A Study of Function Spaces through a Functor

Dr. Pravanj Kumar Rana
Assistant Professor of Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara, Kol.118, India

Abstract: Let X be a locally compact Hausdorff space and let \( F_n(X) \) = limit of the function spaces of maps of X into certain spaces of type \( K(\pi, n) \)

1. Introduction
Throughout this paper we assume that all spaces are locally compact Hausdorff space, also all spaces are of type \( K(\pi, n) \).

Now we recall the following the following definitions and statements:-

**Definition 1.1:**
Let \( \pi \) be a discrete group. A based topological space X is called an Eilenberg-MacLane space of type \( K(\pi, n) \), where \( n \geq 1 \); if all the homotopy groups \( \pi_k(X) \) are trivial except for \( \pi_n(X) \), which is isomorphic to \( \pi \).

A pointed CW complex X is a \( K(\pi, n) \)
(Eilenberg-MacLane space) if
\[
\pi_n(X) = \begin{cases} \pi, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}
\]

**Definition 1.2:**
Let \( f : X \to Y \) be a continuous map, define \( \Sigma f : \Sigma X \to Y \) by \( \Sigma f(x,t) = (f(x),t) \), then \( \Sigma \) is a covariant functor. This implies that since \( \Sigma \) induces homotopic maps into homotopic maps i.e. \( \Sigma \) induces a map \( \Sigma : [X,Y] \to [\Sigma X, \Sigma Y] \).

Define \( \Sigma^{n+1}(X) = \Sigma(\Sigma^nX) \)
\[
\Rightarrow [X,Y] \to [\Sigma X, \Sigma Y] \to ... \to [\Sigma^n X, \Sigma^n Y] \to ...
\]
\[
\Rightarrow \lim_{n \to +\infty} = \{X,Y\}.
\]

In [5] define that S-category same as \( \Sigma \)-category is the category whose objects are topological spaces with base points and whose maps are from X to Y are the elements of \( \{X,Y\} \).

For any space X we define the space \( F_{n,m}(X) = (\Sigma^n \Sigma^{m+1} X) \) topologized by the compact-open topology, then we have the following:

**Lemma 1.3:** Let X be a polyhedron, the map \( F_{n,m}(X) \to F_{n,m+1}(X) \) is a weak homotopy equivalence for each \( m \geq 0 \).

Proof: Since \( \pi_n(F_{n,m}(X)) \approx (\sum^k X, \Omega^n \Sigma^n X) \) and \( \pi_n(F_{n,m+1}(X)) \approx (\sum^k X, \Omega^n \Sigma^{n+1} X) \), it follows that the map \( F_{n,m}(X) \to F_{n,m+1}(X) \) is a weak homotopy equivalence for each \( m \geq 0 \).

**Lemma 1.4:** Each inclusion map \( F_{n,m}(X) \subset F_{n}(X) \) is a weak homotopy equivalence.

Proof:- Since \( F_{n}(X) \) has the weak topology relative to the subsets \( F_{n,m}(X) \), it follows that every subset of \( F_{n,m}(X) \) is contained in \( F_{n}(X) \) for some \( m \geq 0 \) (all the function spaces are easily seen to be Hausdorff). Therefore the inclusion maps \( F_{n,m}(X) \subset F_{n}(X) \) induce the isomorphism \( \lim_n \pi_n(F_{n,m}(X)) \approx \pi_n(F_{n}(X)) \), it follows from **Lemma 1.3** that for any \( m \geq 0 \), \( \pi_n(F_{n,m}(X)) \approx \lim_n \pi_n(F_{n,m}(X)) \)

**Lemma 1.5:** Let \( \lambda : F_{n+1}(\Sigma X) \to F_n(X) \) be defined by \( \lambda(x) \) (\( t_1, t_2, ..., t_n \)) = \( \lambda(x, t_0, t_1, t_2, ..., t_n) \), for \( \alpha \in F_{n+1,m}(\Sigma X) \), then \( \lambda \) is an isomorphism and if \( f : X \to X \), commutativity holds in the diagram
\[
\begin{array}{c}
F_{n+1}(\Sigma X') \to F_{n,m}(\Sigma X) \\
\downarrow \quad \downarrow \\
F_{n,1,m-1}(\Sigma X) \to F_{n,m-1}(\Sigma X)
\end{array}
\]

Proof:
Since \( \lambda : F_{n+1}(\Sigma X) \to F_n(X) \) is induced by the natural isomorphism \( \lambda : F_{n+1,m}(\Sigma X) \to F_{n,m}(X) \) , for every \( m \geq 1 \) and so \( \lambda \) is an isomorphism.

Again since the diagram
\[
\begin{array}{c}
F_{n+1,m-1}(\Sigma X') \to F_{n+1,m}(\Sigma X) \\
\downarrow \quad \downarrow \\
F_{n,m}(\Sigma X') \to F_{n+1,m-1}(\Sigma X)
\end{array}
\]

is commutative and so \( \lambda \) is commutativity.

Let \( \bar{\lambda} : [F_{n+1,m}(\Sigma X), F_{n,m}(X)] \to [F_{n+1,m}(\Sigma X), F_{n,m}(X)] \) be the isomorphism defined by \( \bar{\lambda}(f) = [\lambda_0 : 0_0] \).

Using the above **Lemma 1.3**, it follows that \( \bar{\lambda} : [F_{n+1,m}(\Sigma X), F_{n,m}(X)] \) is commutativity.
Therefore we can extend the functor $F_n$ to a functor $F_{n,m} : \{X, X'\} \rightarrow [F_n(X), F_n(X')]_{\Omega}$ such that the following diagram holds:

\[
\begin{array}{c}
\Sigma^n X \times \Sigma^n X' \\
\downarrow \alpha_{n,m}
\end{array}
\rightarrow
\begin{array}{c}
\{X, X'\}
\end{array}
\xrightarrow{F_{n,m}}
\begin{array}{c}
[F_n(X), F_n(X')]_{\Omega}
\end{array}
\]

Lemma 1.6: $F'_{n,m}$ is a homomorphism

Proof: We prove that $F'_{n,m} : \Sigma^n X \xrightarrow{\Sigma^n X'} \rightarrow [F_n(X), F_n(X')]_{\Omega}$ is a homomorphism for $m \geq 2$.

Let $f : \Sigma^n X \rightarrow \Sigma^n X'$ such that $x_0 \in A \cap B$, $\Sigma^n X = A \cup B$, $f[B] = g[A] = x_0$ and $f \circ \Pi = g \circ \Pi'$. Then $f \circ \Pi = (f \circ \Pi) \circ \Pi = g \circ \Pi'$. Since $(f \circ \Pi) \circ \Pi = g \circ \Pi'$, we conclude that $f \circ \Pi = g \circ \Pi'$.

If $x \in F_{n,m}(X)$ and $x \in \Sigma^n X'$ then $(f(x)) = ((f \circ \Pi)(x)) = (g \circ \Pi')(x)$.

Since $(f \circ \Pi)(x) = (g \circ \Pi')(x)$ is the constant map if $x \in B$ and $(f \circ \Pi)(x) = (g \circ \Pi')(x)$ is the constant map for $x \in A$, we see that $F_{n,m}(f \circ \Pi) = (F_{n,m}(f)) \circ (F_{n,m}(\Pi))$.

Lemma 1.7: Let $Y$ be a space of type $K(\pi, n)$ and let $X$ be a polyhedron such that $H^q(X) = 0$, for $q \geq n$.

Let $A = E' \times \Omega^{n+m} \rightarrow H^q(Y)$ be defined by $A(h) = (e')(\pi h)$, then $A$ is an isomorphism.

Let $X$ be a polyhedron such that $H^q(X) = 0$ for $q \geq n$; then we have isomorphisms $\Delta_n \circ \pi_n : \pi_n(F_n(X)) \rightarrow H^{n+m}(X)$. Let $F_{n,m} : \Sigma^n X \times \Sigma^n X', \Sigma^n \rightarrow \Sigma^n \times \Sigma^n$ and $E_{n,m}$ be defined by $E_{n,m}(F_n(X)) = \Omega^{n+m}$.

Lemma 2.1: If $f : X \rightarrow X'$, then $F_{n,m}(f)(X) \rightarrow F_{n,m}(X)$. We define $F_{n,m}(f) : F_{n,m}(X) \rightarrow F_{n,m}(Y)$ by $F_{n,m}(f)(X) = (f \circ \Pi)(x)$, for $x \in F_{n,m}(X)$, $m \geq 0$.

Lemma 2.2: Let $\{X, X'\}$ be the set of $\Sigma$-homotopy classes of $X$ and $X'$ and $[F_n(X), F_n(X')]_{\Omega}$ denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid $F_n(X)$ into another $F_n(Y)$, then we have a homomorphism $F_n : \{X, X'\} \rightarrow [F_n(X), F_n(X')]_{\Omega}$ such that $F_n(f)(X) = [F_n(f)(X)]_{\Omega}$.

Therefore, we can extend the functor $F_{n,m}$ to a functor $F_{n,m} : \{X, X'\} \rightarrow [F_n(X), F_n(X')]_{\Omega}$ such that the following diagram holds:

\[
\begin{array}{c}
\Sigma^n X \times \Sigma^n X' \\
\downarrow \alpha_{n,m}
\end{array}
\rightarrow
\begin{array}{c}
\{X, X'\}
\end{array}
\xrightarrow{F_{n,m}}
\begin{array}{c}
[F_n(X), F_n(X')]_{\Omega}
\end{array}
\]

Lemma 1.8: Let $f : X \rightarrow X'$, then the diagram $H^{n+m}(X) \rightarrow H^{n+m}(X)$ is commutative.

Proof: To prove the Lemma it suffices to prove the following diagram is commutative:

\[
\begin{array}{c}
H^{n+m}(X) \\
\downarrow \Delta_n
\end{array}
\rightarrow
\begin{array}{c}
H^{n+m}(X)
\end{array}
\xrightarrow{F_{n,m}}
\begin{array}{c}
[F_n(X), F_n(Y)]_{\Omega}
\end{array}
\]

Lemma 1.9: Let $f : X \rightarrow Y$ and $g_1, g_2 : Y \rightarrow Z$, for $i = 1, 2$ be continuous. Then $g_i \circ f_1 = g_i \circ f_2$ is a homomorphism.

In [1], it follows.

In section 2 we construct and investigate functor $F_{n,m}$
Since commutativity holds in the diagram
\[
\begin{array}{ccc}
\Omega^m(\Sigma^mX)^\times & \xrightarrow{h_{\ast}} & \Omega^m(\Sigma^{m+1}Y)^\times \\
\downarrow & & \downarrow \\
\Omega^m(\Sigma^mX)^\times \times 1 & \xrightarrow{h_{\ast}1} & \Omega^m(\Sigma^{m+1}Y)^\times \times 1
\end{array}
\]
the maps $h_{\ast}$ define a continuous map $h^{\ast}: \lim_m(\Omega^m(\Sigma^mX)^\times \times 1) \to F_n\lim_m(\Omega^m(\Sigma^mX)^\times \times 1)$ defines a continuous map $h^{\ast}: F_\ast(X) \times 1 \to F_\ast(X)$
\[
F_\ast(f_\ast) = F_\ast(f_{\ast})
\]
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\section{References}

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Author Profile

Prof. Pravanjan Kumar Rana obtained his MSc in Pure Mathematics and his PhD in Algebraic Topology. He has published, since 2005, more than 23 papers in peer reviewed journals. Formerly, he was the first HOD of Mathematics Department in Berhampore Girls’ College, Berhampore, Murshidabad and latterly he is HOD of Mathematics Department in Ramakrishna Mission Vivekananda Centenary College, Rahara, Kol. 700118, and performs his research at Algebraic Topology and Category Theory.