

# Approximate Solutions of Fractional Biological Population Model by Homotopy Analysis Sumudu Transform Method

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**Abstract:** In this paper, the Homotopy Analysis Sumudu transform method is presented to find the exact solution of a more general biological population model. The homotopy analysis Sumudu transform method is a combined form of Sumudu transform and homotopy analysis method. The fractional derivatives are described in Caputo sense, some examples are provided.

**Keywords:** Fractional calculus, Sumudu transforms, homotopy analysis method, homotopy analysis Sumudu transforms method, biological population model

## 1. Introduction

In recent year's fractional calculus have been given considerable popularity due mainly to its various applications in fluid mechanics, visco-elasticity, biology, electrical network, optics and signal processing and so on. Except in a limited number of these problems, we have difficulty to find their exact analytic solutions. An effective an easy to use method for solving such equations is needed. Various powerful methods such as differential transform method [1-3], Adomian decomposition method [4-8], Variational iteration method [9-11], homotopy perturbation method [12-16], homotopy perturbation transforms method [17-18] etc., have been proposed to obtain the exact and approximate analytic solutions of fractional differential equations. Another analytical approach that can be applied to solve many types of nonlinear fractional differential equation is Homotopy analysis method (HAM)[19-27]. A systematic and clear exposition on HAM is given in [23-24].

The objective of present paper is to apply the homotopy analysis Sumudu transform method, which is an elegant combination of Sumudu transform method and homotopy analysis method, to find the solution of time-fractional biological population model [28], a representative biological population diffusion equation is  $u_t = u_{xx}^2 + u_{yy}^2 + f(u)$ , where  $u(x, y, t)$  denotes the population density and  $f(u)$  represents the population supply due to birth and death. In this paper, we propose a generalized time-fractional nonlinear biological population diffusion equation as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + f(u), \quad t > 0, \quad x, y \in R \quad (1.1)$$

with given initial conditions  $u(x, y, 0)$ , and according to Malthusian law and Verhulst law, we consider a more general form of  $f(x) = hu^a(1 - ru^b)$ , where  $h, a, b, r$  are real numbers. When choose special values, they change to Malthusian law and Verhulst law.

The derivatives in Eq. (1.1) are the Caputo derivative. Linear and Nonlinear population systems were solved by using

Variational iteration method [28], Adomian Decomposition method [29], and Homotopy perturbation method [30]. However, one of the disadvantages ADM is the inherent difficulty in calculating the Adomian polynomial. This paper considers the effectiveness of the homotopy analysis Sumudu transform method (HASTM) in solving fractional biological population system.

## 2. Basic Definitions

For the concept of fractional derivatives, we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variable and their integral order which is the case in most physical processes. Some basic definitions and properties of fractional calculus theory which we have used in this paper are given in this section.

**Definition 2.1** A real function  $f(x)$ ,  $x > 0$  is said to be in the space  $C_\mu$ ,  $\mu \in R$ , if there exist a real number  $p (> \mu)$  such that  $f(x) = x^p f_1(x)$ , where  $f_1(x) \in C[0, \infty)$ , and it is said to be in the space  $C_\mu^m$  iff  $f^{(m)} \in C_\mu$ ,  $m \in N \cup \{0\}$ .

**Definition 2.2.** The Riemann-Liouville fractional integral operator of order  $\alpha \geq 0$  of a function  $f \in C_\mu$ ,  $\mu \geq -1$  is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, x > 0 \quad (2.1)$$

$$J^0 f(x) = f(x) \quad (2.2)$$

Properties of the operator  $J^\alpha$  can be found in [31], we mention only the following:

- (i)  $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$
- (ii)  $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

$$(iii) J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$$

For  $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$  and  $\gamma > -1$ .

**Definition 2.3.** The fractional derivative of  $f(x)$  in the Caputo sense is defined as [31]

$$D_t^\alpha f(x) = J^{m-\alpha} D_*^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt \quad (2.3)$$

For  $m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0, f \in C_{-1}^m$ .

Also, we need here three basic properties

$$(i) D_t^\alpha J^\alpha f(x) = f(x)$$

(ii)

$$J^\alpha D_t^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0$$

$$(iii) D_t^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}; \quad x > 0, \gamma > 0.$$

For  $m-1 < \alpha \leq m, m \in \mathbb{N}, \mu \geq -1$  and  $f \in C_\mu^m$ .

**Lemma 2.1.** If  $m-1 < \alpha \leq m, m \in \mathbb{N}$ , then the

Laplace transform of the fractional derivative  $D_t^\alpha f(t)$  is

$$L(D_t^\alpha f(t)) = s^\alpha \bar{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-k-1}, \quad t > 0 \quad (2.4)$$

Where  $\bar{f}(s)$  is the Laplace transform of  $f(t)$ .

### 3. Sumudu Transform

In early 90's Watugala [32], introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of functions

$$A = \{f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_j}, \text{ if } t \in (-i)^j \times [0, \infty)\} \quad (3.1)$$

by the following formula

$$G(u) = S[F(t)] = \int_0^\infty F(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2) \quad (3.2)$$

The existence and uniqueness of this transformation is discussed in [33]. For further details and properties of this transformation, see [34-36].

### 4. Basic Idea of Homotopy Analysis Method

To give the basic idea of Homotopy Analysis Method [23], let us consider a nonlinear differential equation in the form:

$$\square [u(x, t)] = 0 \quad t \geq 0 \quad (4.1)$$

Where  $\square$  is a fractional differential operator and  $u(x, t)$  is unknown function of the independent variable  $x$  and  $t$ . For

the simplicity we ignore all boundary or initial conditions, which can be treated in the similar way.

In the frame of HAM [22-23], we can construct the following zeroth-order deformation equation:

$$(1-q)L[\phi(x, t; q) - u_0(x, t)] = q\hbar H(t) \square [\phi(x, t; q)] \quad (4.2)$$

Where  $q \in [0, 1]$  is an embedding parameter,  $\hbar \neq 0$  is an auxiliary parameter,  $H(t) \neq 0$  is an auxiliary function,  $L$  is an auxiliary linear operator,  $\square$  is fractional differential operator,  $\phi(x, t; q)$  is an unknown function, and  $u_0(t)$  is an initial guess of  $u(x, t)$ , which satisfies the initial conditions. It should be emphasized that one has great freedom to choose the initial guess  $u_0(t)$ , the auxiliary linear operator  $L$ , the auxiliary parameter  $\hbar$  and the auxiliary function  $H(t)$ . Obviously, when the embedding parameter  $q=0$  and  $q=1$ , it holds

$$\phi(x, t; 0) = u_0(x, t), \quad \phi(x, t; 1) = u(x, t)$$

respectively. Thus as  $q$  increases from 0 to 1, the solution  $\phi(x, t; q)$  varies from the initial guess  $u_0(x, t)$  to  $u(x, t)$ .

Expanding  $\phi(x, t; q)$  in Taylor series with respect to  $q$ , we have

$$\phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) q^m \quad (4.3)$$

where

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t; q)}{\partial q^m} \Big|_{q=0} \quad (4.4)$$

Assume that the auxiliary parameter, the auxiliary function  $H(t)$ , the initial approximation and the auxiliary linear operator  $L$  are properly chosen, the series (4.3) converges at  $q=1$ , then we have

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \quad (4.5)$$

which must be one of the solution of the original nonlinear equations. According to the definition (4.5), the governing equation can be deduced from the zero-order deformation (4.2). Define the vectors

$$\vec{u}_m = \{u_0(x, t), u_1(x, t), u_2(x, t), \dots, u_n(x, t)\} \quad (4.6)$$

Differentiating equation (4.2),  $m$ -times with respect to embedding parameter  $q$ , then setting  $q=0$  and dividing them by  $m!$ , we get the so-called  $m$ th-order deformation equation

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar H(t) \mathfrak{R}_m(\vec{u}_{m-1}) \quad (4.7)$$

Where

$$\mathfrak{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \square [\phi(x, t; q)]}{\partial q^{m-1}} \Big|_{q=0} \quad (4.8)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (4.9)$$

The so-called  $m$ th-order deformation equation (4.7) is linear which can be easily solved using Mathematica package.

## 5. Homotopy Analysis Sumudu Transform Method

To illustrate the basic idea of this method, let us consider a general fractional nonlinear non homogeneous differential equation

$$D_t^\alpha u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t), \quad 0 < \alpha \leq 1 \quad (5.1)$$

Subject to the initial conditions

$$u(x, 0) = f(x) \quad (5.2)$$

Using the differentiation properties of the Sumudu transform and above initial condition, we have

$$\frac{S[u]}{u^\alpha} - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{u^{n-k}} + S[Ru(x, t)] + S[Nu(x, t)] = S[g(x)] \quad (5.4)$$

$$S[u(x, t)] - u^\alpha \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{u^{n-k}} + u^\alpha [S[Ru(x, t)] + S[Nu(x, t)] - S[g(x)]] \quad (5.5)$$

We define the nonlinear operator

$$N[\phi(x, t; q)] = S[\phi(x, t; q)] - u^\alpha \sum_{k=0}^{n-1} \frac{\phi^{(k)}(0)}{u^{n-k}} + u^\alpha [S[R\phi(x, t; q)] + S[N\phi(x, t; q)] - S[g(x)]] \quad (5.6)$$

Where  $q \in [0, 1]$  and  $\phi(x, t; q)$  is a real function of  $x, t, q$ .

The so-called zero-order deformation equation of the Eq. (5.6) has the form

$$(1 - q)S[\phi(x, t; q) - u_0(x, t)] = q\hbar H(x, t) [S[\phi(x, t; q)] - f(x) - u^\alpha S[R\phi(x, t; q) + N\phi(x, t; q) - g(x, t)]] \quad (5.7)$$

Where  $S$  is the Sumudu transform,  $q \in [0, 1]$  is the embedding parameter,  $H(x, t)$  denotes a nonzero auxiliary function,  $\hbar \neq 0$  is an auxiliary parameter,  $u_0(x, t)$  is an initial guess of  $u(x, t)$  and  $\phi(x, t; q)$  is an unknown function. Obviously, when the parameter  $q=0$  and  $q=1$ , it holds

$$\phi(x, t; 0) = u_0(x, t), \quad \phi(x, t; 1) = u(x, t) \quad (5.8)$$

respectively. Thus as  $q$  increases from 0 to 1, the solution  $\phi(x, t; q)$  varies from the initial guess  $u_0(x, t)$  to the solution  $u(x, t)$ . Expanding  $\phi(x, t; q)$  in Taylor series with respect to  $q$ , we have

$$\phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) q^m \quad (5.9)$$

Where

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t; q)}{\partial q^m} \Big|_{q=0} \quad (5.10)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter  $\hbar$ , and the auxiliary function are properly chosen, the series (5.9) converges at  $q=1$ , then we has

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \quad (5.11)$$

Which must be one of the solution of the original nonlinear equations. According to the definition (5.11), the governing equation can be deduced from the zero- order deformation (5.7). Define the vectors

$$\vec{u}(x, t) = \{u_0(x, t), u_1(x, t), u_2(x, t), \dots, u_m(x, t)\} \quad (5.12)$$

Where  $g(x, t)$  is the source term,  $N$  represent the general nonlinear differential operator and  $R$  is the linear differential operator,  $D_t^\alpha$  is the Caputo fractional derivative of the function  $u(x, t)$ .

Now taking the Sumudu transform of both sides of (5.1), we get

$$S[D_t^\alpha u(x, y, t)] + S[R(u, t)] + S[Nu(x, t)] = S[g(x)] \quad (5.3)$$

Differentiating the zero- order deformation equation (5.7) m-times with respect to  $q$  and then dividing by  $m!$  and finally setting  $q=0$  we get the following  $m^{\text{th}}$ - order deformation equation:

$$S[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar H(x, t) \mathfrak{R}_m(\vec{u}_{m-1}(x, t)) \quad (5.13)$$

Applying the inverse Sumudu transform, we have

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + S^{-1}[\hbar H(x, t) \mathfrak{R}_m(\vec{u}_{m-1}(x, t))] \quad (5.14)$$

where

$$\mathfrak{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x, t; q)]}{\partial q^{m-1}} \Big|_{q=0} \quad (5.15)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (5.16)$$

## 6. Numerical Results

In this section we use the Homotopy Analysis Sumudu transform method to solve nonlinear fractional biological population equations:

**Example1.** Consider the Eq.(1.1) with  $a=1$ ,  $r=0$ , corresponding to Malthusian law, we have the following biological population equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu \quad (6.1)$$

Subject to the initial condition

$$u_0 = \sqrt{xy} \quad (6.2)$$

Taking the Sumudu transform of both sides of Eq. (6.1), and using (6.2)

$$S[u] - \sqrt{xy} - u^\alpha \left[ S \left[ \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu \right] \right] = 0 \quad (6.3)$$

Then

$$\mathfrak{R}(\bar{u}_{m-1}) = S[u_{m-1}] - (1 - \chi_m) \sqrt{xy} - u^\alpha \left[ S \left[ \frac{\partial^2}{\partial x^2} \sum_{j=0}^{m-1} u_j u_{m-1-j} \right] + S \left[ \frac{\partial^2}{\partial y^2} \sum_{j=0}^{m-1} u_j u_{m-1-j} \right] + S[hu_{m-1}] \right] \quad (6.4)$$

The  $m^{\text{th}}$ - order deformation is given by

$$S[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \mathfrak{R}_m(\bar{u}_{m-1}) \quad (6.5)$$

Applying the inverse Sumudu transform, we have

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + S^{-1}[\hbar \mathfrak{R}_m(\bar{u}_{m-1})] \quad (6.6)$$

Solving eq. (6.6) for  $m=1, 2, 3, \dots$  we have

$$u_1(x, y, t) = -\hbar \sqrt{xy} \frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2(x, y, t) = -\hbar(1+\hbar) \sqrt{xy} \frac{t^\alpha}{\Gamma(1+\alpha)} + \hbar^2 \sqrt{xy} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$u_3(x, y, t) = -\hbar(1+\hbar)^2 \sqrt{xy} \frac{t^\alpha}{\Gamma(1+\alpha)} + 2\hbar^2(1+\hbar) \sqrt{xy} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \hbar^3 \sqrt{xy} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$

$$u_4(x, y, t) = -\hbar(1+\hbar)^3 \sqrt{xy} \frac{t^\alpha}{\Gamma(1+\alpha)} + 3\hbar^2(1+\hbar)^2 \sqrt{xy} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - 3\hbar^3(1+\hbar) \sqrt{xy} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \hbar^4 \sqrt{xy} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

substituting  $u_0, u_1, u_2, u_3, u_4, \dots$ , into Eq. (5.11) gives the solution in series form by:

$$u(x, y, t) = \sqrt{xy} \left[ 1 - \hbar \{ 1 + (1+\hbar) + (1+\hbar)^2 + (1+\hbar)^3 + \dots \} \frac{t^\alpha}{\Gamma(1+\alpha)} + \hbar^2 \{ 1 + 2(1+\hbar) + 3(1+\hbar)^2 + \dots \} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \hbar^3 \{ 1 - 3(1+\hbar) + \dots \} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \hbar^4 \{ 1 + \dots \} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \dots \right] \quad (6.7)$$

Setting  $\hbar = -1$ ,

$$u(x, y, t) = \sqrt{xy} \left[ 1 + h \frac{t^\alpha}{\Gamma(1+\alpha)} + h^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + h^3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + h^4 \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \dots \right] \quad (6.8)$$

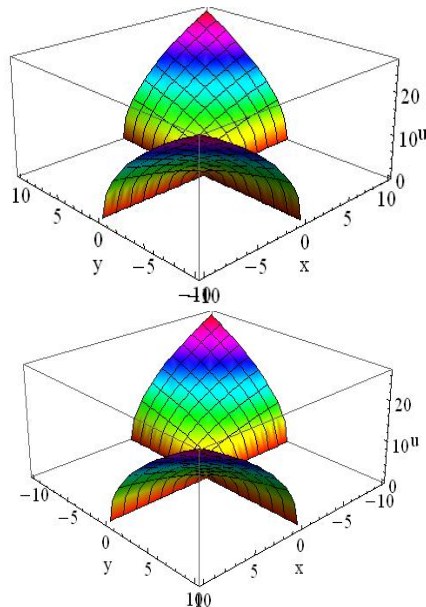
as  $\alpha \rightarrow 1$ , we have

$$u(x, y, t) = \sqrt{xy} \left[ 1 + ht + \frac{h^2 t^2}{2!} + \frac{h^3 t^3}{3!} + \frac{h^4 t^4}{4!} + \dots \right] \quad (6.9)$$

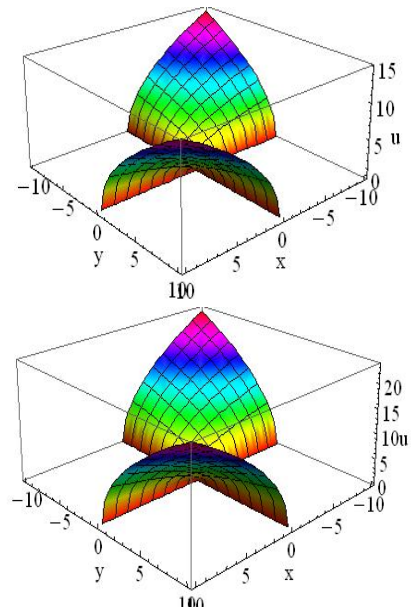
$$u(x, y, t) = \sqrt{xy} e^{ht} \quad (6.10)$$

which is an exact solution to the standard form biological population equation. The evolution result for the exact solution (6.10) and the approximate solution (6.9) for the case  $\alpha=1$ , are shown in Fig.(1). It can be seen from Fig.1 that the solutions obtained by the HASTM is nearly identical with the exact solution. Fig.2 show the approximate solutions. It also be concluded that the approximate solution of fractional biological model is continuous with the parameter  $\alpha$ .





**Figure 1:** The surface shows the solution  $u(x, y, t)$  for (6.8):  
(i) exact solution (6.10); (ii) numerical solution (6.9) when  $h=0.1, t=10$



**Figure 2:** The surface shows the solution  $u(x, y, t)$  for (6.7):  
(i)  $\hbar = -1.4, \alpha = 0.5$  (ii)  $\hbar = -1, \alpha = 0.9$  when  $h=0.1, t=10$

**Example2.** Consider the Eq. (1.1) with  $a=1, b=1$ , this leads to Verhulst law, and we have the following fractional biological population equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu(1-hu) \quad (6.11)$$

subject to the initial condition

$$u_0 = e^{\sqrt{\frac{hr}{8}}(x+y)} \quad (6.12)$$

Taking the Sumudu transform of Eq. (6.11) and using the Eq. (6.12), we have

$$S[u] - e^{\sqrt{\frac{hr}{8}}(x+y)} - u^\alpha \left[ S \left[ \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + (hu - hru^2) \right] \right] = 0 \quad (6.13)$$

$$\begin{aligned} \Re(\bar{u}_{m-1}) &= S[u_{m-1}] - (1 - \chi_m) e^{\sqrt{\frac{hr}{8}}(x+y)} \\ &- u^\alpha \left[ S \left[ \frac{\partial^2}{\partial x^2} \sum_{j=0}^{m-1} u_j u_{m-1-j} \right] + S \left[ \frac{\partial^2}{\partial y^2} \sum_{j=0}^{m-1} u_j u_{m-1-j} \right] + S \left[ hu_{m-1} - hr \sum_{j=0}^{m-1} u_j u_{m-1-j} \right] \right] \end{aligned} \quad (6.14)$$

The  $m^{\text{th}}$ - order deformation is given by

$$S[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \Re_m(\bar{u}_{m-1}) \quad (6.15)$$

Applying the inverse Sumudu transform, we have

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + S^{-1}[\hbar \Re_m(\bar{u}_{m-1})] \quad (6.16)$$

Solving eq. (6.16) for  $m=1, 2, 3, \dots$ , we have

$$\begin{aligned} u_1 &= -\hbar e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^\alpha}{\Gamma(1+\alpha)} \\ u_2 &= -\hbar(1+\hbar) e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^\alpha}{\Gamma(1+\alpha)} + \hbar^2 e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ u_3 &= -\hbar(1+\hbar)^2 e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^\alpha}{\Gamma(1+\alpha)} + 2\hbar^2(1+\hbar) e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \hbar^3 e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \end{aligned}$$

$$u_4 = -h\hbar(1+\hbar)^3 e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^\alpha}{\Gamma(1+\alpha)} + 3h^2\hbar^2(1+\hbar)^2 e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ - 3h^3\hbar^3(1+\hbar) e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + h^4\hbar^4 e^{\sqrt{\frac{hr}{8}}(x+y)} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}$$

substituting  $u_0, u_1, u_2, u_3, u_4, \dots$ , into Eq. (5.11) gives the solution in series form by:

$$u = e^{\sqrt{\frac{hr}{8}}(x+y)} [1 - h\hbar\{1 + (1+\hbar) + (1+\hbar)^2 + (1+\hbar)^3 + \dots\} \frac{t^\alpha}{\Gamma(1+\alpha)} + h^2\hbar^2\{1 + 2(1+\hbar) + 3(1+\hbar)^2 + \dots\} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \\ h^3\hbar^3\{1 - 3(1+\hbar) + \dots\} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + h^4\hbar^4\{1 + \dots\} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \dots] \quad (6.17)$$

Set  $\hbar = -1$ ,

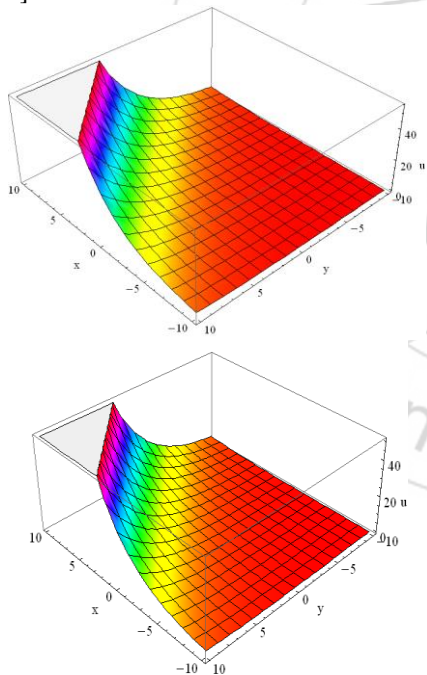
$$u(x, y, t) = e^{\sqrt{\frac{hr}{8}}(x+y)} [1 + h \frac{t^\alpha}{\Gamma(1+\alpha)} + h^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + h^3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + h^4 \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \dots] \quad (6.18)$$

as  $\alpha \rightarrow 1$ , we have

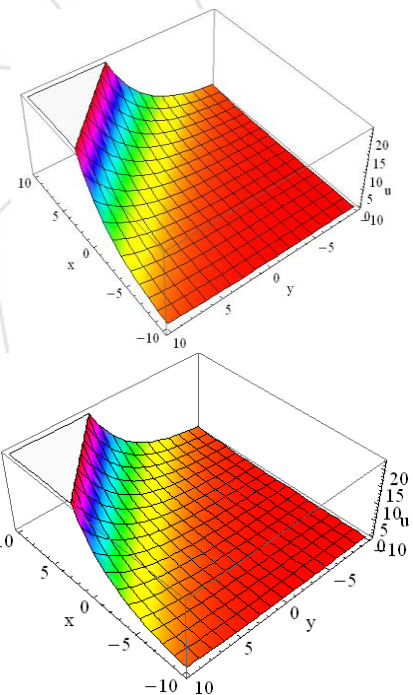
$$u(x, y, t) = e^{\sqrt{\frac{hr}{8}}(x+y)} [1 + ht + \frac{h^2 t^2}{2!} + \frac{h^3 t^3}{3!} + \frac{h^4 t^4}{4!} + \dots] \quad (6.19)$$

$$u(x, y, t) = e^{\sqrt{\frac{hr}{t}}(x+y)+ht} \quad (6.20)$$

which is an exact solution to the standard form biological model [6.11]



**Figure 3:** The surface shows the solution  $u(x, y, t)$  for (6.18): (i) exact solution (6.20);(ii) numerical solution (6.19) when  $h=0.01, t=10, r=48$



**Figure 4:** The surface shows the solution  $u(x, y, t)$  for (6.17): (i)  $\hbar = -1.4, \alpha = 0.95$  (ii)  $\hbar = -1, \alpha = 0.95$  when  $h=0.01, t=10, r=48$

**Example3.** Consider the Eq. (1.1) with  $h=1, r=0, a=1$ , we have the following fractional biological population equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u \quad (6.21)$$

subject to the initial conditions

$$u_0 = \sqrt{\sin x \sin y} \quad (6.22)$$

now taking the Sumudu transform of both sides of (6.21), and using (6.22) we have

$$S[u] - \sqrt{\text{Sinx Sinhy}} - u^\alpha \left[ S \left\{ \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u \right\} \right] = 0 \quad (6.23)$$

$$\begin{aligned} \Re(\bar{u}_{m-1}) &= S[u_{m-1}] - (1 - \chi_m) \sqrt{\text{Sinx Sinhy}} \\ &- u^\alpha \left[ S \left[ \frac{\partial^2}{\partial x^2} \sum_{j=0}^{m-1} u_j u_{m-1-j} \right] + S \left[ \frac{\partial^2}{\partial y^2} \sum_{j=0}^{m-1} u_j u_{m-1-j} \right] + S[u_{m-1}] \right] \end{aligned} \quad (6.24)$$

The  $m^{\text{th}}$ - order deformation is given by

$$S[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \Re_m(\bar{u}_{m-1}) \quad (6.25)$$

Applying the inverse Sumudu transform of above equation, we have

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + S^{-1}[\hbar \Re_m(\bar{u}_{m-1})] \quad (6.26)$$

Solving eq. (6.26) for  $m=1, 2, 3, \dots$  we have

$$\begin{aligned} u_1 &= -\hbar \sqrt{\text{Sinx Sinhy}} \frac{t^\alpha}{\Gamma(1+\alpha)} \\ u_2 &= -\hbar(1+\hbar) \sqrt{\text{Sinx Sinhy}} \frac{t^\alpha}{\Gamma(1+\alpha)} + \hbar^2 \sqrt{\text{Sinx Sinhy}} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ u_3 &= -\hbar(1+\hbar)^2 \sqrt{\text{Sinx Sinhy}} \frac{t^\alpha}{\Gamma(1+\alpha)} + 2\hbar^2(1+\hbar) \sqrt{\text{Sinx Sinhy}} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \hbar^3 \sqrt{\text{Sinx Sinhy}} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ u_4 &= -\hbar(1+\hbar)^3 \sqrt{\text{Sinx Sinhy}} \frac{t^\alpha}{\Gamma(1+\alpha)} + 3\hbar^2(1+\hbar)^2 \sqrt{\text{Sinx Sinhy}} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &- 3\hbar^3(1+\hbar) \sqrt{\text{Sinx Sinhy}} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \hbar^4 \sqrt{\text{Sinx Sinhy}} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \end{aligned}$$

substituting  $u_0, u_1, u_2, u_3, u_4, \dots$ , into Eq. (5.11) gives the solution in series form by,  $\hbar = -1$ , we have

$$u(x, y, t) = \sqrt{\text{Sinx Sinhy}} \left[ 1 + h \frac{t^\alpha}{\Gamma(1+\alpha)} + h^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + h^3 \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + h^4 \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \dots \right] \quad (6.27)$$

as  $\alpha \rightarrow 1$ , we have

$$u(x, y, t) = \sqrt{\text{Sinx Sinhy}} \left[ 1 + ht + \frac{h^2 t^2}{2!} + \frac{h^3 t^3}{3!} + \frac{h^4 t^4}{4!} + \dots \right] \quad (6.28)$$

$$u(x, y, t) = \sqrt{\text{Sinx Sinhy}} e^t \quad (6.29)$$

which is an exact solution.

**Example4.** Consider the Eq. (1.1) with  $a=-1, b=1$ , we have the following fractional biological population equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu^{-1} - hr \quad (6.30)$$

subject to the initial conditions

$$u_0 = \sqrt{\frac{hr}{4} x^2 + \frac{hr}{4} y^2 + y + 5} \quad (6.31)$$

now taking the Sumudu transform of both sides of (26), and using (27) we have

$$S[u] - \sqrt{\frac{hr}{4} x^2 + \frac{hr}{4} y^2 + y + 5} - u^\alpha \left[ S \left[ \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + \left( \frac{h}{u_{m-1}} - hr \right) \right] \right] = 0 \quad (6.32)$$

$$\begin{aligned} \Re(\bar{u}_{m-1}) &= S[u_{m-1}] - (1 - \chi_m) \sqrt{\frac{hr}{4} x^2 + \frac{hr}{4} y^2 + y + 5} \\ &- u^\alpha \left[ S \left[ \frac{\partial^2}{\partial x^2} \sum_{j=0}^{m-1} u_j u_{m-1-j} \right] + S \left[ \frac{\partial^2}{\partial y^2} \sum_{j=0}^{m-1} u_j u_{m-1-j} \right] + S \left[ \frac{h}{u_{m-1}} - hr \right] \right] \end{aligned} \quad (6.33)$$

The  $m^{\text{th}}$ - order deformation is given by

$$S[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \mathfrak{R}_m(\vec{u}_{m-1}) \quad (6.34)$$

Applying the inverse Sumudu transform of above equation, we have

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + S^{-1}[\hbar \mathfrak{R}_m(\vec{u}_{m-1})] \quad (6.35)$$

Solving eq. (6.35) for  $m=1, 2, 3, \dots$ , we have

$$\begin{aligned} u_1 &= -\hbar \sqrt{\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5} \frac{t^\alpha}{\Gamma(1+\alpha)} \\ u_2 &= -\hbar(1+\hbar) \sqrt{\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5} \frac{t^\alpha}{\Gamma(1+\alpha)} + h^2 \hbar^2 \left(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\right)^{-1/2} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ u_3 &= -\hbar(1+\hbar)^2 \sqrt{\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5} \frac{t^\alpha}{\Gamma(1+\alpha)} + 2h^2 \hbar^2 (1+\hbar) \left(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\right)^{-1/2} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \\ &\quad h^3 \hbar^3 \left(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\right)^{-3/2} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \dots \end{aligned}$$

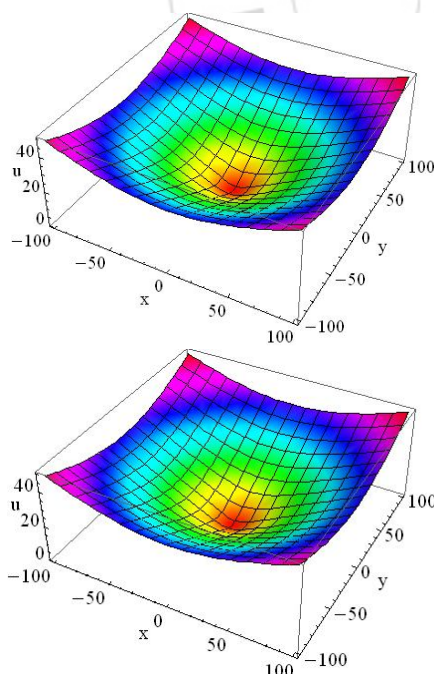
substituting  $u_0, u_1, u_2, u_3, \dots$ , into Eq. (5.11) gives the solution in series form by,  $\hbar = -1$ , Then the approximate solution in series form is

$$u(x, y, t) = u_0 + \frac{ht^\alpha}{u_0} \sum_{n=0}^{\infty} \frac{n+1}{\Gamma(1+(n+1)\alpha)} \left(\frac{-ht^\alpha}{u_0^2}\right)^n \quad (6.36)$$

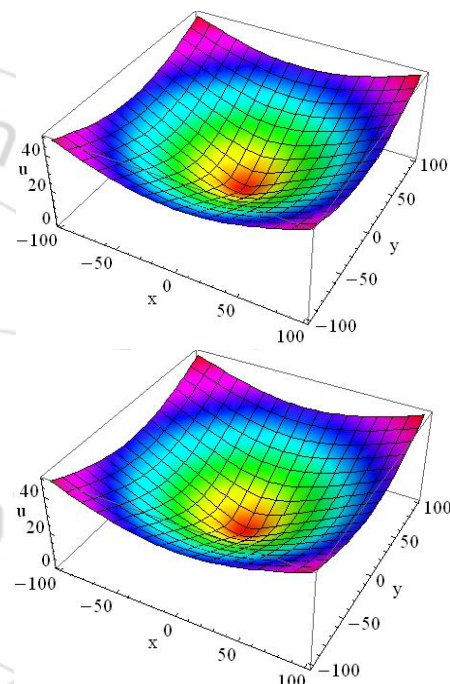
as  $\alpha \rightarrow 1$ , we have

$$u(x, y, t) = u_0 + \left(\frac{ht}{u_0}\right) \exp\left(\frac{-ht}{u_0^2}\right) \quad (6.37)$$

which is an exact solution of the integer order biological population.



**Figure 5:** The surface shows the solution  $u(x, y, t)$  for (6.30): (i) exact solution (6.37); (ii) numerical solution (6.36) when  $h=0.01, t=10, r=48$



**Figure 6:** The surface shows the solution  $u(x, y, t)$  for (6.30): (i)  $\hbar = -1, \alpha = 0.9$  (ii)  $\hbar = -1, \alpha = 0.5$  when  $h=0.01, t=10, r=48$

## 7. Conclusion

We employ the homotopy analysis Sumudu transform method (HASTM) for finding the approximate analytical solutions of time fractional degenerate parabolic equations arising in the spatial diffusion of biological populations subject to the some initial conditions. The results obtained by using this method agree well with the results obtained by ADM [29], VIM [28], HPM [30]. The reliability of HASTM and reduction in computation gives this method a wider applicability. Finally we can conclude that the HASTM is very powerful and efficient in finding analytic as well as numerical solutions for wider classes of linear and nonlinear fractional differential equations. Mathematica has been used for calculation and plot 3D graphs.



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