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Almost gζ*- Normal Spaces and ggζ*-Closed Sets

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Abstract: ***In this paper, we introduce the notion of ggζ*-closed and rggζ*-closed sets in topological spaces and investigate some of their properties. Further, utilizing ggζ*-closed and rggζ*-closed sets, we obtain characterizations and preservation theorems for gζ*-normal, almost gζ*-normal and mildly gζ*-normal spaces.***

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1. Introduction

In 1965, Njastad [**7**] introduced the concept of α-open sets in topological spaces. In 1970, Levine [**5**] initiated the study of so called generalized closed (briefly g-closed) sets in order to extend many of the most important properties of closed sets to a large family. In 1970, Singal and Arya [**9**] introduced the concept of almost normal spaces. Various properties of new classes of topological spaces have been studied and the relations of these new concepts with the concepts of almost regularity have also been investigated. In 1973, Singal and Singal [**10**] introduced the notion of mildly normal spaces in topological spaces. In 1985, Jankovic [**3**] introduced the concept of rc-continuous functions. In 2004, Nono et al. [**8**] introduced the notion of g#α-closed sets in topological spaces. In 2009, Devi et al. [**2**] introduced the concept of #gα-closed sets. In 2012, Kokilavani [**4**] introduced the notion of gζ*-closed sets in topological spaces and investigated some of their properties. In 2013, Balasubramanian [**1**] defined rg-normality, almost rg-normality and mildly rg-normality, continue the study of further properties of rg-normality and show that these three axioms are regular open hereditary.

2. Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A is said to be **α-open** [7] if A ⊂ int(cl(int(A))). The complement of a α-open set is said to be **α-closed** [1]. The intersection of all α-closed sets containing A is called **α-closure** [2] of A, and is denoted by α-cl(A)). The **α-interior** [2] of A, denoted by α-int(A), is defined as union of all α-open sets contained in A.

2.1 Definition. A subset A of a space (X, τ) is said to be
(1) **generalized closed** (briefly **g-closed**) [**5**] if cl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(2) **rg-closed** if cl(A) ⊂ U whenever A ⊂ U and U is regular open in X.
(3) **α-generalized closed** (briefly **αg-closed**) [**6**] if α-cl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(4) **generalized #α-closed** (briefly **g#α-closed**) [**8**] if αc-l(A) ⊂ U whenever A ⊂ U and U is g-open in X.
(5) **#generalized α-closed** (briefly **#gα-closed**) [**2**] if α-cl(A) ⊂ U whenever A ⊂ U and U is g#α-open in X.
(6) **rαg-closed [3]** if α-cl(A) ⊂ U whenever A ⊂ U and U is regular open in X.
(7) **gζ*-closed**[**4**] if α-cl(A) ⊂ U whenever A ⊂ U and U is #gα-open in X.
(8) **g-open** (resp**. rg-open**, **αg-open, rαg-open, gζ*-open**) if the complement of A is g-closed (resp. rg-closed, αg-closed, rαg-closed, gζ*-closed).

The intersection of all gζ*-closed sets containing A is called **gζ*-closure of A**, and is denoted by **gζ*-cl(A)**. The **gζ*-interior of A**, denoted by **gζ*-int(A)**, is defined as union of all gζ*-open sets contained in A. The family of all gζ*-closed (resp. gζ*-open) sets of a space X is denoted by gζ*-C(X) (resp. gζ*-O(X))

2.2 Definition. A subset A of a space (X, τ) is said to be
(1) **generalized gζ*-closed** (briefly **ggζ*-closed**) if gζ*-cl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(2) **regular generalized gζ*-closed** (briefly **rggζ*-closed**) if gζ*-cl(A) ⊂ U whenever A ⊂ U and U is regular open in X.

2.3 Remark. We have the following implications for the properties of subsets:

Where none of the implications is reversible as can be seen from the following examples:

closed	⇒	g-closed	⇒	rg-closed
⇓		⇓		⇓
α-closed	⇒	αg-closed	⇒	rαg-closed
⇓		⇓		⇓
gζ*-closed	⇒	ggζ*-closed	⇒	rggζ*-closed

2.4 Example. Let X = {a, b, c} and τ = {ϕ, {a}, X}. Then A = {b} is g-closed as well as αg-closed. Hence A is ggζ*-closed. But it is not closed.

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2.5 Example. Let X = {a, b, c, d} and τ = {ϕ, {c}, {d}, {a, c}, {c, d}, {a, c, d}, X}. Then A = {a} is α-closed as well as αg-closed. Hence A is gζ*-closed. But it is not closed.

2.6 Example. Let X = {a, b, c} and τ = {ϕ, {a}, {b}, {a, b}, X}. Then A = {a, b} is an rg-closed as well as rαg-closed. Hence A is rggζ*-closed. But it is not αg-closed.

2.7 Example. Let X = {a, b, c} and τ = {ϕ, {a}, X}. Then A = {a, b} is g-closed as well as αg-closed. But it is not α-closed.

2.8 Example. Let X = {a, b, c, d} and τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X}. Then A = {a, b} is rαg-closed as well as rggζ*-closed. But it is not closed.

2.9 Theorem. Union of two rggζ*-closed sets is rggζ*-closed.

Proof. Let A and B are two rggζ*-closed sets. Let A ∪ B ⊂ U and U is regular open. Hence, A ⊂ U, B ⊂ U. Also gζ*-cl(A) ⊂ U, gζ*-cl(B) ⊂ U.
gζ*-cl(A ∪ B) = gζ*-cl(A) ∪ gζ*-cl(B) ⊂ U. So, A ∪ B is rggζ*-closed.

2.10 Remark. Intersection of two rggζ*-open sets is rggζ*-open.

2.11 Theorem. If A ⊂ Y ⊂ X and A is rggζ*-closed in X, then A is rggζ*-closed relative to Y.

Proof. A ⊂ Y ⊂ X. A is rggζ*-closed in X. Let us prove A is rggζ*-closed relative to Y. Let A ⊂ Y ∩ U where U is regular open in X. A ⊂ U implies gζ*-cl(A) ⊂ U.
Y ∩ gζ*-cl(A) ⊂ Y ∩ U. So, A is rggζ*-closed relative to Y.

2.12 Theorem. If A is both regular open and rggζ*-closed in X, then A is rggζ*-closed.

Proof. A ⊂ A. A is regular open. A is rggζ*-closed. Hence gζ*-cl(A) ⊂ A. But
A ⊂ gζ*-cl(A). So A = gζ*-cl(A), Hence A is gζ*-closed.

2.13 Theorem. For x ∈ X, X – {x} is rggζ*-closed or regular open.

Proof. Let X – {x] be not regular open. X is the only regular open set containing X – {x}. So, gζ*-cl(X – {x}) ⊂ X. Hence X – {x} is rggζ*-closed.

2.14 Theorem. Let A be a rggζ*-closed set if and only if gζ*-cl(A) – A contains no non empty regular closed set.

Proof. Let F be a regular closed set such that F ⊂ gζ*-cl(A) – A. Then, A ⊂ X – F. A is rggζ*-closed and X – F is regular open. So, gζ*-cl(A) ⊂ X – F. Hence F ⊂ X – gζ*-cl(A).
F ⊂ (X – gζ*-cl(A)) ∩ (gζ*-cl(A) – A)). This implies F = ϕ.
Let gζ*-cl(A) – A contain no nonempty regular closed set. Let A ⊂ U and U is regular open. Let gζ*-cl(A) ⊄ U. So gζ*-cl(A) ∩ U^c is non empty, regular closed set of gζ*-cl(A) – A, a contradiction. Hence gζ*-cl(A) ⊂ U. So A is rggζ* -closed.

2.15 Theorem. If A is rggζ*-closed and A ⊂ B ⊂ gζ*-cl(A), then B is rggζ*-closed.

Proof: Let B ⊂ U, where U is regular open. Then A ⊂ B implies A ⊂ U. Since A is
rggζ*-closed. gζ*-cl(A) ⊂ U. B ⊂ gζ*-cl(A). Hence, gζ*-cl(B) ⊂ gζ*-cl(A) ⊂ U. So, B is rggζ*-closed.

2.16 Remark. gζ*-cl(X – A) = X – gζ*-int(A).

2.17 Theorem. A ⊂ X is rggζ*-open if and only if F ⊂ gζ*-int (A) whenever F is regular closed and F ⊂ A.

Proof. Let A be rggζ*-open. Let F be regular closed and F ⊂ A. Then X – A ⊂ X – F.
X – F is regular open. X – A is rggζ*-closed. gζ*-cl(X – A) ⊂ X – F. X – gζ*-int(A) ⊂ X – F. So, F ⊂ gζ*-int(A). Let F be regular closed and F ⊂ A imply F ⊂ gζ*-int(A). Let X – A ⊂ U, where U is regular open. X – U ⊂ A, where X – U is regular closed. Hence X – U ⊂ gζ*-int(A). So, X – gζ*-int(A) ⊂ U. That is gζ*-cl(X – A) ⊂ U. Hence X – A is rggζ*-closed. This implies A is rggζ*-open.

2.18 Theorem. If gζ*-int(A) ⊂ B ⊂ A and A is rggζ*-open, B is rggζ*-open.

Proof. X – A ⊂ X – B ⊂ X – gζ*-int(A). That is, X – A ⊂ X – B ⊂ gζ*-cl(X – A). X – A is rggζ*-closed. By **Theorem 2.15**, X – B is rggζ*-closed. B is rggζ*-open.

2.19 Remark. gζ*-int(gζ*-cl(A) – A)) = ϕ.

2.20 Theorem. If A ⊂ X is rggζ*-closed, gζ*-cl(A) – A is rggζ*-open.

Proof. A is rggζ*-closed. Let F be a regular closed set such that F ⊂ gζ*-cl(A) – A. Then by **Theorem 2.14**, F = ϕ. So F ⊂ gζ*-int (gζ*-cl(A) – A)). By **Theorem 2.17,** gζ*cl(A) – A is rggζ*-open.

3. Almost gζ*-normal

3.1 Definition. A topological space X is said to be **gζ*-normal** (resp. **rg-normal** [**23**]) if for every pair of disjoint closed subsets H, K of X, there exist disjoint gζ*-open (resp. rg-open) sets U, V of X such that H ⊂ U and K ⊂ V.

3.2 Definition. A topological space X is said to be **almost gζ*-normal** (resp. **almost normal** [**20**], **almost rg-normal** [**10**]) if for any two disjoint closed subsets A and B of X, one of which is regular closed, there exist disjoint gζ*-open (resp. open, rg-open) sets U and V of X such that A ⊂ U and B ⊂ V.

3.3 Definition. A topological space X is said to be **mildly gζ*-normal** (resp. **mildly normal** [**19**, **22**], **mildly rg-normal** [**10**]) if for every pair of disjoint regular closed subsets H, K, there exist disjoint gζ*-open (resp. open, rg-open) sets U, V of X such that H ⊂ U and K ⊂ V.

3.4 Example. Let X = {a, b, c,} and τ = {ϕ, {a}, {b, c}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {b, c}. Also U = {a} and V = {b, c} are open sets such that A ⊂ U and B ⊂ V. Hence X is normal as well as almost normal.

3.5 Example. Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, X}. Then X is rg-normal.

3.6 Example. Let X = {a, b, c} and τ = {ϕ, {a}, {a, b}, {a, c}, X}. Then X is almost normal but it is not normal.
By the definitions and examples stated above, we have the following diagram:

rg-normal	$\Rightarrow$	almost rg-normal	$\Rightarrow$	mildly rg-normal
$\Uparrow$		$\Uparrow$		$\Uparrow$
normal	$\Rightarrow$	almost normal	$\Rightarrow$	mildly normal
$\Downarrow$		$\Downarrow$		$\Downarrow$
$g\zeta^*$-normal	$\Rightarrow$	almost $g\zeta^*$-normal	$\Rightarrow$	mildly $g\zeta^*$-normal.

Where none of the implications is reversible as can be seen from the above examples:

3.7 Theorem. For a topological space X, the following are equivalent:

a) X is almost $g\zeta^*$-normal.
b) For every closed set A and every regularly open set B containing A, there is a $g\zeta^*$-open set U such that $A \subset U \subset g\zeta^*\text{-cl}(U) \subset B$.
c) For every regularly closed set A and every open set B containing A, there is a $g\zeta^*$-open set U such that $A \subset U \subset g\zeta^*\text{-cl}(U) \subset B$.
d) For every pair consisting of disjoint sets A and B, one of which is closed and other is regularly closed, there exist $g\zeta^*$-open sets U and V such that $A \subset U$, $B \subset V$ and $g\zeta^*\text{-cl}(U) \cap g\zeta^*\text{-cl}(V) = \phi$.

Proof. (a) $\Rightarrow$ (b). Let A be a closed set and let B be a regularly open set containing A. Thus $A \cap (X - B) = \phi$, where A is closed and X – B is regularly closed. Therefore, there exist $g\zeta^*$-open sets U and V such that $A \subset U$, $X - B \subset V$ and $U \cap V = \phi$. Thus, $A \subset U \subset X - V \subset B$. Now, X – V is closed. Therefore, $A \subset U \subset g\zeta^*\text{-cl}(U) \subset B$.
(b) $\Rightarrow$ (c). Let A be regularly closed set and let B be an open set containing A. Then,
$X - B \subset X - A$, whence X – A is a regularly open set containing the closed set X – B. Therefore, there is a $g\zeta^*$-open set M such that $X - B \subset M \subset g\zeta^*\text{cl}(M) \subset X - A$. Thus, $A \subset X - g\zeta^*\text{-cl}(M) \subset X - M \subset B$. Let $X - g\zeta^*\text{-cl}(M) = U$. Then U is $g\zeta^*$-open and
$A \subset U \subset g\zeta^*\text{-cl}(U) \subset B$.
(c) $\Rightarrow$ (d). Let A be a regularly closed set and B be a closed set such that $A \cap B = \phi$. Then, $A \subset X - B$ which is open. Therefore, there exists a $g\zeta^*$-open set M such that
$A \subset M \subset g\zeta^*\text{-cl}(M) \subset X - B$. Again, M is a $g\zeta^*$-open set containing the regularly closed set A. Therefore, there is a $g\zeta^*$-open set U such that $A \subset U \subset g\zeta^*\text{-cl}(U) \subset M$. Let
$X - g\zeta^*\text{-cl}(M) = V$. Then, $A \subset U$, $B \subset V$ and $g\zeta^*\text{-}(U) \cap g\zeta^*\text{-}(V) = \phi$.
(d) $\Rightarrow$ (a) is obvious.

3.8 Theorem. For a topological space X, the Following are equivalent:
(a) X is almost $g\zeta^*$-normal.
(b) For every closed set A and every regularly closed set B, there exist disjoint $gg\zeta^*$-open sets U and V such that $A \subset U$ and $B \subset V$.
(c) For every closed set A and every regularly closed set B, there exist disjoint $rgg\zeta^*$-open sets U and V such that $A \subset U$ and $B \subset V$.
(d) For every closed set A and every regularly open set B containing A, there exists a $gg\zeta^*$-open set U of X such that $A \subset U \subset g\zeta^*\text{-cl}(U) \subset B$.
(e) For every closed set A and every regularly open set B containing A, there exists a $rgg\zeta^*$-open set U of X such that $A \subset U \subset g\zeta^*\text{-cl}(U) \subset B$.
(f) For every pair of disjoint sets A and B, one of which is closed and other is regularly closed, there exist $g\zeta^*$-open sets U and V such that $A \subset U$ and $B \subset V$ and $U \cap V = \phi$.

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (d) $\Rightarrow$ (e), (c) $\Rightarrow$ (d), (e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (a).
(a) $\Rightarrow$ (b). Let X be an almost $g\zeta^*$-normal. Let A be a closed and B be a regularly closed sets in X. by assumption, there exist disjoint $g\zeta^*$-open sets U and V such that $A \subset U$ and $B \subset V$. Since every $g\zeta^*$-open set is $gg\zeta^*$-open set, U, V are $gg\zeta^*$-open sets such that $A \subset U$ and $B \subset V$.
(b) $\Rightarrow$ (c). Let A be a closed and B be a regularly closed sets in X. By assumption, there exist disjoint $gg\zeta^*$-open sets U and V such that $A \subset U$ and $B \subset V$. Since every $gg\zeta^*$-open set is $rgg\zeta^*$-open set, U, V are $rgg\zeta^*$-open sets such that $A \subset U$ and $B \subset V$.
(d) $\Rightarrow$ (e). Let A be any closed set and B be any regularly open set containing A. By assumption, there exists a $gg\zeta^*$-open set U of X such that $A \subset U \subset g\zeta^*\text{-cl}(U) \subset B$. Since every $gg\zeta^*$-open set is $rgg\zeta^*$-open set, there exists a $rgg\zeta^*$-open set U of X such that
$A \subset U \subset g\zeta^*\text{-cl}(U) \subset B$.
(c) $\Rightarrow$ (d). Let A be any closed set and B be a regularly open set containing A. By assumption, there exist disjoint $rgg\zeta^*$-open sets U and W such that $A \subset U$ and $X - B \subset W$. By **Theorem 2.17,** we get, $X - B \subset g\zeta^*\text{-int}(W)$ and $g\zeta^*\text{-cl}(U) \cap g\zeta^*\text{-int}(W) = \phi$. Hence,
$A \subset U \subset g\zeta^*\text{-cl}(U) \subset X - g\zeta^*\text{-int}(W) \subset B$.
(e) $\Rightarrow$ (f). For any closed set A and any regularly open set B containing A. Then
$A \subset X - B$ and X – B is a regularly closed. By assumption, there exists a $rgg\zeta^*$-open set G of X such that $A \subset G \subset g\zeta^*\text{-cl}(G) \subset X - B$. Put $U = g\zeta^*\text{-int}(G)$, $V = X - g\zeta^*\text{-cl}(G)$. Then U and V are disjoint $g\zeta^*$-open sets of X such that $A \subset U$ and $B \subset V$.
(f) $\Rightarrow$ (a) is obvious.

3.9 Definition. A function $f : X \to Y$ is called **rc-continuous** [3] if for each regular closed set F in Y, $f^{-1}(F)$ is regularly closed in X.

3.10 Definition. A function $f : X \to Y$ is called **M-$g\zeta^*$-open** (resp. **M-$g\zeta^*$-closed**) if $f(U) \in g\zeta^*\text{-O}(Y)$ ((resp. $f(U) \in g\zeta^*\text{-C}(Y)$) for each $U \in g\zeta^*\text{-O}(X)$ (resp. $U \in g\zeta^*\text{-C}(X)$).

3.11 Definition. A function $f : X \to Y$ is called **almost $g\zeta^*$-irresolute** if for each $x \in X$ and each $g\zeta^*$ - neighbourhood V of $f(x)$, $g\zeta^*\text{-cl}(f^{-1}(V))$ is a $g\xi^*$-neighbourhood of x.

3.12 Definition. A topological space X is called **weakly gζ*-regular** if for each point x and a regularly open set U containing {x}, there is a gζ*-open set V such that
$x \in V \subset gζ^*\text{-cl}(V) \subset V$.

3.13 Remark. One can prove that almost gζ*-normality is also regularly closed hereditary.

3.14 Definition. A topological space X is called **almost gζ*-regular** if for every regularly closed set F and each point $x \notin F$, there exist disjoint gζ*-open sets U and V such that
$x \in V$ and $F \subset V$.

3.15 Remark. Almost gζ*-normality does not imply almost gζ*-regular.
Next, we prove the invariant of almost gζ*-normality in the following.

3.16 Theorem. If $f : X \to Y$ is continuous M- gζ*-open rc-continuous and almost
gζ*-irresolute surjection from an almost gζ*-normal space X onto a space Y, then Y is almost gζ*-normal.
Proof. Let A be a closed set and B be a regularly open set containing A. Then by rc-continuity of f, $f^{-1}(A)$ is a closed set contained in the regularly open set $f^{-1}(B)$. Since X is almost gζ*-normal, there exist a gζ*-open set V in X such that $f^{-1}(A) \subset V \subset gζ^*cl(V) \subset f^{-1}(B)$ by **Theorem 3.8**.Then, $f(f^{-1}(A)) \subset f(V) \subset f(gζ^*\text{-cl}(V)) \subset f(f^{-1}(B))$. Since f is M-gζ*-open and almost gζ*-irresolute surjection, we obtain $A \subset f(V) \subset gζ^*\text{-cl}(f(V)) \subset B$. Then again by **Theorem 3.8**, Y is almost gζ*-normal.

3.17 Theorem. If $f : X \to Y$ is rc-continuous M- gζ*-closed map from an almost gζ*-normal space X onto a space Y, then Y is almost gζ*-normal.
Proof. Easy to verify.

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