A Study of Function Spaces through a Functor

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Abstract: ***Let X be a locally compact Hausdorff space and let $F_n(X)$ = limit of the function spaces of maps of X into certain spaces of type $K(\pi, n)$***
$\Rightarrow$ each of the spaces of sequences
$SP^{\infty}\sum^{n}, \Omega SP^{\infty}\sum^{n+1}, \Omega SP^{\infty}\sum^{n+2},\ldots,\Omega SP^{\infty}\sum^{n+m},\ldots$ is a space of type $K(\pi, n)$,
$\Rightarrow SP^{\infty}\sum^{n} \rightarrow \Omega SP^{\infty}\sum^{n+1} \rightarrow \Omega^2 SP^{\infty}\sum^{n+2} \rightarrow \ldots \rightarrow \Omega^m SP^{\infty}\sum^{n+m} \rightarrow \Omega^{m+1}SP^{\infty}\sum^{n+m+1} \rightarrow \ldots$.
For any space X ,we define the space $F_{n,m}(X) = (\Omega^m SP^{\infty}\sum^{n+m})^X$ topologized by the compact-open topology.
The aim of this paper is i) to investigate the properties of $F_{n,m}(X)$; ii) to study of the object $F_{n,m}$.

Keywords: Eilenberg-MacLane space, function spaces, Σ-homotopy classes, contravariant functor, compact open topology

1. Introduction

Throughout this paper we assume that all spaces are locally compact Hausdorff space, also all spaces are of type $K(\pi, n)$.

Now we recall the following the following definitions and statements:-

Definition 1.1:
Let π be a discrete group. A based topological space X is called an **Eilenberg-MacLane space** of type $K(\pi; n)$, where $n \geq 1$; if all the homotopy groups $\pi_k(X)$ are trivial except for $\pi_n(X)$; which is isomorphic to π.

A pointed CW complex X is a $K(\pi, n)$ (Eilenberg-MacLane space) if

$$\pi_k(X) = \begin{cases} \pi, k = n \\ 0, k \neq n \end{cases}$$

Definition 1.2:
Let $f : X \rightarrow Y$ be a continuous map, define $\Sigma f: \Sigma X \rightarrow \Sigma Y$ by $\Sigma f(x,t) = (f(x),t)$, then Σ is a covariant functor. This implies that Σ induces homotopic maps into homotopic maps i.e. Σ induces a map
$\Sigma : [X,Y] \rightarrow [\Sigma X, \Sigma Y]$.

Define $\Sigma^{n+1}(X) = \Sigma(\Sigma^n X)$
$\Rightarrow [X,Y] \rightarrow [\Sigma X, \Sigma Y] \rightarrow \ldots \rightarrow [\Sigma^n X, \Sigma^n Y] \rightarrow \ldots$
$\Rightarrow \lim_{n\rightarrow\infty}[\Sigma^n X, \Sigma^n Y] = \{X,Y\}$.

In [5] define that S-category same as Σ-category is the category whose objects are topological spaces with base points and whose maps are from X to Y are the elements of $\{X,Y\}$

For any space X we define the space
$F_{n,m}(X) = (\Omega^m SP^{\infty}\Sigma^{n+m})^X$ topologized by the compact-open topology, then we have the following:

Lemma1.3: Let X be a polyhedron, the map $F_{n,m}(X) \rightarrow F_{n,m+1}(X)$ is a weak homotopy equivalence for each $m \geq 0$.

Proof:

Since $\pi_k(F_{n,m}(X)) \approx [\Sigma^k X, \Omega^m SP^{\infty}\Sigma^{n+m}]$ and $\pi_k(F_{n,m+1}(X)) \approx [\Sigma^k X, \Omega^{m+1} SP^{\infty}\Sigma^{n+m+1}]$,
it follows that the map $F_{n,m}(X) \rightarrow F_{n,m+1}(X)$ is a weak homotopy equivalence for each $m \geq 0$.

Lemma1.4: Each inclusion map
$F_{n,m}(X) \subset F_n(X)$ is a weak homotopy equivalence.

Proof:-
Since $F_n(X)$ has the weak topology relative to the subsets $F_{n,m}(X)$, it follows that every subset of $F_n(X)$ is contained in $F_{n,m}(X)$ for some $m \geq 0$ (all the function spaces are easily seen to be Hausdorff). Therefore the inclusion maps
$F_{n,m}(X) \subset F_n(X)$ induce the isomorphism
$\lim_m \pi_q(F_{n,m}(X)) \approx \pi_q(F_n(X))$, it follows from **Lemma1.3** that for any $m \geq 0$,
$\pi_q(F_n(X)) \approx \lim_m \pi_q(F_{n,m}(X))$

Lemma 1.5: Let $\lambda : F_{n+1}(\Sigma X) \longrightarrow F_n(X)$ be defined by $\lambda(\alpha)(x)(t_1, t_2,\ldots,t_m) = \lambda(x, t_m)((t_1, t_2,\ldots,t_{m-1})$, for $\alpha \in F_{n+1,m-1}(\Sigma X)$, then λ is an isomorphism and if $f: X \rightarrow X'$, commutativity holds in the diagram

$$F_{n+1}(\Sigma X') \xrightarrow{F_{n+1}(\Sigma f)} F_{n+1}(\Sigma X)$$
$$\downarrow\downarrow\ F_n(X') \xrightarrow{F_n(f)} F_n(X)$$

Proof:
Since $\lambda : F_{n+1}(\Sigma X) \longrightarrow F_n(X)$ is induced by the natural isomorphism
$\lambda' : F_{n+1,m-1}(\Sigma X) \longrightarrow F_{n,m}(X)$, for every $m \geq 1$ and so λ is an isomorphism.
Again since the diagram

$$F_{n+1,m-1}(\Sigma X') \xrightarrow{F_{n+1,m-1}(\Sigma f)} F_{n+1,m-1}(\Sigma X)$$
$$\downarrow\downarrow\ F_{n,m}(X') \xrightarrow{F_{n,m}(f)} F_{n,m}(X)$$

is commutative and so λ is commutativity.

Let $\bar{\lambda}: [F_{n+1}(\Sigma X'),F_{n+1}(\Sigma X)] \longrightarrow [F_n(X'),F_n(X)]_H$ be the isomorphism defined by
$\bar{\lambda}[f]_H = [\lambda f \lambda^{-1}]_H$.
Using the above **Lemma1.3**, it follows that
$\bar{\lambda}F_{n+1}(\Sigma) = F_n: \{X,X'\} \rightarrow [F_n(X'),F_n(X)]_H$.

Therefore we can extend the functor F_n to a functor $F_n^{/}$: $\{X,X^{/}\} \longrightarrow [F_n(X^{/}),F_n(X)]_H$ such that the following diagram

$$[\Sigma^m X, \Sigma^m X^{/}] \longrightarrow \{X, X^{/}\}$$
$$\overline{\lambda}^{\,m} F_{n+m} \searrow \swarrow F_n^{/}$$
$[F_n(X^{/}),F_n(X)]_H$ is commutative.

Lemma 1.6: $F_n^{/}$ is a homomorphism

Proof: We prove that

F_{n+m} : $[\Sigma^m X, \Sigma^m X^{/}] \rightarrow [F_{n+m}(\Sigma^m X), F_{n+m}(\Sigma^m X^{/})]_H$ is a homomorphism for $m \geq 2$.

Let $f,g : \Sigma^m X \rightarrow \Sigma^m X^{/}$ such that $x_0 \in A \cap B$, $\Sigma^m X = A \cup B$, $f^{/}|B = g^{/}|A = x_0^{/}$ and $f \simeq f^{/}$, $g \simeq g^{/}$.

Then $f^{/}+g^{/} : \Sigma^m X \rightarrow \Sigma^m X^{/}$ is defined by

$f^{/} + g^{/}|A = f^{/}|A$ and $f^{/} + g^{/}|B = g^{/}|B$ and

$[f] + [g] = [f^{/}+ g^{/}]$.

If $\lambda^{/} \in F_{n+p,m}(\Sigma^p X^{/})$ and $x \in \Sigma^p X^{/}$ then

$((F_{n+p,m}(f^{/}+g^{/}))\lambda^{/})x=\lambda^{/}(f^{/}+g^{/})(x)$

$$=\begin{cases} \lambda^{/} f^{/} x = ((F_{n+p} f^{/})\lambda^{/})x, \; if\, x \in A \\ \lambda^{/} g^{/} x = ((F_{n+p} g^{/})\lambda^{/})x, \; if\, x \in B \end{cases}$$

Since $(F_{n+p,m}f^{/})\lambda^{/})x$ is the constant map if $x\in B$ and $((F_{n+p,m} g^{/})\lambda^{/})x$ is the constant map for $x \in A$, we see that

$(F_{n+p,m}(f^{/} + g^{/}))\lambda^{/} = ((F_{n+p,m}f^{/})\lambda^{/}).((F_{n+p,m}\, g^{/})\lambda^{/}$,

so $F_{n+p,m}(f^{/} + g^{/}) = F_{n+p,}m(f^{/}) \,.\, F_{n+p,m}(g^{/})$,

Lemma1.7: Let Y be a space of type $K(\pi, n)$ and let X be a polyhedron such that $H^q(X) = 0$,

for $q \geq n$.

Let $\Delta : \pi_q(Y^X) \rightarrow H^{n-q}(X;\pi)$ be defined by

$\Delta\alpha = E^*(\tau)/h\alpha$, then Δ is an homomorphism

$\Delta : \pi_q(Y^X) \approx H^{n-q}(X;H)$

Let X be a polyhedron such that $H^q(X) = 0$ for $q \geq n$, then we have isomorphisms

$\Delta_{n,m} : \pi_q(F_{n,m}(X)) \approx H^{n-q}(X)$ defined by

$\Delta_{n,m}\alpha = E^*_{n,m}(\tau_{n,m})$ / $h\alpha$, where

$E_{n,m} : F_{n,m}(X \times X) \rightarrow \Omega^m\, SP^\infty \sum^{n+m}$ is a evaluation map.

From the commutativity of the diagram

$$\begin{array}{ccc} F_{n,m}\,(X \times X) & \xrightarrow{E_{n,m}} & \Omega^m\, SP^\infty\Sigma^{n+m} \\ \downarrow & & \downarrow \Omega^m\rho \\ F_{n,m+1}(X \times X) & \xrightarrow{E_{n,m+1}} & \Omega^m\, SP^\infty\Sigma^{n+m+1} \end{array}$$

and the fact that $(\Omega^m\rho)^*\tau_{n,m+1} = \tau_{n,m}$,

we get the commutative diagram

$$\begin{array}{ccc} \pi_q(F_{n,m}(X)) & \rightarrow & \pi_q(F_{n,m+1}(X)) \\ & \Delta_{n,m}\searrow \quad \swarrow \Delta_{n,m+1} & \\ & H^{n-q}(X) & \end{array}$$

$\Rightarrow$ Δ_n: $\pi_q(F_n(X)) \approx H^{n-q}(X)$ is an isomorphism and commutativity holds in the diagram

$$\begin{array}{ccc} \pi_q(F_{n,m}(X)) & \rightarrow & \pi_q(F_n(X)) \\ & \Delta_{n,m}\searrow \quad \swarrow \Delta_n & \\ & H^{n-q}(X) & \end{array}$$

Lemma 1.8:

Let $f: X \rightarrow X^{/}$, then the diagram

$$\begin{array}{ccc} H^{n-q}(X^{/}) & \xrightarrow{f^*} & H^{n-q}(X) \\ \Delta_n \downarrow & & \downarrow \Delta_n \\ \pi_q(F_n(X^{/})) & \xrightarrow{(F_n f)_\#} & \pi_q(F_n(X)) \end{array}$$

is commutative

Proof: To prove the Lemma it suffices to prove the following diagram is commutative

$$\begin{array}{ccc} H^{n-q}(X^{/}) & \xrightarrow{f^*} & H^{n-q}(X) \\ \Delta_{n,m} \downarrow & & \downarrow \Delta_{n,m} \\ \pi_q(F_{n,m}(X^{/})) & \xrightarrow{(F_{n,m} f)_\#} & \pi_q(F_{n,m}(X)) \end{array}$$

Let i: $F_{n,m}(X^{/}) \subset F_{n,m}(X^{/})$, j : $X \subset X$. By definition of $F_{n,m}(f)$ we have the commutative diagram

$$\begin{array}{ccc} F_{n,m}(X^{/} \times X) & \xrightarrow{i \times f} & F_{n,m}(X^{/} \times X^{/}) \\ F_{n,m}(f \times j) \downarrow & & \downarrow E^{/} \\ F_{n,m}(X \times X) & \xrightarrow{i \times f} & \Omega^m SP^\infty \Sigma^{n+m} \end{array}$$

E, $E^{/}$ are the respective evaluation maps.

Lemma1.9 Let $f_i : X \rightarrow Y$ and g_i: $Y \rightarrow Z$, for i = 1,2 be continuous. If $f_1 \simeq f_2$ and $g_1 \simeq g_2$, then $g_1 o f_1 \simeq g_2 o f_2$; that is $[g_1 o f_1] = [g_2 o f_2]$.

In [1] , it follows.

In section 2 we construct and investigate functor $F_{n,\, m}$

Theorem 2.1 If $f: X \rightarrow X^{/}$,

then $F_{n,\, m}(f)$: $F_{n,\, m}(X^{/}) \rightarrow F_{n,\, m}(X)$ is a continuous homomorphism.

Proof: We define $F_{n,\, m}(f)$: $F_{n,\, m}(X^{/})$: $\rightarrow F_{n,\, m}(X)$ by

$(F_{n,\, m}(f(\lambda^{/}))(x) = \lambda^{/}(f(x))$, for $\lambda^{/} \in F_{n,\, m}(X^{/})$, $m \geq 0$.

Since for every m , $F_n(f)$: $F_n(X^{/})$: $\rightarrow F_n(X)$ is a continuous homomorphism and

$F_n(f)_{/}\, F_{n\,,m}(X^{/}) = F_{n,\, m}(f)$ is continuous.

Theorem 2.2 Let $\{X,X^{/}\}$ is the set of Σ-homotopy classes from X to $X^{/}$ and $[F_n(X^{/}),F_n(X)]_H$ denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid $F_n(X^{/})$ into another $F_n(X)$, then we have a homomorphism

F_n : $\{X,X^{/}\} \longrightarrow [F_n(X^{/}),F_n(X)]_H$ such that $F_n[f] = [F_n(f)]_H$.

Proof: Let $h: X \times I \rightarrow X^{/}$ be a homotopy from f_0 to f_1. Then for each m we have a continuous homomorphism $F_{n,m}(h):(\Omega^m SP^\infty \Sigma^{n+m})^{X^{/}} \rightarrow (\Omega^m\, SP^\infty\Sigma^{n+m})^{X \times I}$, which corresponds to a continuous map

h_m : $(\Omega^m SP^\infty \Sigma^{n+m})^{X^{/}} \times I \rightarrow (\Omega^m\, SP^\infty\Sigma^{n+m})^X$ which is a continuous homomorphism for every $t \in I$.

Since commutativity holds in the diagram

$$\begin{array}{ccc} (\Omega^m SP^\infty \Sigma^{n+m})^{X'} \times I & \xrightarrow{h_m} & (\Omega^m SP^\infty \Sigma^{n+m})^X \\ \downarrow & & \downarrow \\ (\Omega^{m+1} SP^\infty \Sigma^{n+m+1})^{X'} \times I & \xrightarrow{h_{m+1}} & (\Omega^{m+1} SP^\infty \Sigma^{n+m+1})^X \end{array}$$

$\Rightarrow$ the maps h_m define a continuous map $h': \lim_m((\Omega^m SP^\infty \Sigma^{n+m})^{X'} \times I) \longrightarrow F_n$

$\Rightarrow \lim_m((\Omega^m SP^\infty \Sigma^{n+m})^{X'} \times I) \approx \lim_m(\Omega^m SP^\infty \Sigma^{n+m})^{X'} \times I \Rightarrow$

h' defines a continuous map $h'': F_n(X') \times I \longrightarrow F_n(X)$

$\Rightarrow F_n(f_0) \simeq F_n(f_1)$

Theorem 2.3 Let $\{X,X'\}$ is the set of Σ-homotopy classes from X to X' . The set of all Σ-homotopy classes and their homomorphisms forms a category , it is denoted by $'\mathcal{HC}'$

Proof: We take all the Hausdorff spaces are the set of object and the set of Σ-homotopy classes are set of morphisms and the composition is the usual composition of mappings.

Theorem2.4 $[F_n(X'),F_n(X)]_H$ denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid $F_n(X')$ into another $F_n(X)$. The set of all monoid of homotopy classes of homomorphisms, homotopic through homomorphisms forms a category , it is denoted by $'\mathcal{FNHC}'$

Proof: We take all the abelian monoid are the set of object and the set of all monoid of homotopy classes of homomorphisms, homotopic through homomorphisms are set of morphisms and the composition is the usual composition of mappings.

Theorem2.5 Let $'\mathcal{HC}'$ be the category of homotopy classes of homomorphism and $'\mathcal{FNHC}'$ be the monoid of homotopy classes of homomorphisms, there exists a contravariant n- homotopy functor $\mathcal{F}_n$: $'\mathcal{HC}' \to '\mathcal{FNHC}'$

Proof:
Let $\{X,X'\}$ be the set of Σ-homotopy classes from X to X' in $'\mathcal{HC}'$ then $[F_n(X'),F_n(X)]_H$ denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid $F_n(X')$ into another $F_n(X)$ in $'\mathcal{FNHC}'$.

Let $\{X_1, X_2\}$be the set of Σ-homotopy classes from $f : X_1 \to X_2$ and $\{X_2,X_3\}$be the set of Σ-homotopy classes from $g: X_2 \to X_3$, then by **Definition1.2** and **Lemma1.9**, $\{X_1,X_3\}$be the set of Σ-homotopy classes from $gf: X_1 \to X_3$ in $'\mathcal{HC}'$ and also for $\{X_1,X_3\}$be the set of Σ-homotopy classes from $gf :X_1 \to X_3$ in $'\mathcal{HC}'$, then $[F_n(X_3),F_n(X_1)]_H$ denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid $F_n(X_3)$ into another $F_n(X_1)$ in $'\mathcal{FNHC}'$.

$F_n : \{ X_1 ,X_3\} \longrightarrow [F_n(X_3),F_n(X_1)]_H$ such that $F_n[g\ f] = [F_n(g\ f)]_H$. Using the **Lemma 1.9**, we have if $f \simeq g \Rightarrow F_n(f) \simeq F_n(g) \Rightarrow [F_n(f)] = [F_n(g)]$. Using **Theorem2.2**,

$F_n[g \circ f] = [F_n(g \circ f)]_H = [F_n(f) \circ F_n(g)]_H = [F_n(f)]_H \circ [F_n(g)]_H$.
If $\{X,X\}$ be the set of Σ-homotopy classes from X to X in $'\mathcal{HC}'$ then $[F_n(X),F_n(X)]_H$ denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid $F_n(X)$ into another $F_n(X)$ in $'\mathcal{FNHC}'$ that is
$F_n[I_{\{X,X\}}] = I_{[F_n(X),F_n(X)]_H}$

Theorem 2.6 The set of all monoid of homotopy classes of continuous homomorphism forms a category , it is denoted by $'\mathcal{FNMHC}'$.

Proof: We take all the abelian monoid are the set of object and the set of all monoid of homotopy classes of continuous homomorphisms, homotopic through continuous homomorphisms are set of morphisms and the composition is the usual composition of continuous mappings.

Theorem 2.7 Let $'\mathcal{HC}'$ be the category of homotopy classes of homomorphism and $'\mathcal{FNMHC}'$ be the category of homotopy classes of continuous homomorphisms, then there exists a contravariant (n,m) functor $\mathcal{F}_{n,m}$: $'\mathcal{HC}' \to '\mathcal{FNMHC}'$

Proof: Using the **Theorem2.1**, **Theorem 2.2** and **Theorem 2.5**, it follows

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