On Sandwich Theorem for Certain Subclasses of Symmetric Analytic Functions Associated with Noor Integral Operator

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Abstract: In this paper, we obtain some interesting properties of differential subordination and superordination for the classes of symmetric analytic functions in the unit disk, by applying Noor integral operator. We investigate several sandwich theorems on basis of this theory.

Keywords: Convex functions, Differential subordination and superordination, Noor integral operator, Best dominant

1. Introduction

Let H(U) denote the class of analytic functions in the open unit disk $U = \{z : |z| < 1\}$ and let H[a,1] denote the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = a + a_1 z + a_2 z^2 + \dots (a \in \mathbb{C})$$

Also, let A be the class of functions $f \in H(U)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

For two functions f(z) given by (1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z'$$

The Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Let $f, g \in H(U)$, we say that the function f is subordinate to g, if there exist a Schwarz function w, analytic in U, with w(0) = 0 and $|w(z)| < 1(z \in U)$, such that f(z) = g(w(z)) for all $z \in U$.

This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$. It is well known that, if the function g is univalent in U, then $f(z) \prec g(z)$ if and only if f(0) = g(0) and $f(U) \subset g(U)$.

Let
$$p(z), h(z) \in H(U)$$
, and let
 $\Phi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If $p(z)$ and
 $\Phi(z, t) = \frac{1}{2} H(z) = \frac{1}{2} H(z)$

 $\Phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent functions, and if p(z) satisfies the second-order superordination

 $h(z) \prec \Phi(p(z), zp'(z), z^2 p''(z); z)$ (2)

then p(z) is called to be a solution of the differential superordination (2). (If f(z) is subordinatnate to g(z), then g(z) is called to be superordinate to f(z)). An analytic function q(z) is called a subordinant if $q(z) \prec p(z)$ for all p(z) satisfies (2). An univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants q(z) of (2) is said to be the best subordinant.

Recently, Miller and Mocanu [13] obtained conditions on h(z), q(z) and Φ for which the following implication holds true:

$$h(z) \prec \Phi(p(z), zp'(z), z^2 p''(z); z) \Longrightarrow q(z) \prec p(z)$$

Using these results, the authors in [3] considered certain classes of first- order differential superordinations, see also [7], as well as superordination-preserving integral operators [6]. Aouf et al. [3, 4], obtained sufficient conditions for certain normalized analytic functions f(z) to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in U with $q_1(0) = 1$ and $q_2(0) = 1$.

In [18], Sakaguchi defined the class of starlike functions with respect to symmetrical points as follows:

Let $f \in A$. Then f is said to be starlike with respect to symmetrical points in U if, and only if,

$$\mathsf{R}\frac{zf'(z)}{f(z)-f(-z)} > 0, \quad (z \in \mathsf{U}).$$

Obviously, it forms a subclass of close-to-convex functions and hence univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see [18].

Let A denote by $D^{\alpha} : A \to A$ the operator defined by

$$D^{\alpha}f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z) \qquad (\alpha > -1)$$

or equiavalently,

$$D^{k}f(z) = \frac{z(z^{k-1}f(z))^{(k)}}{k!} \qquad k \in \mathsf{N}_{0} = 0, 1, 2, \dots$$

where the symbol (*) stands for the Hadamard product (or Convolution). We note that $D_0 f(z) = f(z)$ and $D^{1}f(z) = zf'(z)$. The operator $D^{k}f$ is called the Ruscheweyh derivative of kth order of f, see [17]. Analogous to $D^k f$, Noor [14] and Noor et al. [15] defined an integral operator $I_k : A \to A$ as follows.

Let
$$f_k(z) = \frac{z}{(1-z)^{k+1}}, k \in \mathbb{N}_0$$
, and let $f_k^{(\tau)}$ be defined
such that

such tha

$$f_k(z) * f_k^{(\tau)}(z) = \frac{z}{(1-z)^2}$$

Then

$$I_k f(z) = f_k(z) * f_k^{(r)}(z) = \left[\frac{z}{(1-z)^{k+1}}\right]^r * f(z).$$
(3)

From (3) it is easy to verify that

$$z(I_{k+1}f(z))' = (k+1)I_kf(z) - kI_{k+1}f(z).$$
 (4)

We note that $I_0 f(z) = z f'(z)$ and $I_1 f(z) = f(z)$. The operator $I_k f(z)$ defined by (3) is called the Noor Integral operator of kth order of f, see [8]. Moreover, Liu [8] introduced some new subclasses of strongly starlike functions defined by using the Noor integral operator and studied their properties. Liu and Noor [9] investigated some interesting properties of the Noor integral operator.

Definition 1.1 A function $f \in A$ is said to be in the class $\mathsf{B}^{\lambda,\mu}(A,B)$, if it satisfies the following subordination condition:

$$\left(1-\lambda\right)\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} + \lambda\frac{z\{I_{k}f(z)-I_{k}f(-z)\}}{I_{k+1}f(z)-I_{k+1}f(-z)}\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} \prec \frac{1+Az}{1+Bz}$$
(5)

then

where and throughout this paper unless otherwise mention the parameters λ, μ , A and B are constrained as follows:

 $\lambda \in \mathbb{C}: \mathbb{R}(\mu) > 0: -1 \le B \le 1, A \ne B, A \in \mathbb{R},$ and all powers are understood as principal values.

In this paper, we prove such results as subordination and superordination properties, convolution properties, distortion theorems, and inequality properties of the class $\mathsf{B}^{\lambda,\mu}(A,B)$. For interested readers see the work done by the authors [1, 5].

2. Preliminary Results

Definition 2.1 Let Q be the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \xi \in \partial \mathbf{U} : \lim_{z \to \xi} f(z) = \infty \right\},\$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$.

To establish our main results we need the following Lemmas.

Lemma 2.1 (Miller and Mocanu [12, 13]). Let the function h(z) be analytic and convex (univalent) in U with h(0) = 1. Suppose also that the function $\Phi(z)$ given b

$$\Phi(z) = 1 + c_1 z + c_2 z^2 + \dots$$
 (6)

is analytic in **U**,

$$\Phi(z) + \frac{z\Phi'(z)}{\gamma} \prec h(z) \quad (z \in \mathsf{U}, \mathsf{R}\gamma \ge 0; \gamma \ne 0), \quad (7)$$

$$\Phi(z) \prec \Psi(z) = \frac{\gamma}{z^{\gamma}} \int_0^z t^{\gamma-1} h(t) dt \prec h(z),$$

 $(z \in U)$, and $\Psi(z)$ is the best dominant of (1).

Lemma 2.2 (Shanmugam et al. [19]). Let $\sigma \in \mathbf{C}, \eta \in \mathbf{C}^* = \mathbf{C} \setminus 0$ and let q be a convex univalent function in U with

$$\mathsf{R}\left(1+\frac{zq''(z)}{q'(z)}\right) > max\left\{0;-\mathsf{R}\left(\frac{\sigma}{\eta}\right)\right\}, \quad (z \in \mathsf{U}),$$

If p is analytic in **U** and

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z)$$
(8)
$$n(z) \prec q(z) \text{ and } a \text{ is the best dominant of } (2)$$

then $p(z) \prec q(z)$, and q is the best dominant of (3).

Lemma 2.3 ([13]). let q(z) be a convex univalent function **U** and let $m \in \mathbf{C}, m > 0$. Further assume that in $\mathsf{R}m > 0$. If $g(z) \in \mathsf{H}[q(0), 1] \cap Q$, and

$$g(z) + mzq'(z) \prec g(z) + mzg'(z),$$

implies $q(z) \prec g(z)$, and q(z) is the best subordinant.

Lemma 2.4 ([10]). let F be a analytic and convex in \bigcup . If $f,g \in A$ and $f,g \prec F$ Then

$$\lambda f + (1 - \lambda)g \prec F, \quad (0 \le \lambda \le 1).$$

Lemma 2.5 ([16]). let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ be analytic

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www.ijsr.net Licensed Under Creative Commons Attribution CC BY and convex in \bigcup and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be analytic and convex in \bigcup . If $f(z) \prec g(z)$, then $|a_n| < |b_1|$, $(n \in \mathbb{N})$.

3. Main Results

Theorem 3.1 Let $f(z) \in \mathsf{B}^{\lambda,\mu}(A, B)$ with $\mathsf{R}\lambda > 0$. Then

$$\left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu} \prec \Psi(z) = \frac{\mu(k+1)}{\lambda} \int_{0}^{1} \frac{1 + Azu}{1 + Bzu} u^{\frac{\mu(k+1)}{\lambda} - 1} du$$
$$\prec \frac{1 + Az}{1 + Bz} \tag{9}$$

and $\Psi(z)$ is the best dominant. *Proof.* Set

$$\left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu} = h(z), \qquad (z \in \mathsf{U}).$$
(10)

Then h(z) is analytic in **U** with h(0) = 1. Logarithmic differentiation of (5) and simple computations yield

$$(1-\lambda)\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} + \lambda \frac{z\{I_kf(z)-I_kf(-z)\}}{I_{k+1}f(z)-I_{k+1}f(-z)}\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu}$$
 sat

$$=h(z)+\frac{\lambda}{\mu(k+1)}zh'(z)\prec\frac{1+Az}{1+Bz}.$$
(11)

Applying Lemma 2.2 to (11) with $\gamma = \frac{\mu(n+1)}{\lambda}$, we have

$$\begin{bmatrix} \underline{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \end{bmatrix}^{\mu} \prec \Psi(z)$$

$$= \frac{\mu(k+1)}{\lambda} z^{-\frac{\mu(k+1)}{\lambda}} \int_{0}^{z} \frac{1+At}{1+Bt} t^{\frac{\mu(k+1)}{\lambda}-1} dt$$

$$= \frac{\mu(k+1)}{\lambda} \int_{0}^{1} \frac{1+Azu}{1+Bzu} u^{\frac{\mu(n+1)}{\lambda}-1} du \prec \frac{1+Az}{1+Bz}, \quad (12)$$

and $\Psi(z)$ is the best dominant. This completes the proof.

Theorem 3.2 Let q(z) be univalent in $\bigcup, \lambda \in \mathbb{C}^*$. Suppose also that q(z) satisfies the following inequality:

$$\mathsf{R}\left(1+\frac{zq^{''}(z)}{q'(z)}\right) > max\left\{0; -(k+1)\mathsf{R}\left(\frac{\mu}{\lambda}\right)\right\}. (13)$$

If $f \in A$ satisfies the following subordination: $(z) - I_{k+1} f(-z)]^{\mu}$

$$(1-\lambda)\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} + \lambda \frac{z\{I_{k}f(z)-I_{k}f(-z)\}}{I_{k+1}f(z)-I_{k+1}f(-z)}\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} \prec q(z) + \frac{\lambda}{\mu(k+1)}zq'(z), (14)$$

then

$$\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} \prec q(z), \quad (z \in U).$$

and q(z) is the best dominant.

Proof. Let the function h(z) be defined by (10). We know that the first part of (11) holds true. Combining (11) and (14), we have

$$h(z) + \frac{\lambda}{\mu(k+1)} z h'(z) \prec q(z) + \frac{\lambda}{\mu(k+1)} z q'(z) \quad (15)$$

By using Lemma 2.3 and (7), we easily get the assertion of Theorem 3.2.

Corollary 3.3 Let $\lambda \in \mathbb{C}^*$ and $-1 \leq B < A \leq 1$. Suppose also that

$$\mathsf{R}\left(\frac{1-Bz}{1+Bz}\right) > max\left\{0; -(k+1)\mathsf{R}\left(\frac{\mu}{\lambda}\right)\right\}.$$

If $f \in A$ satisfies the following subordination:

$$\begin{split} &(1-\lambda) \bigg[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \bigg]^{\mu} + \lambda \frac{z \langle I_k f(z) - I_k f(-z) \rangle}{I_{k+1}f(z) - I_{k+1}f(-z)} \bigg[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \bigg]^{\mu} \\ &\prec \frac{1 + Az}{1 + Bz} + \lambda \frac{(A - B)z}{(1 + Bz)^2}, \text{ then} \end{split}$$

$$\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} \prec \frac{1+Az}{1+Bz}, \ (z \in U).$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

If f is subordinate to F, then F is superordinate to f. We now derive the following superordination result for the class $\mathsf{B}^{\lambda,\mu}(A,B)$.

Theorem 3.4 let q(z) be convex univalent function in U and let $\lambda \in \mathbb{C}$ with $\mathbb{R}\lambda > 0$. Also let

$$\begin{bmatrix} I_{k+1}f(z) - I_{k+1}f(-z) \\ 2z \end{bmatrix}^{\mu} \in \mathsf{H}[q(0), 1] \cap \mathsf{Q}, \text{ and} \\ (1 - \lambda \Big[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \Big]^{\mu} + \lambda \frac{z \{I_k f(z) - I_k f(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \Big[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \Big]^{\mu}$$

be univalent in **U**. If

$$q(z) + \frac{\lambda}{\mu(k+1)} zq'(z) \\ \prec (1 - \lambda \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} + \lambda \frac{z(I_kf(z) - I_kf(-z))}{I_{k+1}f(z) - I_{k+1}f(-z)} \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu}$$
then

Volume 5 Issue 5, May 2016 www.ijsr.net

$$q(z) \prec \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu}$$

and q is the best subordinant.

Proof. Let the function h(z) be defined by (10). Then

$$q(z) + \frac{\lambda}{\mu(k+1)} zq'(z)$$

 $\prec (1-\lambda) \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} + \lambda \frac{z\{I_kf(z) - I_kf(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu}$
 $= h(z) + \frac{\lambda}{\mu(k+1)} zh'(z).$

An application of Lemma 2.4 yields the assertion of Theorem 3.4.

Taking $q(z) = \frac{1 + Az}{1 + Bz}$ in Theorem 3.4, we obtain the following corollary.

Corollary 3.5 *let* q(z) *be convex univalent function in* **U** *and let* $-1 \le B < A \le 1, \lambda \in \mathbb{C}$ *with* $\mathbb{R}\lambda > 0$. *Also let*

$$0 \neq \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu} \in \mathsf{H}[q(0), 1] \cap Q,$$

and

$$(1-\lambda) \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} + \lambda \frac{z\{I_kf(z) - I_kf(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu}$$

be univalent in U. If
$$\frac{1+Az}{1+Bz} + \lambda \frac{(A-B)z}{(1+Bz)^2}$$

then

$$\frac{1+Az}{1+Bz} \prec \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu}, \quad (z \in U).$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

 $< (1-\lambda) \frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} + \lambda \frac{I_{k+1}f(z) - I_{k+1}f(-z)}{I_{k+1}f(z) - I_{k+1}f(-z)}$

Combining the above results of subordination and superordination, we easily get the following sandwich- type result.

Corollary 3.6 let q_1 be convex univalent and let q_2 be univalent in U, $\lambda \in C$ with $R\lambda > 0$. Let q_2 satisfy(12). If

$$0 \neq \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu} \in \mathsf{H}[q(0), 1] \cap Q,$$

and

$$(1-\lambda)\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu}+\lambda\frac{z\{I_{k}f(z)-I_{k}f(-z)\}}{I_{k+1}f(z)-I_{k+1}f(-z)}\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu}$$

be univalent function in U, also

$$q_{1}(z) + \frac{\lambda}{\mu(n+1)} zq_{1}'(z)$$

$$\prec (1 - \lambda \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} + \lambda \frac{z \{I_{k}f(z) - I_{k}f(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu}$$

$$q_{2}(z) + \frac{\lambda}{\mu(k+1)} zq_{2}'(z)$$
then

$$q_1(z) \prec \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu} \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and dominant.

Theorem 3.7 If $\lambda \in \mathbb{C}.\mu > 0$ and $f(z) \in B^{0,\mu} (1-2\rho,-1)$, $(0 \le \rho < 1)$, then $f(z) \in \mathbb{B}^{\lambda,\mu} (1-2\rho,-1)$ for |z| < R, where $R = \left[\left(\sqrt{\left(\frac{|\lambda|}{\mu(k+1)}\right)^2} + 1 \right) - \frac{|\lambda|}{\mu(k+1)} \right]$ (16)

The bound R is the best possible.

Proof. Set

$$\left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu} = (1 - \rho)h(z) + \rho, \quad z \in \mathsf{U}, \quad (0 \le \rho < 1). \quad (17)$$

Then, clearly the function h(z) is analytic in **U** with h(0) = 1. Proceeding as an Theorem 3.1, we have

$$\frac{1}{1-\rho} \left\{ (1-\lambda) \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} + \lambda \frac{z \{I_k f(z) - I_k f(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} - \rho \right\} = h(z) + \frac{\lambda}{\mu(k+1)} z h'(z).$$
(18)

Using the following well-known estimate, see [11]

$$\frac{|zh'|}{\mathsf{R}[h(z)]} \le \frac{2r}{1-r^2} \quad (|z|=r<1)$$

in (18), we obtain that

$$R\frac{1}{1-\rho}\left\{ (1-\lambda) \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} + \lambda \frac{z\{I_{k}f(z) - I_{k}f(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} - \rho \right\}$$

$$\geq R \quad h(z) \left\{ 1 - \frac{2|\lambda|r}{\mu(k+1)(1-r^{2})} \right\}.$$
(19)

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Right hand side of (19) is positive, provided that r < R, where R is given by (16).

In order to show that the bound R is best possible, we consider the function $f(z) \in A$ defined by

$$\left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu} = \rho + (1 - \rho)\frac{1 + z}{1 - z} \quad (z \in \mathsf{U}, \quad 0 \le \rho < 1).$$

We note that

$$\frac{1}{1-\rho} \left\{ (1-\lambda) \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} + \lambda \frac{z \{I_k f(z) - I_k f(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} - \rho \right\}$$
$$= \frac{1+z}{1-z} + \frac{2|\lambda|z}{\mu(k+1)(1-z)^2} = 0,$$

for |z| = R, we conclude that the bound is the best possible (12) of Theorem 3.8 holds and this completes the proof. and this proves the theorem.

Theorem 3.8 If
$$0 \le \lambda_1 \le \lambda_2$$
 and
 $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$. Then
 $\mathbf{B}^{\lambda_2,\mu}(A_2, B_2) \subset \mathbf{B}^{\lambda_1,\mu}(A_1, B_1)$. (20)

Proof. Suppose that $f \in \mathsf{B}^{\lambda_2,\mu}(A_2, B_2)$. We know that $\left(1-\lambda_{2}\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu}+\lambda_{2}\frac{z\{I_{k}f(z)-I_{k}f(-z)\}}{I_{k+1}f(z)-I_{k+1}f(-z)}\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu}\right)$

$$\frac{1+A_2z}{1+B_2z}.$$

Since $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, we easily find that

this is $f \in \mathsf{B}^{\lambda_1,\mu}(A_1, B_1)$. Thus the assertion (20) holds true for $0 \le \lambda_1 = \lambda_2$. If $\lambda_2 > \lambda_1$, by Theorem 3.1 and (21), we know that , $f \in \mathsf{B}^{0,\mu}(A_2, B_2)$ that is,

$$\left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z}\right]^{\mu} \prec \frac{1 + A_1 z}{1 + B_1 z}.$$
 (22)

At the same time, we have

$$(1 - \lambda_1 \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} + \lambda_1 \frac{z \{I_k f(z) - I_k f(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu}$$

$$= \frac{\lambda_1}{\lambda_2} \left\{ (1 - \lambda) \left[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \right]^{\mu} + \lambda_2 \frac{z \{I_k f(z) - I_k f(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \right]^{\mu} \right\}$$

$$\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} + \left(1-\frac{\lambda_1}{\lambda_2}\right)\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu}.$$
(23)

Moreover,

$$0 \le \frac{\lambda_1}{\lambda_2} < 1,$$

and the function $\frac{1+A_1z}{1+B_1z}, -1 \le B_1 < A_1 \le 1, z \in U$ is

analytic and convex in U. Combining (21 - 23) and Lemma 2.4, we find that

$$\begin{split} & (1-\lambda_1) \bigg[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \bigg]^{\mu} + \lambda_1 \frac{z \{I_k f(z) - I_k f(-z)\}}{I_{k+1}f(z) - I_{k+1}f(-z)} \bigg[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \bigg]^{\mu} \\ & \prec \frac{1+A_1 z}{1+B_1 z}. \end{split}$$

this is $f \in \mathsf{B}^{\lambda_1,\mu}(A_1,B_1)$, which implies that the assertion 1.110.

Theorem 3.9 Let
$$f(z) \in \mathsf{B}^{\lambda,\mu}(A,B)$$
 with $\lambda > 0$ and
 $-1 \leq B_1 < A_1 \leq 1$. Then

$$\frac{\mu(k+1)}{\lambda} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\mu(k+1)}{\lambda}-1} du$$
 $< \mathsf{R} \bigg[\frac{I_{k+1}f(z) - I_{k+1}f(-z)}{2z} \bigg]^{\mu} < \frac{\mu(k+1)}{\lambda} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\mu(k+1)}{\lambda}-1} du$
(24)
The extremel function of (24) is defined by

The extremal function of (24) is defined by

$$F_{\lambda,\mu,A,B}(z) = 2z \left(\frac{\mu(k+1)}{\lambda} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu(k+1)}{\lambda}-1} du\right)^{\frac{1}{\mu}}.$$
 (25)

Proof. Let $f(z) \in \mathsf{B}^{\lambda,\mu}(A,B)$ with $\lambda > 0$. From Theorem 1, we know that (1) holds, which implies that

$$\mathbb{R}\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} < \sup_{z \in U} \mathbb{R}\left\{\frac{\mu(k+1)}{\lambda}\int_{0}^{1}\frac{1+Azu}{1+Bzu}u^{\frac{\mu(k+1)}{\lambda}-1}du\right\}$$

$$\leq \left\{\frac{\mu(k+1)}{\lambda}\int_{0}^{1}\sup_{z \in U} \mathbb{R}\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu(k+1)}{\lambda}-1}du\right\}$$

$$< \frac{\mu(k+1)}{\lambda}\int_{0}^{1}\frac{1+Au}{1+Bu}u^{\frac{\mu(k+1)}{\lambda}-1}du, \quad (26)$$

and

$$\mathbb{R}\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} > \inf_{z \in \mathbb{U}} \mathbb{R}\left\{\frac{\mu(k+1)}{\lambda}\int_{0}^{1}\frac{1+Azu}{1+Bzu}u^{\frac{\mu(n+1)}{\lambda}-1}du\right\}$$

$$\geq \left\{\frac{\mu(k+1)}{\lambda}\int_{0}^{1}\inf_{z \in \mathbb{U}} \mathbb{R}\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu(k+1)}{\lambda}-1}du\right\}$$

$$> \frac{\mu(k+1)}{\lambda}\int_{0}^{1}\frac{1+Au}{1+Bu}u^{\frac{\mu(k+1)}{\lambda}-1}du. \tag{27}$$

Combining (26) and (27), we obtain (20). Noting that the function $F_{\lambda,\mu,A,B}(z)$ defined by (25) belongs to the class

Volume 5 Issue 5, May 2016 www.ijsr.net Licensed Under Creative Commons Attribution CC BY $\mathsf{B}^{\lambda,\mu}(A,B)$, we get that inequality (24) is sharp. This completes the proof.

In view of Theorem 9, we have the following distortion theorems for the class $\mathsf{B}^{\lambda,\mu}(A,B)$.

Corollary 3.10 Let $f(z) \in \mathsf{B}^{\lambda,\mu}(A,B)$ with $\lambda > 0$ and $-1 \leq B_1 < A_1 \leq 1$. Then for |z| = r < 1, we have $2r\left(\frac{\mu(k+1)}{\lambda}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\mu(k+1)}{\lambda}-1}du\right)^{\frac{1}{\mu}}$ $<|I_{k+1}f(z)-I_{k+1}f(-z)| < 2r\left(\frac{\mu(k+1)}{\lambda}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\mu(k+1)}{\lambda}-1}du\right)^{\frac{1}{\mu}}.$ (28)

The extremal function of (28) is defined by (25). By noting that

$$\left(\mathsf{R}(v)\right)^{\frac{1}{2}} \leq \mathsf{R}\left(v^{\frac{1}{2}}\right) \leq |v|^{\frac{1}{2}}, \quad v \in \mathsf{C}; \mathsf{R}(v) \geq 0.$$

From Theorem 9, we can easily derive the following result.

Corollary 3.11 Let
$$f(z) \in B^{\lambda,\mu}(A, B)$$
 with $\lambda > 0$ and
 $-1 \le B_1 < A_1 \le 1$. Then
 $\left(\frac{\mu(k+1)}{\lambda} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\mu(k+1)}{\lambda}-1} du\right)^{\frac{1}{2}}$
 $< R\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\frac{\mu}{2}} < \left(\frac{\mu(k+1)}{\lambda} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\mu(k+1)}{\lambda}-1} du\right)^{\frac{1}{2}}$.
Theorem 3.12 Let $f(z) \in B^{\lambda,\mu}(A, B)$ with $\lambda > 0$ and
 $-1 \le B_1 < A_1 \le 1$. Then

$$\leq B_{1} < A_{1} \leq 1. \text{ Then}$$

$$\left|a_{1}\right| \leq \left|\frac{2(A-B)}{\lambda + 2\mu(k+1)}\right|. \tag{29}$$

The inequality (29) is sharp, with the extremal function defined by (25).

Proof. Combining (1) and (5), we have

$$(1-\lambda)\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu} + \lambda \frac{z\{I_kf(z)-I_kf(-z)\}}{I_{k+1}f(z)-I_{k+1}f(-z)}\left[\frac{I_{k+1}f(z)-I_{k+1}f(-z)}{2z}\right]^{\mu}$$

= 1+ $\left[1+\frac{\lambda}{2\mu(k+1)}\right]\mu(k+1)a_1z + \cdots \prec \frac{1+Az}{1+Bz}$
= 1+ $(A-B)z + \cdots$ (30)

An application of Lemma 2.4 to (30) yields

$$\left[1 + \frac{\lambda}{2\mu(k+1)}\right]\mu(k+1)a_1 \le |A-B|. \tag{31}$$

Thus, from (31), we easily arrive at (29) asserted by Theorem 3.12.

Theorem 3.13 Let $f(z) \in \mathsf{B}^{\lambda,\mu}(A,0)$ with $\mathsf{R}\lambda > 0$ and A > 0 and $|\lambda| \left(1 + \mathsf{R} \frac{\mu(k+1)}{\lambda} \right) > A\mu(k+1)$. Then $\left| \frac{z\{I_k f(z) - I_k f(-z)\}}{I_{k-1} f(z) - I_{k-1} f(-z)} - 1 \right| < \frac{A\left\{ \left| \lambda \left(1 + \mathsf{R} \left(\frac{\mu(k+1)}{\lambda} \right) \right) + \mu(k+1) \right\} \right\}}{\left(- \left(- \left(- \left(\frac{\mu(k+1)}{\lambda} \right) \right) - 1 \right) - 1} \right)$

$$I_{k+1}f(z) - I_{k+1}f(-z) + |\lambda| \left\{ |\lambda| \left(1 + \mathsf{R}\left(\frac{\mu(k+1)}{\lambda}\right) \right) - A\mu(k+1) \right\}$$

Proof. let h(z) be defined by (9). It follows from (10) that

$$h(z) + \frac{\lambda}{\mu(k+1)} z h'(z) = 1 + Aw(z), \qquad (32)$$

where

$$w(z) = \sum_{n=1}^{\infty} w_n z_n$$

is analytic in **U** with $|w(z)| < 1, z \in U$. From (32), we can get

$$h(z) = 1 + A \frac{\mu(k+1)}{\lambda} \int_{0}^{1} t^{\frac{\mu(k+1)}{\lambda} - 1} w(tz) dt$$

= $1 + A \frac{\mu(k+1)}{\lambda} \sum_{n=1}^{\infty} \frac{1}{n + \frac{\mu(k+1)}{\lambda}} w_{n} z^{n}.$ (33)

It follows from (33) that

$$(zh(z))' = 1 + A \frac{\mu(k+1)}{\lambda} \sum_{n=1}^{\infty} \frac{n+1}{n + \frac{\mu(k+1)}{\lambda}} w_n z^n.$$

$$= 1 + A \frac{\mu(k+1)}{\lambda} \sum_{n=1}^{\infty} \frac{1}{n + \frac{\mu(k+1)}{\lambda}} w_n z^n$$

$$+ A \frac{\mu(k+1)}{\lambda} \left(w(z) - \frac{\mu(k+1)}{\lambda} \int_0^1 \frac{\mu(k+1)}{\lambda} w(tz) dt \right).$$
(34)

We now find from (33) and (34) that

$$(zh(z))' = A \frac{\mu(k+1)}{\lambda} \left(w(z) - \frac{\mu(k+1)}{\lambda} \int_0^1 t^{\frac{\mu(k+1)}{\lambda}} w(tz) dt \right).$$
(35)

Combining (33) and (35), we can get

$$\frac{zh'(z)}{h(z)} < \frac{A\left\{\left|\lambda\right|\left(1 + \mathsf{R}\left(\frac{\mu(k+1)}{\lambda}\right)\right) + \mu(k+1)\right\}}{\left|\lambda\right|\left\{\left|\lambda\right|\left(1 + \mathsf{R}\left(\frac{\mu(k+1)}{\lambda}\right)\right) - A\mu(k+1)\right\}}$$
(36)

Thus, from (10) and (36), we easily arrive at the assertion of Theorem 3.13.

Remark 3.1 If $\mu = 1$, we obtain the results of [2], Theorems 3.1, 4.1 and 4.4.

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2319

References

- [1] Ali Muhammad, Some differential subordination and superordinations properties of symmetric functions, Rend. Sem. Mat. Univ. Politec. Torino 69, 3, 247-259, 2011.
- [2] Ali Muhammad and Amjadullak Khattak, Some differential subordination and superordination properties of symmetric analytic functions involving Noor integral operator, Le Mathamatiche Vol. 67 (2012) Fasc. II, 77-92.
- [3] M. K. Aouf- T. Bulboac \ddot{a} , Subordination and superordination properties of multivalent functions defined by certain integral operator, J. Franklin Inst, 347, 641-653, 2010.
- [4] M. K. Aouf- F. M. Al-Oboudi- M. M. Haidan, On some results for λ -spirallike and λ -Robertson functions of complex order, Publ. Inst. Math. Belgrade 77, 91, 93-98, 2005.
- [5] M. K. Aouf T. M. Seoudy, Some properties of a class of multivalent analytic functions involving the Liu-Owa operator, J. Com. Math. App. 60, 1525-1535, 2010.
- [6] T. Bulboac ä, A class of superordination preserving integral opeators, Indag. Math. (New Ser.) 13, 3, 301-311, 2002.
- T. Bulboac a, Classes of first-order differential subordination, Demonstratio Math. 35, 2, 287-292,2002.
- [8] J. L. Liu, The Noor integral and strongly starlike functions, J. Math. Anal. Appl. 261, 441- 447, 2001.
- [9] J. L. Liu K. I. Noor, Some properties of Noor integral operator, J. Nat. Geom. 21, 81- 90, 2002.
- [10] M. S. Liu, On certain subclass of analytic functions, J. South China Normal Univ. 4, 15- 20, 2002.
- [11] T. H. Macgregor, The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 14, 514-520, 1963.
- [12] S. S. Miller P. T. Mocanu, Differential subordination Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker Inc., New York, Basel, 2000.
- [13] S. S. Miller- P. T. Mocanu, Subordinations of differential superordinations, Complex Variables 48, 10, 815-826, 2003.
- [14] K. I. Noor, On new classes of integral operators, J. Natur. Geom. 16, 71- 80, 1999.
- [15] K. I. Noor M. A. Noor, On integral operators, J. Math. Anal. Appl. 238, 341- 352, 1999.
- [16] W. Rogosinski, On the coefficient of subordinate functions, Proc. Lond. Math. Soc. Ser.2 48, 48- 82, 1943.
- [17] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49, 109-115, 1975.
- [18] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan, 11, 72-75, 1959.
- [19] T. N. Shanmugam, V. Ravichandran and S. Sivasubbramanian, Differential sandwich theorems for subclasses of analytic functions, Aust. J. Math. Anal. Appl. 3, 1- 11, Art. 8, 2006.