International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2015): 6.391

A Class of QT Bézier Curve with Two Shape Parameters

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Abstract: ***In this paper a class of quartic trigonometric Bézier curve, called QT Bézier curve, with two shape parameters is presented. These curves not only inherit most properties of the usual quartic Bézier curves in the polynomial space, but also enjoy some other advantageous properties for shape modelling. The shape of the curve can be adjusted by altering the values of shape parameters while the control polygon is kept unchanged. With the increase of the shape parameter, the curve approaches to the control polygon. When the shape parameters are chosen properly, the QT Bézier curves can be used to represent ellipse precisely.***

Keywords: Trigonometric Bézier basis function, CAGD, trigonometric Bézier curve, shape parameters, approximability.

1. Introduction

Bézier curves and surfaces are the basic tools for modelling in Computer Aided Geometric Designing(CAGD) and Computer Graphics(CG). Bézier polynomial has several applications in the fields of engineering, science and technology such as highway or railway rout designing, networks, Computer aided design system, animation, environment design, robotics, communications and many other disciplines because it is easy to compute and is also very stable. The classical Bézier curves have some limitations that their shape and position are fixed relative to their control polygon. Thus people attempt to find a solution of the problem in the non-polynomial function space. During the last few years, a major research focus has been the use of trigonometric functions or the blending of polynomial and trigonometric functions.

Trigonometric B-splines were first presented in [1] and the recurrence relation for the trigonometric B-splines of arbitrary order was established in [2]. In recent years, several new trigonometric splines have been studied in the literature; see [3], [4] and [5]. In [6] cubic trigonometric Bézier curve with two shape parameters were presented. In [7], a novel generalization of Bézier curve and surface with n shape parameters are presented. In [8], the cubic trigonometric polynomial spline curve of G^3 continuity is constructed, which can be G^5 continuity under special condition. In [9], uniform T-B-spline basis function of $(n+1)^{th}$ order and its solution is presented. In [10], quartic splines with C^2 continuity are presented for a non-uniform knot vectors which are C^2 and G^3 continuous under special case. Algebraic-Trigonometric blended spline curves are presented in [11] which can represent some transcendental curves. Cubic trigonometric Bézier curve with two shape parameters is presented in [12]. Recently in [13], a quadratic trigonometric Bézier curve with shape parameter is constructed which is G^1 continuous. In [14], the generalized basis functions of degree $n+1$ with two shape parameters are presented. The cubic trigonometric polynomial spline curve of G^1 continuity is constructed in [15], which can be G^3 continuity under special condition. In [16], the cubic trigonometric polynomial curve similar to the cubic Bézier curves is constructed. In [17], the shape features of the cubic trigonometric polynomial curves with a shape parameter are analyzed. An extension of the Bézier model is studied in [18]. In [19] and [20] quartic and cubic trigonometric Bézier curve respectively with shape parameter is presented and the effect of shape parameter is studied. A new rational cubic trigonometric Bézier curve with four shape parameters are defined in [21]. The main purpose of this work is to present a class of quartic trigonometric Bézier Curve, called QT Bézier Curve, with two shape parameters.

The paper is organized as follows. In section 2, QT Bézier basis functions with two shape parameters are established and the properties of the basis functions are shown. In section 3, QT Bézier curves are given and some properties are discussed. By using shape parameter, shape control of the QT Bézier curves are studied. In section 4, the representation of ellipse has been shown. In section 5, the approximability of the QT Bézier curves and the classical quartic Bézier curves corresponding to their control polygon are shown. Conclusion is given in section 6.

2. QT Bézier Basis Functions

Firstly, the definition of QT Bézier basis functions is given as follows.

2.1 The construction of the basis functions

Definition 1. For an arbitrarily selected real values of λ and μ, where $\lambda,\ \mu \in$ [0, 1], the following five functions of $t\ (t \in [0, \pi/2])$ are defined as QT Bézier basis functions with two shape parameters λ and μ:

$$\begin{cases} b_0(t) = (1-\sin t)(1+\frac{\lambda}{2}\sin t) \\ b_1(t) = (1-\lambda)\sin t(1-\sin t) \\ b_2(t) = \frac{\lambda}{2}\sin t(1-\sin t) + \frac{\mu}{2}\cos t(1-\cos t) \\ b_3(t) = (1-\mu)\cos t(1-\cos t) \\ b_4(t) = (1-\cos t)(1+\frac{\mu}{2}\cos t) \end{cases} \quad (1)$$

2.2. The properties of the basis functions

Theorem 1. The basis functions (1) have the following properties:
(a) *Non-negativity*: $b_i(t) \geq 0$ for $i = 0, 1, 2, 3, 4$.
(b) *Partition of unity*: $\sum_{i=0}^{4} b_i(t) = 1$
(c) *Symmetry*: $b_i(t; \lambda, \mu) = b_{4-i}(\frac{\pi}{2} - t; \mu, \lambda)$,
for $i = 0, 1, 2, 3, 4$.
(d) *Monotonicity*: For a given parameter t, as the shape parameter λ increases (or decreases), $b_0(t)$ decreases (or increases) and $b_1(t)$, $b_2(t)$ increases (or decreases). Similarly as the shape parameter μ increases (or decreases), $b_4(t)$ decreases (or increases) and $b_2(t)$, $b_3(t)$ increases (or decreases).

Proof (a) For $t \in [0, \frac{\pi}{2}]$ and $\lambda, \mu \in [0,1]$, then
$0 \leq (1 - \mathrm{sin}t) \leq 1$, $0 \leq (1 - \mathrm{cos}t) \leq 1$,
$0 \leq \mathrm{sin}t \leq 1$, $0 \leq \mathrm{cos}t \leq 1$
$0 \leq (1 + \frac{\lambda}{2}\mathrm{sin}t) \leq 1$, $0 \leq (1 + \frac{\mu}{2}\mathrm{cos}t) \leq 1$,
It is obvious that $b_i(t) \geq 0$ for $i = 0, 1, 2, 3, 4$.
(b) $\sum_{i=0}^{4} b_i(t) = (1 - \mathrm{sin}t)\left(1 + \frac{\lambda}{2}\mathrm{sin}t\right) + (1 - \lambda)\mathrm{sin}t(1 - \mathrm{sin}t + \lambda 2 \mathrm{sin}t 1 - \mathrm{sin}t + \mu 2 \mathrm{cos}t 1 - \mathrm{cos}t + 1 - \mu \mathrm{cos}t(1 - \mathrm{cos}t) + (1 - \mathrm{cos}t)(1 + \frac{\mu}{2}\mathrm{cos}t) = 1$

The remaining cases follow obviously.

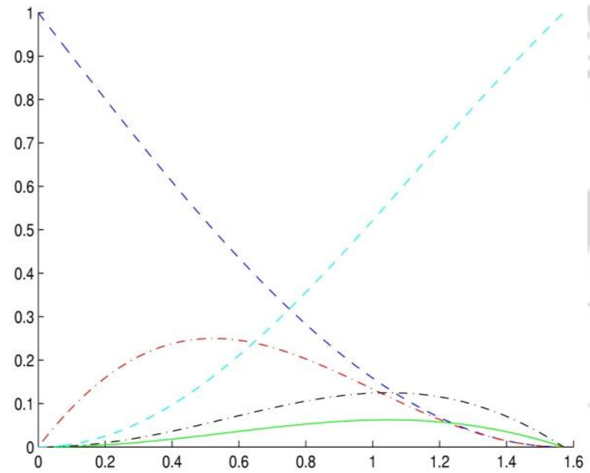

Figure 1: QT Bézier basis functions for $\lambda = 0$, $\mu = 0.5$

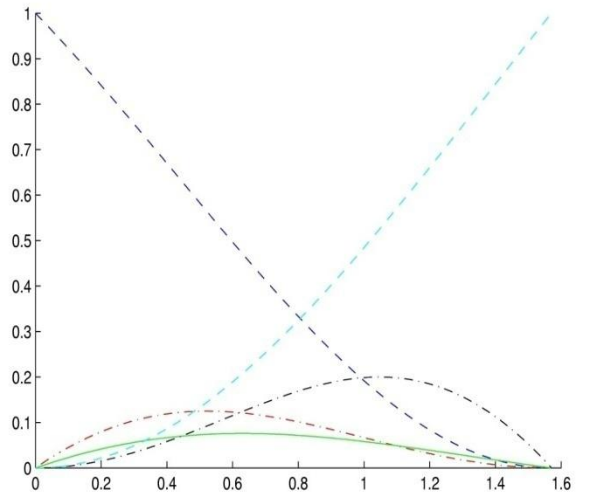

Figure 2: QT Bézier basis functions for $\lambda = 0.5$, $\mu = 0.2$

The curves of the QT Bézier basis functions for $\lambda = 0$, $\mu = 0.5$ are shown in Figure 1 and for $\lambda = 0.5$, $\mu = 0.2$ are shown in Figure 2.

3. QT Bézier Curve

3.1 The construction of the QT Bézier curve

Definition 2. Given control points P_i (i=0,1,2,3,4) in R^2 or R^3, then
$r(t) = \sum_{i=0}^{4} b_i(t) P_i, t \in [0,1], \lambda, \mu \in [0,1]$ (2) (2)
is called a QT Bézier curve with two shape parameters.
From the definition of the basis function, some properties of the QT Bézier curve can be obtained as follows:

Theorem 2. The QT Bézier curves (2) have the following properties:
(a) *Terminal Properties*:
$r(0) = P_0, r(\frac{\pi}{2}) = P_4$, (3)
$r'(0) = (1 - \frac{\lambda}{2})(P_1 - P_0) + \frac{\lambda}{2}(P_2 - P_1)$,
$r'(\frac{\pi}{2}) = \frac{\mu}{2}(P_3 - P_2) + (1 - \frac{\mu}{2})(P_4 - P_3)$ (4)

(b) *Symmetry*:
P_0, P_1, P_2, P_3, P_4 and P_4, P_3, P_2, P_1, P_0 define the same QT Bézier curve in different parametrizations, i.e.,

$r(t; \lambda, \mu; P_0, P_1, P_2, P_3, P_4) = r(\frac{\pi}{2} - t; \mu, \lambda; P_4, P_3, P_2, P_1, P_0)$; (5)
for $t \in [0, \frac{\pi}{2}], \lambda, \mu \in [0,1]$
(c) *Geometric invariance*:
The shape of a QT Bézier curve is independent of the choice of coordinates, i.e. (2) satisfies the following two equations:
$r(t; \lambda, \mu; P_0 + q, P_1 + q, P_2 + q, P_3 + q, P_4 + q) = r(t; \lambda, \mu; P_0, P_1, P_2, P_3, P_4) + q$
$r(t; \lambda, \mu; P_0 * T, P_1 * T, P_2 * T, P_3 * T, P_4 * T) = r(t; \lambda, \mu; P_0, P_1, P_2, P_3, P_4) * T$ (6)
for $t \in \left[0, \frac{\pi}{2}\right], \lambda, \mu \in [0,1]$
where q is arbitrary vector in R^2 or R^3, and T is an arbitrary $d \times d$ matrix, d=2 or 3.

(d) *Convex hull property*:
The entire QT Bézier curve segment lies inside its control polygon spanned by P_0, P_1, P_2, P_3, P_4.

3.2 Shape Control of the QT Bézier Curve

3.2.1 For $t \in [0, \pi/2]$, we rewrite (2) as follows:

$r(t) = \sum_{i=0}^{4} P_i c_i(t) + \frac{\lambda}{2}\mathrm{sin}t(1 - \mathrm{sin}t)[P_0 - 2P_1 + P_2] + \frac{\mu}{2}\mathrm{cos}t(1 - \mathrm{cos}t)[P_4 - 2P_3 + P_2]$ (7)
where $c_0(t) = (1 - \mathrm{sin}t)$, $c_1(t) = \mathrm{sin}t(1 - \mathrm{sin}t)$, $c_2(t) = 0, c_3(t) = \mathrm{cos}t(1 - \mathrm{cos}t)$, $c_4(t) = (1 - \mathrm{cos}t)$.

Obviously, shape parameter λ affects the curve on the control edges $(P_0 - P_1)$ and $(P_2 - P_1)$ and μ affects the curve on the control edges $(P_4 - P_3)$ and $(P_2 - P_3)$. The shape parameters λ and μ serve to effect local control in the curves: as λ increases, the curve moves in the direction of edges $(P_0 - P_1)$ and $(P_2 - P_1)$; as λ decreases, the curve moves in the opposite direction to the edges $(P_0 - P_1)$ and $(P_2 - P_1)$. The same effect on the edges $(P_4 - P_3)$ and $(P_2 - P_3)$ are produced by the shape parameter μ. As the shape parameter $\lambda\,(=\mu)$ increases or decreases, the curve moves in the direction of point P_2 or the opposite direction to the point P_2.

Figure 3 shows some computed examples of QT Bézier with different values of shape parameters λ and μ. These curves are generated by setting $\lambda = \mu = 1$ (magenta dotted lines), $\lambda = \mu = 0.5$ (red solid lines) and $\lambda = \mu = 0$ (blue dash dotted lines). The corresponding classical Quartic Bézier curve is shown by solid black lines.

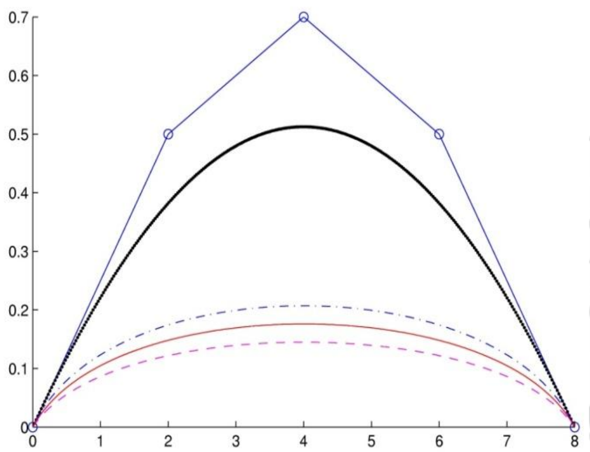

Figure 3: QT Bézier Curves with different values of shape parameters λ and μ

4. The Representation of Ellipse

The basis functions (1) have the following properties:
For $\lambda = \mu = 0$, the basis functions reduces to the cubic trigonometric Bézier curve (since $b_2(t) = 0$). So in this case, we redefine the basis functions with $\lambda = \mu = 0$ as follows:

$$\begin{cases} bc_0(t) = (1 - \sin t) \\ bc_1(t) = \sin t(1 - \sin t) \\ bc_2(t) = \cos t(1 - \cos t) \\ bc_3(t) = (1 - \cos t) \end{cases} \quad (3)$$

(8)

Let P_0, P_1, P_2, P_3 with respective coordinates (a, b), (2a, 2b), (2a, 2b), (a, 3b). From (3) we obtain

$$\begin{cases} x(t) = a(\sin t + \cos t) \\ y(t) = 2b + b(\sin t - \cos t) \end{cases}$$

This gives the intrinsic equation: $(\frac{x}{a})^2 + (\frac{y-2b}{b})^2 = 2$
This is an arc of an ellipse.

5. Approximability

Suppose P_0, P_1, P_2, P_3 and P_4 are not collinear; the relationship between QT Bézier curve $r(t)$ and the quartic Bézier curve $B(t) = \sum_{i=0}^{4} P_i \binom{4}{i} (1-t)^{4-i} t^i$ with the same control points P_i is given by

$$r\left(\frac{\pi}{4}\right) - P_2 = \frac{\sqrt{2}-1}{4}[P_0(2\sqrt{2} + \lambda) + P_1(1 - \lambda) + P_2(\lambda + \mu - \frac{4}{\sqrt{2}-1}) + P_3(1 - \mu) + P_4(2\sqrt{2} + \mu)] \quad (9)$$

and

$$B(\tfrac{1}{2}) - P_2 = \frac{1}{16}[P_0 + 4P_1 - 10P_2 + 4P_3 + P4] \quad (10)$$

6. Conclusion

As mentioned above QT Bézier curve have all the properties that quartic Bézier curves have. Since there is nearly no difference in structure between a QT Bézier curve and a quartic Bézier curve, it is not difficult to adapt a QT Bézier curve to a CAD/CAM system that already uses the quartic Bézier curves.

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