems that are similar to Lemma 3.1, Lemma 3.2 and Theorem 4.1,Theorem 4.2, where $\bar{f}(x)$ and $\bar{g}(x)$ are defined as follow. The corresponding value functions are defined by $V_{A_1}(x)$ and $V_{B_1}(x)$, respectively.

$$\bar{f}(x)=\begin{cases} \bar{f}_1(x)=\left(C_1(x_{1^*})+C_2(x_{1^*})\right)e^{-\int_x^{x_0-d}\frac{1}{G^{-1}(y+d)}dy}, \\ \qquad m\le x<x_0-d, \\ \bar{f}_2(x)=C_1(x_{1^*})e^{b_1(x+d-x_0)}+C_2(x_{1^*})e^{b_2(x+d-x_0)}, \\ \qquad x_0-d\le x<x_{1^*}, \\ \bar{f}_3(x)=\beta_1(x-x_{1^*})+\bar{f}_2(x_{1^*}), \\ \qquad x\ge x_{1^*}, \end{cases}$$

where

$$C_1(x_{1^*})=\frac{\beta_1 b_2}{e^{b_1(x_{1^*}-x_0+d)}b_1(b_2-b_1)},\ C_2(x_{1^*})=\frac{\beta_1 b_1}{e^{b_2(x_{1^*}-x_0+d)}b_2(b_1-b_2)};$$

$$\bar{g}(x)=\begin{cases} \bar{g}_1(x)=\left(C_1(x_{1^{**}})+C_2(x_{1^{**}})\right)e^{-\int_x^{x_0-d}\frac{1}{G^{-1}(y+d)}dy}, \\ \qquad m\le x<x_0-d, \\ \bar{g}_2(x)=C_1(x_{1^{**}})e^{b_1(x+d-x_0)}+C_2(x_{1^{**}})e^{b_2(x+d-x_0)}, \\ \qquad x_0-d\le x<x_{1^{**}}, \\ \bar{g}_3(x)=\beta_1(x-x_{1^{**}})+\bar{g}_2(x_{1^{**}}), \\ \qquad x\ge x_{1^{**}}, \end{cases}$$

where

$$C_1(x_{1^{**}})=\frac{\beta_1 b_2}{e^{b_1(x_{1^{**}}-x_0+d)}b_1(b_2-b_1)},\ C_2(x_{1^{**}})=\frac{\beta_1 b_1}{e^{b_2(x_{1^{**}}-x_0+d)}b_2(b_1-b_2)}.$$

(II) $\lambda\le 2\mu$.

In this case, the company does not need to reinsure, i.e. $a(x)\equiv 1$ for all $x\ge m$. By the same argument as in (i), we can get the Lemmas and Theorems that are similar to Lemma 3.1, Lemma 3.2 and Theorem 4.1, Theorem 4.2, where $\tilde{f}(x)$ and $\tilde{g}(x)$ are defined as follow. The corresponding value functions are defined by $V_{A_2}(x)$ and $V_{B_2}(x)$, respectively.

$$\tilde{f}(x)=\begin{cases} \tilde{f}_1(x)=C_{11}(x_{1_*})e^{b_1(x+d)}+C_{21}(x_{1_*})e^{b_2(x+d)}, & m\le x<x_{1_*}; \\ \tilde{f}_2(x)=\beta_1(x-x_{1_*})+\tilde{f}_1(x_{1_*}), & x\ge x_{1_*}, \end{cases}$$

where $C_{11}(x_{1_*})=\dfrac{\beta_1 b_2}{e^{b_1(x_{1_*}+d)}b_1(b_2-b_1)},\ C_{21}(x_{1_*})=\dfrac{\beta_1 b_1}{e^{b_2(x_{1_*}+d)}b_2(b_1-b_2)}$.

$$\tilde{g}(x)=\begin{cases} \tilde{g}_1(x)=C_{11}(x_{1_{**}})e^{b_1(x+d)}+C_{21}(x_{1_{**}})e^{b_2(x+d)}, & m\le x<x_{1_{**}}; \\ \tilde{g}_2(x)=\beta_1(x-x_{1_{**}})+\tilde{g}_1(x_{1_{**}}), & x\ge x_{1_{**}}, \end{cases}$$

where

$$C_{11}(x_{1_{**}})=\frac{\beta_1 b_2}{e^{b_1(x_{1_{**}}+d)}b_1(b_2-b_1)},\ C_{21}(x_{1_{**}})=\frac{\beta_1 b_1}{e^{b_2(x_{1_{**}}+d)}b_2(b_1-b_2)}.$$

5. The Solution to the General Problem

We now study the optimal control problem without any restriction on the issuance of equity.

Theorem 5.1. Let concave function $\Gamma(x)\in C^2$ satisfy the following HJB equation and boundary condition : for $x\ge m$,

$$\max\left\{\max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2a^2\Gamma''(x)+[a\lambda-(\lambda-\mu)]\Gamma'(x)-\delta\Gamma(x)\right\},\beta_1-\Gamma'(x),\right.$$
$$\left.\Im\Gamma(x)-\Gamma(x),\ \Gamma'(x)-\beta_2\right\}=0, \tag{5.1}$$
$$\max\{-\Gamma(m),\Gamma'(m)-\beta_2\}=0. \tag{5.2}$$

Then $\Gamma(x)\ge V(x,\pi)$ for any admissible policy π.

Proof. Since $\Gamma(x)$ is a concave, increasing and continuous function on $[m,\infty)$. From $\Im\Gamma(x)\le\Gamma(x)$, we know

$$\Gamma(x-\xi)-\Gamma(x)\le K-\beta_1\xi,\ \forall x\ge m,\xi>0. \tag{5.3}$$

For a policy $\pi\in\Pi$, we note

$\Theta=\{s:L_s^\pi\ne L_{s^-}^\pi\}=\{\tau_1^\pi,\tau_2^\pi,......\}$ and $\Theta'=\{s:G_s^\pi\ne G_{s^-}^\pi\}$.

$G_t^{\pi,c}=G_t^\pi-\sum\limits_{s\in\Theta',\,s\le t}(G_s^\pi-G_{s^-}^\pi)$ is the continuous part of G_t^π.

By Itô formula,

$$\begin{aligned} e^{-\delta(t\wedge\tau^\pi)}\Gamma(R_{t\wedge\tau^\pi}^\pi)=\Gamma(x)&+\int_0^{t\wedge\tau^\pi}e^{-\delta s}\vartheta\Gamma(R_s^\pi)ds \\ &+\int_0^{t\wedge\tau^\pi}e^{-\delta s}\sigma a_s^\pi\Gamma'(R_s^\pi)dBs \\ &+\int_0^{t\wedge\tau^\pi}e^{-\delta s}\Gamma'(R_s^\pi)dG_s^{\pi,c} \\ &+\sum_{s\in\Theta\cup\Theta',\,s\le t\wedge\tau^\pi}e^{-\delta s}\left[\Gamma(R_s^\pi)-\Gamma(R_{s^-}^\pi)\right], \end{aligned}$$

where $\vartheta\Gamma(x)=\frac{1}{2}\sigma^2a^2\Gamma''(x)+[a\lambda-(\lambda-\mu)]\Gamma'(x)-\delta\Gamma(x)$,

$$\sum_{s\in\Theta\cup\Theta',\,s\le t\wedge\tau^\pi} e^{-\delta s}[\Gamma(R_s^\pi)-\Gamma(R_{s-}^\pi)]=\sum_{s\in\Theta,\,s\le t\wedge\tau^\pi} e^{-\delta s}[\Gamma(R_s^\pi)-\Gamma(R_{s-}^\pi)]$$
$$+\sum_{s\in\Theta',\,s\le t\wedge\tau^\pi} e^{-\delta s}[\Gamma(R_s^\pi)-\Gamma(R_{s-}^\pi)]$$
$$\le \sum_{s\in\Theta',\,s\le t\wedge\tau^\pi} e^{-\delta s}\beta_2(G_s^\pi-G_{s-}^\pi)$$
$$-\sum_{i=1}^{\infty} e^{-\delta\tau_i^\pi}(-K+\beta_1\xi_i^\pi)I_{\{\tau_i^\pi\le t\wedge\tau^\pi\}}.$$

By substituting this inequality into the above equation and taking expectation on both sides, we obtain

$$E\{e^{-\delta(t\wedge\tau^\pi)}\Gamma(R_{t\wedge\tau^\pi}^\pi)\}\le\Gamma(x)+E\left\{\int_0^{t\wedge\tau^\pi} e^{-\delta s}\beta_2 dG_s^\pi\right\}$$
$$-E\left\{\sum_{i=1}^{\infty} e^{-\delta\tau_i^\pi}(-K+\beta_1\xi_i^\pi)I_{\{\tau_i^\pi\le t\wedge\tau^\pi\}}\right\}.$$

By the definition of τ^π and $\beta_1\le\Gamma'(x)\le\beta_2$, it is easy to prove that

$$\liminf_{t\to\infty} e^{-\delta(t\wedge\tau^\pi)}\Gamma(R_{t\wedge\tau^\pi}^\pi)+e^{-\delta\tau^\pi}\Gamma(m)I_{\{\tau^\pi<\infty\}}+\liminf_{t\to\infty} e^{-\delta t}\Gamma(R_t^\pi)I_{\{\tau^\pi=\infty\}}\ge 0.$$

So we deduce that

$$V(x,\pi)=E\left[\sum_{i=1}^{\infty} e^{-\delta\tau_i^\pi}(-K+\beta_1\xi_i^\pi)I_{\{\tau_i^\pi\le\tau^\pi\}}-\int_0^{\tau^\pi} e^{-\delta s}\beta_2 dG_s^\pi\right]\le\Gamma(x).$$

The main results of this paper are the following.

Theorem 5.2. (I) $\mu<\lambda<2\mu$.

(i) $x_1>d>x_0$.

x_{1_*}, $x_{1_{**}}$ are given in Lemmas, $V(x)$, $f(x)$ and $g(x)$ are defined by (2.4), (4.1.1) and (4.2.1) respectively. $V_A(x)$ and $V_B(x)$ are defined in Section 4.

If $x_{1_*}\le x_{1_{**}}$, then $V(x)=f(x)=V_A(x)$. The optimal policy $\pi^*=(a^{\pi^*},L^{\pi^*},G^{\pi^*})$ satisfies the following

$$\begin{cases} R_t^{\pi^*}=x+\int_0^t[a_s^{\pi^*}\lambda-(\lambda-\mu)]d_s+\int_0^t a_s^{\pi^*}dB_s-\sum_{n=1}^{\infty}I_{\{\tau_i^{\pi^*}\le t\}}\xi_i^{\pi^*}, \\ R_t^{\pi^*}\le x_{1_*}, \\ \int_0^\infty I_{\{t:R_t^{\pi^*}<x_{1_*}\}}(t)dL_t^{\pi^*}=0, \\ G_t^{\pi^*}=0, \end{cases} \quad (5.4)$$

where $a_t^{\pi^*}=a(R_t^{\pi^*})$.

If $x_{1_*}\ge x_{1_{**}}$, then $V(x)=f(x)=V_B(x)$. The optimal policy $\pi^{**}=(a^{\pi^{**}},L^{\pi^{**}},G^{\pi^{**}})$ satisfies the following

$$\begin{cases} R_t^{\pi^{**}}=x+\int_0^t[a\pi^{**}\lambda-(\lambda-\mu)]d_s+\int_0^t a_s^{\pi^{**}}dB_s-\sum_{n=1}^{\infty}I_{\{\tau_i^{\pi^{**}}\le t\}}\xi_i^{\pi^{**}}+G_t^{\pi^{**}}, \\ m\le R_t^{\pi^{**}}\le x_{1_{**}}, \\ \int_0^\infty I_{\{t:R_t^{\pi^{**}}<x_{1_{**}}\}}(t)dL_t^{\pi^{**}}=0, \\ \int_0^\infty I_{\{t:R_t^{\pi^{**}}<m\}}(t)dG_t^{\pi^{**}}=0, \end{cases} \quad (5.5)$$

where $a_t^{\pi^{**}}=a(R_t^{\pi^{**}})$.

According to Lions and Sznitman [14] we know that the processes $\pi^*=(a^{\pi^*},L^{\pi^*},G^{\pi^*})$ and $\pi^{**}=(a^{\pi^{**}},L^{\pi^{**}},G^{\pi^{**}})$ are uniquely determined by (5.4) and (5.5).

(ii) $0\le d<x_0$.

All the results are similar to $x_1>d>x_0$.

(II) $\lambda\le 2\mu$.

All the results are same to the case $x_1>d>x_0$. In this case, the insurance company doesn't need to reinsure.

Proof. (I) If $x_{1_*}\le x_{1_{**}}$, the function $f(x)$ satisfies the HJB equation and boundary conditions. And $f(x)$ also satisfies conditions (5.1) and (5.2) in Theorem 5.1. So $f(x)\ge V(x)\ge V_A(x)$ by Theorem 5.1. We will prove $f(x)=V(x)$ corresponding to π^*. Applying generalized Itô formula, we obtain $\vartheta f(R_{t\wedge\tau^{\pi^*}}^{\pi^*})=0$ and

$$e^{-\delta(t\wedge\tau^{\pi^*})}f(R_{t\wedge\tau^{\pi^*}}^{\pi^*})=f(x)+\sigma\int_0^{t\wedge\tau^{\pi^*}} e^{-\delta s}a(R_s^{\pi^*})f'(R_s^{\pi^*})dB_s$$
$$-\sum_{i=1}^{\infty} e^{-\delta\tau_i^{\pi^*}}(-K+\beta_1\xi_i^{\pi^*})I_{\{\tau_i^{\pi^*}\le t\wedge\tau^{\pi^*}\}}. \quad (5.6)$$

where $\tau^{\pi^*}=\inf\{t\ge 0:R_t^{\pi^*}<m\}$. Because $\liminf_{t\to\infty} e^{-\delta(t\wedge\tau^{\pi^*})}f(R_{t\wedge\tau^{\pi^*}}^{\pi^*})=e^{-\delta\tau^{\pi^*}}f(m)=0$, taking expectation at both sides of (5.6), we get

$$f(x)=E\left[\sum_{i=1}^{\infty} e^{-\delta\tau_i^{\pi^*}}(-K+\beta_1\xi_i^{\pi^*})I_{\{\tau_i^{\pi^*}\le\tau^{\pi^*}\}}\right]=V(x,\pi^*).$$

So $f(x)$ is the value function corresponding to π^*, and $f(x)\le V_A(x)$. Using the results $f(x)\ge V(x)\ge V_A(x)$, we have $f(x)=V(x)=V_A(x)$.

If $x_{1_*}\ge x_{1_{**}}$, $g(x)$ defined in (4.2.1) satisfies the HJB equation and boundary conditions. Thus $g(x)$ satisfies conditions (5.1) and (5.2) in Theorem 5.1. So $g(x)\ge V(x)\ge V_B(x)$ by Theorem 5.1. We will prove $g(x)=V(x)$ corresponding to π^{**}. Applying generalized Itô formula, we obtain $\vartheta f(R_{t\wedge\tau^{\pi^{**}}}^{\pi^{**}})=0$ and

$$e^{-\delta(t\wedge\tau^{\pi^{**}})}g(R_{t\wedge\tau^{\pi^{**}}}^{\pi^{**}})=g(x)+\int_0^{t\wedge\tau^{\pi^{**}}} e^{-\delta s}\beta_2 dG_s^{\pi^{**}}$$
$$+\sigma\int_0^{t\wedge\tau^{\pi^{**}}} e^{-\delta s}a(R_s^{\pi^{**}})f'(R_s^{\pi^{**}})dB_s$$
$$-\sum_{i=1}^{\infty} e^{-\delta\tau_i^{\pi^{**}}}(-K+\beta_1\xi_i^{\pi^{**}})I_{\{\tau_i^{\pi^{**}}\le t\wedge\tau^{\pi^{**}}\}}, \quad (5.7)$$

where $\tau^{\pi^{**}}=\inf\{t\ge 0:R_t^{\pi^{**}}<m\}$. Since $\liminf_{t\to\infty} e^{-\delta(t\wedge\tau^{\pi^{**}})}g(R_{t\wedge\tau^{\pi^{**}}}^{\pi^{**}})=0$, by taking expectation at both sides of (5.7), we get

$$g(x)=E\left[\sum_{i=1}^{\infty} e^{-\delta\tau_i^{\pi^{**}}}(-K+\beta_1\xi_i^{\pi^{**}})I_{\{\tau_i^{\pi^{**}}\le\tau^{\pi^{**}}\}}-\int_0^{\tau^{\pi^{**}}} e^{-\delta s}\beta_2 dG_s^{\pi^{**}}\right]$$
$$=V(x,\pi^{**}).$$

So $g(x)$ is the value function corresponding to π^{**}, and $g(x)\le V_B(x)$. Using the results $g(x)\ge V(x)\ge V_B(x)$, we have $g(x)=V(x)=V_B(x)$. The proof of (ii) and (II) is similar to (i), so we omit it here.

6. Conclusion

In this paper, we consider the optimal dividend and financing control problem in the risk model with non-cheap proportional reinsurance. The management of the company controls the reinsurance rate, dividends payout and the equity issuance to maximize the expected present value of the dividends payout minus the equity issuance until the ruin time. To be more realistic, we assume the minimal reserve restrictions and consider the fixed and proportional transaction costs. The former cost is generated by the advisory and consulting as well as the latter is generated by the tax. It is the first time to study non-cheap proportional reinsurance in an insurance model with this method to solve the optimal control problem, which construct two categories of suboptimal control problems, one is the classical model without equity issuance, the other never goes bankrupt by equity issuance. We verify the optimal strategy and the value function with the corresponding solution in either category of suboptimal models, tightly connecting with the relationships among the parameters.

7. Acknowledgements

We are very gratrful to the careful reading of the manuscript, correction of errors, valuable suggestions which improved this paper very much.

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