

# Common Fixed Points of Four Mapping in Metric Spaces

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**Abstract:** In this paper, we prove some common fixed point theorems for four mappings satisfying fractional inequalities using common limit property in metric spaces. Our results generalize and improve upon, among several results of fixed point arena including the results of Fisher [5], Jungck [7] and Badshah et al. [2], Lohani and Badshah [11].

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## 1. Introduction

Jungck [6] generalized Banach contraction principle [3] for a pair of commuting mappings. Afterward, study of common fixed points of mappings satisfying some contractive type condition has been center of vigorous research activity and a number of interesting results have been obtained using commutativity and its weaker forms such as weak commutativity [18], compatibility [7], R-weak commutativity [12], semi-compatibility [4], compatibility of type (A) [8], compatibility of type (B) [16], compatible mappings of type (T) [17], biased maps [9] and weak compatibility [10] etc. Pant [12, 13, 14, 15] studied fixed point results for the class of non-compatible mappings.

On the other hand Amari and Moutawakil [1] introduced the notion of property (E.A) which contains the classes of compatible as well non-compatible mappings. Sintunavarat and Kumam [19] defined the notion of (CLR<sub>g</sub>) property. It has been noticed that (CLR<sub>g</sub>) property never requires completeness (or closedness) of subspaces (also see [20, 21]).

Recently, Badshah et al. [2] proved common fixed point theorem by using a fractional inequality and compatible mappings instead of commuting mappings. In this paper, we prove results of Badshah et al. [2] Using (CLR<sub>g</sub>) property and (E. A) property. Our results generalize and improve upon, among several results of fixed point arena including the results of Fisher [5], Jungck [7] and Badshah et al. [2], Lohani and Badshah [11].

## 2. Preliminaries

Sessa [18] introduced the notion of weak commutativity:

**Definition 2.1.** [18] Two self-mappings  $S$  and  $T$  of a metric space  $(X, d)$  are said to be weakly commuting if

$$d(STx, TSx) \leq d(Sx, Tx), \text{ for all } x \in X,$$

It is clear that two commuting mappings are weakly

commuting but the converse is not true as is shown in [18].

**Definition 2.2.** [7] Two self-mappings  $S$  and  $T$  of a metric space  $(X, d)$  are said to be compatible if

$$\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0,$$

whenever  $\{x_n\}$  is a sequence in  $X$  such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t,$$

for some  $t \in X$ .

Obviously, two weakly commuting mappings are compatible, but the converse is not true as shown in [7].

**Definition 2.3.** [10] Two self-mappings  $S$  and  $T$  of a metric space  $(X, d)$  are said to be weakly compatible if they commute at their coincidence points, i.e. if  $Su = Tu$  for some  $u \in X$ , then  $STu = T Su$ . It is easy to see that two compatible mappings are weakly compatible.

**Definition 2.4** [1] Two self-mappings  $S$  and  $T$  of a metric space  $(X, d)$  are said to satisfy the property (E.A) if there exists a sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t,$$

for some  $t \in X$ .

**Definition 2.5** [19] Two self-mappings  $S$  and  $T$  of a metric space  $(X, d)$  are said to satisfy the common limit in the range of  $T$  property if there exists a sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = Tu,$$

for some  $u \in X$ .

In what follows, the common limit in the range of  $g$  property will be denoted by the (CLRT) property.

Now, we give examples of mappings  $f$  and  $g$  which satisfy the (CLRT) property.

**Example 2.6.** Let  $X = [0, \infty)$  with the usual metric on  $X$ . Define  $S, T: X \rightarrow X$  by

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$Sx = x/2$  and  $Tx = 2x$  for all  $x \in X$ . Consider the sequence  $\{x_n\} = \{1/n\}$ . Since  $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = 0 = T0$ , Therefore  $S$  and  $T$  satisfy the (C LRT) property.

**Example 2.7.** Let  $X = [0, \infty)$  with the usual metric on  $X$ . Define  $S, T: X \rightarrow X$  by  $Sx = x + 2$  and  $Tx = 3x$  for all  $x \in X$ . Consider the sequence  $\{x_n\} = \{1 + 1/n\}$ . Since  $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = 3 = T1$ , Therefore  $S$  and  $T$  satisfy the (C LRT) property.

**Remark 2.8.** It is clear from the Jungck's definition [6] that two self-mappings  $S$  and  $T$  of a metric space  $(X, d)$  will be non-compatible if there exists atleast one sequence  $\{x_n\}$  in  $X$  such that

$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ , for some  $t \in X$ , but  $\lim_{n \rightarrow \infty} d(STx_n, TSx_n)$  is either non-zero or non-existent.

Thus, two non-compatible self-mappings of a metric space  $(X, d)$  satisfy the property (E.A).

### 3. Main Results

**Lemma 3.1.** Let  $A, B, S$  and  $T$  be self-mappings from a metric space  $(X, d)$  into itself satisfying following conditions:

- (3.1) The pairs  $\{A, S\}$  and  $\{B, T\}$  are weakly compatible;  
 (3.2)

$$d(Ax, By) \leq a \frac{[d(Sx, Ax)]^3 + [d(Ty, By)]^3}{[d(Sx, Ax)]^2 + [d(Ty, By)]^2} + bd(Sx, Ty)$$

for all  $x, y \in X$  where  $a, b \geq 0$  and  $a + b < 1$ . If there exists  $u, v \in X$  such that  $Au = Su = Bv = Tv = t$  for some  $t$  in  $X$  then  $t$  is the unique fixed point of  $A, B, S$  and  $T$ .

**Proof:** Since  $\{A, S\}$  is weakly compatible and  $Au = Su = t$ , we have  $At = ASu = SAu = St$ . We claim that  $At = t$ , if not then using (3.2), we have

$$\begin{aligned} d(At, t) &= d(At, Bv) \leq a \frac{[d(St, At)]^3 + [d(Tv, Bv)]^3}{[d(St, At)]^2 + [d(Tv, Bv)]^2} + bd(St, Tv) \\ &\leq a[d(St, At) + d(Tv, Bv)] + bd(At, t) \\ &\leq bd(At, t), \end{aligned}$$

which is a contradiction. Hence  $At = t$ . Thus we have  $At = St = t$ . Similarly we can prove that  $Bt = Tt = t$ . Hence  $t$  is common fixed point of mappings  $A, B, S$  and  $T$ .

If possible suppose that  $t$  and  $z$  are two distinct common fixed points of  $A, B, S$  and  $T$ , then using (3.2), we have

$$\begin{aligned} d(t, z) &= d(At, Bz) \leq a \frac{[d(St, At)]^3 + [d(Tz, Bz)]^3}{[d(St, At)]^2 + [d(Tz, Bz)]^2} + bd(St, Tz) \\ &\leq a[d(St, At) + d(Tz, Bz)] + bd(St, Tz) \\ &\leq a[d(t, t) + d(z, z)] + bd(t, z) \\ &\text{i.e. } d(t, z) \leq bd(t, z), \end{aligned}$$

which is a contradiction, hence  $t = z$ . Therefore  $t$  is unique common fixed point of  $A, B, S$  and  $T$ .

**Theorem 3.2.** Let  $A, B, S$  and  $T$  be self-mappings from a metric space  $(X, d)$  into itself satisfying (3.1), (3.2) and following conditions:

- (3.3)  $A(X) \subseteq T(X)$  and  $B(X) \subseteq S(X)$ ,  
 (3.4) One of the pairs  $(A, S)$  or  $(B, T)$  satisfying property (E.A.)  
 (3.5) One of the  $A(X), B(X), S(X)$  and  $T(X)$  is closed subspace of  $X$ .

Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$ .

**Proof:** Suppose that the pair  $(B, T)$  satisfies property (E.A), then there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = \text{for some } t \in X$

Further, since  $B(X) \subseteq S(X)$ , there exists a sequence  $\{y_n\}$  in  $X$  such that  $Bx_n = Sy_n$ . Hence  $\lim_{n \rightarrow \infty} Sy_n = t$ .

Now we claim that  $\lim_{n \rightarrow \infty} Ay_n = t$ . If possible suppose that  $\lim_{n \rightarrow \infty} Ay_n = t_1 \neq t$ , then putting  $x = y_n, y = x_n$  in (3.2) we have.

$$\begin{aligned} d(Ay_n, Bx_n) &\leq a \frac{[d(Sy_n, Ay_n)]^3 + [d(Tx_n, Bx_n)]^3}{[d(Sy_n, Ay_n)]^2 + [d(Tx_n, Bx_n)]^2} + bd(Sy_n, Tx_n) \\ &\leq a[d(Sy_n, Ay_n) + d(Tx_n, Bx_n)] + bd(Sy_n, Tx_n). \end{aligned}$$

Taking limit as  $n \rightarrow \infty$ , we get

$$d(t_1, t) \leq ad(t, t_1)$$

or

$$(1 - a)d(t, t_1) \leq 0$$

as  $0 \leq a < 1$ , we have  $d(t, t_1) = 0$ . Hence  $t = t_1$ , thus we have  $\lim_{n \rightarrow \infty} Ay_n = t$ .

Now suppose that  $S(X)$  is closed subspace of  $X$  then there exists  $u \in X$  such that

$$\lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = t = Su$$

Next we shall claim that  $Au = Su$ . Taking  $x = u, y = x_n$  in (3.2), we get

$$\begin{aligned} d(Au, Bx_n) &\leq a \frac{[d(Su, Au)]^3 + [d(Tx_n, Bx_n)]^3}{[d(Su, Au)]^2 + [d(Tx_n, Bx_n)]^2} + bd(Su, Tx_n) \\ &\leq a[d(Su, Au) + d(Tx_n, Bx_n)] + bd(Su, Tx_n). \end{aligned}$$

Taking limit as  $n \rightarrow \infty$ , we have

$$d(Au, t) \leq ad(t, Au),$$

which is a contradiction. Hence we have  $Au = t$ . Thus  $Au = Su = t$ .

Further, since  $A(X) \subseteq T(X)$ , there exists  $v$  in  $X$  such that  $Au = Tv$ . Thus we have  $Au = Su = Tv = t$ . Now, we show that  $Bv = Tv$ . For taking  $x = u, y = v$  in (3.2), we have

$$\begin{aligned} d(Au, Bv) &\leq a \frac{[d(Su, Au)]^3 + [d(Tv, Bv)]^3}{[d(Su, Au)]^2 + [d(Tv, Bv)]^2} + bd(Su, Tv) \\ &\leq a[d(Su, Au) + d(Tv, Bv)] + bd(Su, Tv) \\ &\quad d(t, Bv) \leq ad(t, Bv), \end{aligned}$$

which is a contradiction, therefore  $t = Bv$ . Hence  $Bv = Tv = t$ . Thus we have

$$Au = Su = Bv = Tv = t.$$

A similar argument works if we assume  $T(X)$  to be closed.

On the other hand if  $A(X)$  is closed, there exists  $u'$  in  $X$  such that  $Au' = t$  since

$A(x) \subseteq T(x)$ , there exists  $v'$  in  $X$  such that  $Tv' = t$ . Using (3.2) with  $x = x_n, y = v'$ , we have

$$d(Ax_n, Bv') \leq a \frac{[d(Sx_n, Ax_n)]^3 + [d(Tv', Bv')]^3}{[d(Sx_n, Ax_n)]^2 + [d(Tv', Bv')]^2} + bd(Sx_n, Tv')$$

or

$$d(Ax_n, Bv') \leq a[d(Sx_n, Ax_n) + d(Tv', Bv')] + bd(Sx_n, Tv')$$

Taking limit as  $n \rightarrow \infty$

$$d(t, Bv') \leq ad(t, Bv')$$

Which is a contraction therefore  $Bv' = t$ . Since  $B(x) \subseteq S(x)$ ,

(3.6) The pair  $\{A, T\}$  is weakly compatible;

$$(3.7) \quad d(Ax, Ay) \leq a \frac{[d(Tx, Ax)]^3 + [d(Ty, Ay)]^3}{[d(Tx, Ax)]^2 + [d(Ty, Ay)]^2} + bd(Tx, Ty)$$

for all  $x, y \in X$ , where  $a, b > 0$  and  $a + b < 1$

(3.8)  $A(X) \subseteq T(X)$

(3.9) The pair  $(A, T)$  satisfy properly (E.A.)

(3.10)  $A(X)$  or  $T(X)$  is closed subspace of  $X$ .

Then  $A$  and  $T$  have a unique common fixed point in  $X$ .

**Corollary 3.4.** Let  $A, B, S$  and  $T$  be self-mappings from a metric space  $(X, d)$  into itself satisfying (3.1), (3.2), (3.3), (3.5) and if either the pair  $(A, S)$  or  $(B, T)$  is non-compatible. Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$ .

**Theorem 3.5.** Let  $(X, d)$  be a metric space  $A, B, S, T : X \rightarrow X$  be self-mappings satisfying (3.1), (3.2), (3.3) and either the pair  $(A, S)$  satisfies  $(CLR_A)$  property or the pair  $(B, T)$  satisfies  $(CLR_B)$  property. Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$ .

**Proof:** Suppose the pair  $(B, T)$  satisfies  $(CLR_B)$

$$d(Au, Bv) \leq a \frac{[d(Su, Au)]^3 + [d(Tv, Bv)]^3}{[d(Su, Au)]^2 + [d(Tv, Bv)]^2} + bd(Su, Tv)$$

$$\leq a[d(Su, Au) + d(Tv, Bv)] + bd(Su, Tv)$$

$$d(Tv, Bv) \leq ad(Tv, Bv),$$

which is a contradiction, therefore  $Bu = Tv$ . Thus we have  $Au = Su = Bv = tv = t$ . Hence from Lemma (3.1),  $A, B, S$  and  $T$  have a unique common fixed point  $t$  in  $X$ .

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there exists  $u''$  in  $X$ , such that  $Su'' = t$  using 3.2 we can easily show that  $Au'' = Su'' = Bv' = Tv' = t$

A similarly argument can be produced if  $B(X)$  is closed. Also if the pair  $(A, S)$  satisfies (E.A) property we will get similar result. Now appealing to Lemma (3.1) in all cases, we conclude that  $A, B, S$  and  $T$  have a unique common fixed point  $t$  in  $X$ .

**Corollary 3.3.** Let  $(X, d)$  be a metric space and  $A, T: X \rightarrow X$  be two self mappings satisfying the following conditions:

property, then there exists a sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = Bx \text{ for some } x \in X$$

Since  $B(X) \subseteq S(X)$ , there exists  $u$  in  $X$  Such that  $Bx = Su$ . We claim that  $Au = Su = t$ . For this using (3.2) with  $x=u, y=x_n$ , we have

$$d(Au, Bx_n) \leq a \frac{[d(Su, Au)]^3 + [d(Tx_n, Bx_n)]^3}{[d(Su, Au)]^2 + [d(Tx_n, Bx_n)]^2} + bd(Su, Tx_n) \\ \leq a[d(Su, Au) + d(Tx_n, Bx_n)] + bd(Su, Tx_n).$$

Letting  $n \rightarrow \infty$ , we have

$$d(Au, Su) \leq ad(Su, Au),$$

which is a contradiction, hence  $Au = Su$ . Thus we have  $Au = Su = Bx = t$ . Further, since  $A(X) \subseteq T(X)$ , there exists  $v$  in  $X$  such that  $Au = Tv$ . Now we show that  $Bv = Tv$ . Using (3.2) with  $x = u, y = v$ , we have

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