A Study of Function Spaces through a Functor

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Abstract: Let $X$ be a locally compact Hausdorff space and let $F_{n}(X)$ = limit of the function spaces of maps of $X$ into certain spaces of type $K(\pi, n)$ for each of the spaces of sequences $S^{\infty}_{+}X = \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, ... is a space of type $K(\pi, n)$, $\Rightarrow$ each of the spaces of sequences $S^{\infty}_{+}X = \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, ... is a space of type $K(\pi, n)$.

For any space $X$, we define the space $F_{n}(X)$ = limit of the function spaces of maps of $X$ into certain spaces of type $K(\pi, n)$, $\Rightarrow$ each of the spaces of sequences $S^{\infty}_{+}X = \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, ... is a space of type $K(\pi, n)$.

Let $f : X \rightarrow Y$ be a continuous map, define $f(X,Y) = \{f(x) | x \in X\}$.

The aim of this paper is to investigate the properties of $F_{n}(X)$; ii) to study of the object $F_{n}(X)$.

Keywords: Eilenberg-MacLane space, function spaces, $\Sigma$-homotopy classes, contravariant functor, compact open topology

1. Introduction

Throughout this paper we assume that all spaces are locally compact Hausdorff space, also all spaces are of type $K(\pi, n)$.

Now we recall the following definitions and statements:-

Definition 1.1:
Let $\pi$ be a discrete group. A based topological space $X$ is called an Eilenberg-MacLane space of type $K(\pi, n)$, where $n \geq 1$; if all the homotopy groups $\pi_{n}(X)$ are trivial except for $\pi_{0}(X)$; which is isomorphic to $\pi$.

A pointed CW complex $X$ is a $K(\pi, n)$ (Eilenberg-MacLane space) if

$\pi_{n}(X) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & n \neq 0,1 \end{cases}$

Definition 1.2:
Let $f : X \rightarrow Y$ be a continuous map, define $\Sigma f : \Sigma X \rightarrow \Sigma Y$ by $\Sigma f(x,t) = (f(x),t)$, then $\Sigma$ is a covariant functor. This implies that $\Sigma$ induces homotopic maps into homotopic maps i.e. $\Sigma$ induces a map $\Sigma : \{X,Y\} \rightarrow \{\Sigma X, \Sigma Y\}$.

Define $S^{\infty}_{+}X = \Sigma(S^{\infty}X)$

$\Rightarrow [X,Y] \rightarrow [\Sigma X, \Sigma Y] \rightarrow \cdots \rightarrow [\Sigma^{n}X, \Sigma^{n}Y] \rightarrow \cdots$

$\Rightarrow \lim_{n \rightarrow \infty} [\Sigma^{n}X, \Sigma^{n}Y] = \{X,Y\}$.

In [5] define that $S$-category same as $\Sigma$-category is the category whose objects are topological spaces with base points and whose maps are from $X$ to $Y$ are the elements of $\{X,Y\}$.

For any space $X$ we define the space $F_{n}(X)$ = $(\Omega \Sigma^{\infty}_{+}X^{\infty}_{+}X)^{\infty}$ topologized by the compact-open topology, then we have the following:

Lemma 1.3: Let $X$ be a polyhedron, the map $F_{n}(X) \rightarrow F_{n+1}(X)$ is a weak homotopy equivalence for each $m \geq 0$.

Proof: Since $\pi_{n}(F_{n}(X)) = \pi_{n}(X)$, $\Rightarrow$ each of the spaces of sequences $S^{\infty}_{+}X = \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, ... is a space of type $K(\pi, n)$, $\Rightarrow$ each of the spaces of sequences $S^{\infty}_{+}X = \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, ... is a space of type $K(\pi, n)$.

Lemma 1.4: Each inclusion map $F_{n}(X) \subset F_{n+1}(X)$ is a weak homotopy equivalence.

Proof: Since $F_{n}(X)$ has the weak topology relative to the subsets $F_{n}(X)$, it follows that every subset of $F_{n}(X)$ is contained in $F_{n}(X)$ for some $n \geq 0$ (all the function spaces are easily seen to be Hausdorff). Therefore the inclusion maps $F_{n}(X) \subset F_{n+1}(X)$ induce the isomorphism $\lim_{n \rightarrow \infty} \pi_{n}(F_{n}(X)) = \pi_{n}(F_{n+1}(X))$, it follows from Lemma 1.3 that for any $m \geq 0$,

$\pi_{n}(F_{n}(X)) \approx \lim_{n \rightarrow \infty} \pi_{n}(F_{n}(X))$.

Lemma 1.5: Let $\lambda : F_{n+1}(\Sigma X) \rightarrow F_{n}(X)$ be defined by $\lambda(x,t) = (x,t)$, then $\lambda$ is a covariant functor.

Since $\pi_{n}(F_{n}(X)) = \pi_{n}(X)$, $\Rightarrow$ each of the spaces of sequences $S^{\infty}_{+}X = \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, ... is a space of type $K(\pi, n)$, $\Rightarrow$ each of the spaces of sequences $S^{\infty}_{+}X = \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, $\Omega S^{\infty}_{+}X \rightarrow \Omega S^{\infty}_{+}X$, ... is a space of type $K(\pi, n)$.

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Therefore we can extend the functor $F_n$ to a functor $F'_n : \{X,X'\} \rightarrow [F_n(X'),F_n(X)]_H$ such that the following diagram

$$\begin{array}{c}
\Sigma \Sigma X & \rightarrow \Sigma X' \\
\lambda \rightarrow F_n \searrow \downarrow \\
\{X,X'\} & \rightarrow [F_n(X'),F_n(X)]_H
\end{array}$$

$F'_n : \{X,X'\} \rightarrow [F_n(X'),F_n(X)]_H$ is commutative.

**Lemma 1.6:** $F'_n$ is a homomorphism

Proof: We prove that

$$F_{n+m} : \{\Sigma \Sigma X, \Sigma \Sigma X'\} \rightarrow [F_{n+m}(\Sigma \Sigma X),F_{n+m}(\Sigma \Sigma X')]_H$$

is a homomorphism for $m \geq 2$.

Let $f,g : \Sigma \Sigma X \rightarrow \Sigma \Sigma X'$ such that $x_0 \in A \cap B$, $\Sigma \Sigma X = A \cup B$, $f[B = g] = \lambda = x' \lambda $ and $f \neq f'$, $g \neq g'$.

Then $f' + g' : \Sigma \Sigma X \rightarrow \Sigma \Sigma X'$ is defined by $f' + g'[A = f][A]$ and $f' + g'[B = g][B]$ and $[f] + [g] = f' + g'$.

If $\lambda' \in F_{n+m}(\Sigma \Sigma X)$ and $x \in \Sigma \Sigma X'$ then

$$(F_{n+m}(f')\lambda')x = \lambda'(f'x),$$

and

$$\lambda' \rightarrow (F_{n+m}(f')\lambda')x, x \in \Sigma \Sigma X.$$  

Since $(F_{n+m}(f')\lambda')x$ is the constant map $x \in B$ and $(F_{n+m}(f')\lambda')x$ is the constant map for $x \in A$, we see that

$$(F_{n+m}(f' + g')\lambda') = ((F_{n+m}(f')\lambda'), (F_{n+m}(g)\lambda'),$$

so $F_{n+m}(f' + g') = F_{n+m}(f'), F_{n+m}(g')$.

**Lemma 1.7:** Let $Y$ be a space of type $K(\pi, n)$ and let $X$ be a polyhedron such that $H^n(X) = 0$, for $q \geq n$.

Let $\Delta : \pi_q(Y) \rightarrow H^n(X)$ be defined by

$$\Delta \alpha = (\pi_q(Y) \times \pi_q(H))/\text{rel},$$

and we have the commutative diagram

$$\begin{array}{c}
\pi_q(Y) & \rightarrow & H^n(X) \\
\Delta \downarrow & & \downarrow \Delta \\
\pi_q(Y) & \rightarrow & H^n(X)
\end{array}$$

and the fact that $(\Delta \alpha)^* \Delta = \Delta$ we get the commutative diagram

$$\begin{array}{c}
\pi_q(F_n(X)) & \rightarrow & \pi_q(F_n(X)) \\
\Delta \downarrow & & \downarrow \Delta \\
\pi_q(F_n(X)) & \rightarrow & \pi_q(F_n(X))
\end{array}$$

**Theorem 2.1:** If $f : X \rightarrow X'$, then $\pi_m(F_n(f)) : F_n(X') \rightarrow F_n(X)$ is a continuous homomorphism.

Proof: We define $F_n(f) : F_n(X') \rightarrow F_n(X)$ by

$$(F_n(f)(\lambda'))(x) = \lambda'(fx),$$

for $x \in F_n(X)$, $m \geq 0$.

Since for every $m$, $F_n(f) : F_n(X') \rightarrow F_n(X)$ is a continuous homomorphism and $F_n(f)F_n(X') = F_n(X)$.

**Theorem 2.2:** Let $\{X,X'\}$ be the set of $\Sigma$-homotopy classes from $X$ to $X'$ and $[F_n(X),F_n(X)]_H$ denote the monoid of $\Sigma$-homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid $F_n(X')$ into another $F_n(X)$, then we have a homomorphism

$F_n : \{X,X'\} \rightarrow [F_n(X),F_n(X)]_H$ such that $F_n[f] = [F_n(f)]_H$.

Proof: Let $h : X \times I \rightarrow X'$ be a homotopy from $f_0$ to $f_1$. Then for each $m$ we have a continuous homomorphism $F_n(h) : (\Omega^m SP^{\sum_{n=0}^{m+1}} X') \rightarrow (\Omega^m SP^{\sum_{n=0}^{m+1}} X')$, which corresponds to a continuous map

$F_n : (\Omega^m SP^{\sum_{n=0}^{m+1}} X') \times I \rightarrow (\Omega^m SP^{\sum_{n=0}^{m+1}} X')$ which is a continuous homomorphism for every $t \in I$.  

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Since commutativity holds in the diagram

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\begin{align*}
\left(\bigoplus_{i=1}^{n} S^{m} \times \bigoplus_{j=1}^{n} S^{m+n}\right) \times X & \xrightarrow{h} \left(\bigoplus_{i=1}^{n} S^{m} \times \bigoplus_{j=1}^{n} S^{m+n}\right) X \\
\left(\bigoplus_{i=1}^{n+1} S^{m} \times \bigoplus_{j=1}^{n} S^{m+n+1}\right) \times \left(\bigoplus_{k=1}^{n} S^{m} \times \bigoplus_{j=1}^{n} S^{m+n}\right) & \xrightarrow{\text{the maps } h_{n+1}} \text{define a continuous map } h' \\
\lim_{m \to \infty} \left(\bigoplus_{i=1}^{n} S^{m} \times \bigoplus_{j=1}^{n} S^{m+n}\right) X \times I & \to F_{n} \\
\lim_{m \to \infty} \left(\bigoplus_{i=1}^{n} S^{m} \times \bigoplus_{j=1}^{n} S^{m+n}\right) X \times I & \to \lim_{m \to \infty} \left(\bigoplus_{i=1}^{n} S^{m} \times \bigoplus_{j=1}^{n} S^{m+n}\right) X \times I \\
\text{Hence, a continuous map } h' & \equiv F_{n}(f_{n}) \times I \to F_{n}(X) \\
\text{Hence, } F_{n}(f_{n}) & \equiv F_{n}(f_{n})
\end{align*}
\]

**Theorem 2.3** Let \(\{X, X\} \) be the set of \(\Sigma\)-homotopy classes from \(X\) to \(X\). The set of all \(\Sigma\)-homotopy classes and their homomorphisms forms a category, it is denoted by \(\mathcal{H}C\).

**Proof:** We take all the Hausdorff spaces are the set of object and the set of \(\Sigma\)-homotopy classes are set of morphisms and the composition is the usual composition of mappings.

**Theorem 2.4** \([F_{n}(X), F_{n}(X)]_{h}\) denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid \(F_{n}(X)\) into another \(F_{n}(X)\). The set of all monoid of homotopy classes of homomorphisms, homotopic through homomorphisms forms a category, it is denoted by \(\mathcal{F}NH\).

**Proof:** We take all the abelian monoid are the set of object and the set of all monoid of homotopy classes of homomorphisms, homotopic through homomorphisms are set of morphisms and the composition is the usual composition of mappings.

**Theorem 2.5** Let \(\mathcal{H}C\) be the category of homotopy classes of homomorphisms and \(\mathcal{F}NH\) be the monoid of homotopy classes of homomorphisms, there exists a contravariant \(n\)-homotopy functor \(\mathcal{F}_{n}: \mathcal{H}C \to \mathcal{F}NH\).

**Proof:** Let \(\{X, X\} \) be the set of \(\Sigma\)-homotopy classes from \(X\) to \(X\) in \(\mathcal{H}C\) then \([F_{n}(X), F_{n}(X)]_{h}\) denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid \(F_{n}(X)\) into another \(F_{n}(X)\) in \(\mathcal{F}NH\).

Let \(\{X_{i}, X_{i}\} \) be the set of \(\Sigma\)-homotopy classes from \(f: X_{i} \to X_{j}\) and \(\{X_{j}, X_{j}\} \) be the set of \(\Sigma\)-homotopy classes from \(g: X_{j} \to X_{j}\), then by Definition 2.1 and Lemma 1.9, \(\{X_{i}, X_{i}\} \) be the set of \(\Sigma\)-homotopy classes from \(gf: X_{j} \to X_{j}\) in \(\mathcal{H}C\) and also for \(\{X_{i}, X_{i}\} \) be the set of \(\Sigma\)-homotopy classes from \(gf: X_{j} \to X_{j}\) in \(\mathcal{H}C\), then \([F_{n}(X_{i}), F_{n}(X_{i})]_{h}\) denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid \(F_{n}(X_{i})\) into another \(F_{n}(X_{i})\) in \(\mathcal{F}NH\).

\(F_{n}: \{X_{i}, X_{i}\} \to [F_{n}(X_{i}), F_{n}(X_{i})]_{h}\) such that \(F_{n}[g f] = [F_{n}(g) F_{n}(f)]_{h}\). Using the Lemma 1.9, we have if \(f \approx g \Rightarrow F_{n}(f) \approx F_{n}(g) \Rightarrow [F_{n}(f)]_{h} = [F_{n}(g)]_{h}\). Using Theorem 2.2.

**References**


Author Profile

Prof. Pravanjan Kumar Rana obtained his MSc in Pure Mathematics and his PhD in Algebraic Topology. He has published, since 2005, more than 23 papers in peer reviewed journals. Formerly, he was the first HOD of Mathematics Department in Berhampore Girls’ College, Berhampore, Murshidabad and latterly he is HOD of Mathematics Department in Ramakrishna Mission Vivekananda Centenary College, Rahara, Kol. 700118, and performs his research at Algebraic Topology and Category Theory.