Fixed Coefficients for Certain Subclass of Univalent Functions using Hypergeometric Function

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Abstract: When object of the present paper is to determine coefficient estimates, distortion bounds, closure theorems and extreme points for functions \( f(z) \) belonging to a new subclass of uniformly starlike functions.

Keywords: Univalent, Uniformly starlike function, Hypergeometric function.

1. Introduction

Let \( A \) denote the class of functions \( f(z) \) of the form
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
which are analytic in the open unit disk \( Y = \{ z \in X : |z| < 1 \} \). Further, by \( \Sigma \) we shall denote the class of functions \( f \in A \) which are univalent in \( Y \).

For \( f \in A \) given by (1) and \( g(z) \) given by
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n,
\]
their convolution (or Hadamard product), denoted by \((f * g)\), is defined as
\[
(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in Y)
\]
(3)

Note that \( f * g \in A \).

A function \( f \in A \) is said to be in \( \beta - US(\alpha) \), the class of \( \beta \) - uniformly starlike functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), if and only if
\[
\left| \frac{zf''(z)}{f'(z)} \right| > \beta \left| \frac{zf'(z)}{f(z)} \right| + 1 + \alpha \quad (\beta \geq 0),
\]
(4)

and a function \( f \in A \) is said to be in \( \beta - UC(\alpha) \), the class of \( \beta \) - uniformly convex functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), if and only if
\[
\left| 1 + \frac{zf''(z)}{f'(z)} \right| > \beta \left| \frac{zf'(z)}{f(z)} \right| + \alpha \quad (\beta \geq 0).
\]
(5)

Uniformly starlike and uniformly convex functions were first introduced by [1, 2] and then studied by various authors [3, 4]. It is known that \( \beta - US(\alpha) \) or \( \beta - UC(\alpha) \) if and only if

In [5], Sakaguchi (1959) defined the class \( S_1 \) of starlike functions with respect to symmetric points as follows:

Let \( f \in A \). Then \( f \) is said to be starlike with respect to symmetric points in \( Y \) if and only if
\[
\left| \frac{zf'(z)}{f(z) - f'(-z)} \right| > 0 \quad (z \in Y).
\]
(8)

Recently, Owa et. al. (2007) [6] defined and studied the class \( S_1(\alpha, t) \),

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by (see [8]) and for all
\[ D_k^m(a_1, \beta_1)f(z) = \frac{1}{(n!)} \left( \sum_{j=0}^{n} \frac{(m+j-1)!}{j!(n-j)!} (a_1)_j \right) f(z), \]
where \( n \in \mathbb{N} \) and \( \lambda \geq 0 \).

If \( f \in \mathcal{A} \), then from (10) we may easily deduce that
\[ D_k^m(a_1, \beta_1)f(z) = z + \sum_{n=2}^{\infty} \left( \sum_{j=0}^{n} \frac{(m+j-1)!}{j!(n-j)!} (a_1)_j \right) \frac{z^n}{n^n} \]
where \( \Gamma_n(a_1) = \frac{(a_1)_{n-1}}{(\beta_1)_{n-1}} = \frac{(a_1)_{n-1}}{(\beta_1)_{n-1}} (n-1)! \).

Special cases of the operator \( D_k^m(a_1, \beta_1)f \) includes various other linear operators which were considered in many earlier work on the subject of analytic and univalent functions. If we let \( m = 0 \) in \( D_k^m(a_1, \beta_1)f \), we have
\[ D_0^m(a_1, \beta_1)f(z) = H^1_q(a_1, \beta_1)f(z) \]
where \( H^1_q(a_1, \beta_1) \) is Dziok-Srivastava operator for functions in \( \mathcal{A} \) (see [9]) and for \( q = 2, s = 1, a_1 = \beta_1, a_2 = 1 \) and \( \lambda = 1 \), we get the operator introduced by Salagean (1983)([9]).

**Definition 1.1** A function \( f(z) \in \mathcal{A} \) is said to be in the class \( k - \Sigma \lambda \alpha \beta \gamma \) if it satisfies
\[ \Re \left[ \frac{(1-t)z \left( D_k^m(a_1, \beta_1)f(z) \right)}{D_k^m(a_1, \beta_1)f(z) - D_k^m(a_1, \beta_1)f(tz)} \right] \geq \frac{1}{1+\gamma} \]
(12)
for \( \lambda \geq 0, \ km \geq 0, \ |t| \leq 1, \ t \neq 1 \) and \( 0 \leq \gamma < 1 \).

Furthermore, we say that a function \( f(z) \in k - \Sigma \lambda \alpha \beta \gamma \) is in the subclass \( k - \Sigma \lambda \alpha \beta \gamma \) if \( f(z) \) is of the following form:
\[ f(z) = \sum_{n=2}^{\infty} a_n z^n \quad (a_n > 0, n = \infty). \]

The main objective of this paper is to study the coefficient estimates, extreme points, distortion bounds and closure properties for \( f(z) \in k - \Sigma \lambda \alpha \beta \gamma \) by fixing second coefficients.

Similar other classes of univalent functions with fixed second coefficients have been extensively studied by Aouf (1997)[10, 11], S. M. Khamar et. al., (2010), [12], Darwish (2008)[13], and others see [14].

### 2. Coefficient Estimate

**Lemma 2.1** Let \( \omega = u + iv \). Then \( \Re \omega \geq \alpha \) if and only if
\[ |\omega - (1+\alpha)| \leq |\omega + (1+\alpha)|. \]

**Lemma 2.2** Let \( \omega = u + iv \) and \( \alpha, \gamma \) are real numbers. Then \( \Re \omega > \alpha|\omega| + \gamma \) if and only if
\[ \Re \omega(1+\alpha e^{i\theta}) - \alpha e^{i\theta} > \gamma. \]

**Theorem 2.3** Let the function \( f(z) \) be defined by (13). Then \( f(z) \in k - \Sigma \lambda \alpha \beta \gamma \) if and only if
\[ \sum_{n=2}^{\infty} C_n \left( m, \lambda, \alpha, \beta, \gamma, t \right) \]
where
\[ C_n = \frac{1}{k+1} \left( \gamma \right) \left( n+1 \right) u_n \]
and \( u_n = 1+t+t^2+\cdots+t^n \).

The result is sharp for the function \( f(z) \) is given by
\[ f(z) = \frac{z - \frac{1}{1-\gamma} \sum_{n=2}^{\infty} C_n \left( m, \lambda, \alpha, \beta, \gamma, t \right) z^n}{C_n \left( m, \lambda, \alpha, \beta, \gamma, t \right) \sum_{n=2}^{\infty} C_n \left( m, \lambda, \alpha, \beta, \gamma, t \right) z^n}. \]

**Proof.** By definition \( f(z) \in k - \Sigma \lambda \alpha \beta \gamma \) if and only if the condition (12) is satisfied. Then by Lemma 2.1, we have
\[ \Re \left[ \frac{(1-t)z \left( D_k^m(a_1, \beta_1)f(z) \right)}{D_k^m(a_1, \beta_1)f(z) - D_k^m(a_1, \beta_1)f(tz)} \right] \geq \frac{1}{1+\gamma} \]
(15)
for \( \lambda \geq 0, \ km \geq 0, \ |t| \leq 1, \ t \neq 1 \) and \( 0 \leq \gamma < 1 \).
\[ |1 - i| (2 - \gamma) |z| - \sum_{n=2}^{\infty} (1 + (n - 1)\lambda) \Gamma_n(\alpha_i) \left[ n + (1 - \gamma)u_n \right] a_n |z|^n \]

Also, \[ A(z) - (1 + \gamma) B(z) \leq \]

\[ \left\{ 2 (1 - \gamma) |z| - 2 \sum_{n=2}^{\infty} (1 + (n - 1)\lambda) \Gamma_n(\alpha_i) \left[ n (k + 1) - (k + \gamma)u_n \right] a_n \right\} \geq 0. \]

or

\[ \sum_{n=2}^{\infty} (1 + (n - 1)\lambda) \Gamma_n(\alpha_i) \left[ n (k + 1) - (k + \gamma)u_n \right] a_n \leq (1 - \gamma). \]

which gives the desired estimation.

Conversely, suppose that (14) holds. Then we must show that

\[ \left\{ 1 - 2 \sum_{n=2}^{\infty} (1 + (n - 1)\lambda) \Gamma_n(\alpha_i) \left[ n + (1 - \gamma)u_n \right] a_n \right\} \geq 0. \]

Choosing the values of \( z \) on the positive real axis where \( 0 \leq z = r < 1 \), the above inequality reduces to

\[ (1 - \gamma) - \sum_{n=2}^{\infty} (1 + (n - 1)\lambda) \Gamma_n(\alpha_i) (n + (1 + \epsilon)u_n - (\gamma + k)u_n) a_n |z|^{n-1} \]

\[ \geq 0. \]

Since \( Re(-e^{i\theta}) \geq -e^{i\theta} \) \( = -1 \), the above inequality reduces to

\[ \left\{ 1 - \sum_{n=2}^{\infty} (1 + (n - 1)\lambda) \Gamma_n(\alpha_i) (n + (1 + \epsilon)u_n) a_n \right\} e^{i\theta} \geq 0. \]

Letting \( r \to 1 \), we have desired conclusion.

**Corollary 2.4** Let the function \( f(z) \) defined by (13) be in the class \( k - \Sigma T \), then

**Theorem 2.5** Let the function \( f(z) \) be defined by (18). Then \( f(z) \) in \( k - \Sigma T \) if and only if

\[ \sum_{n=2}^{\infty} (m, \lambda, k, \gamma) a_n \leq (1 - \gamma |1 - c| \\text{ if } \lambda = 1. \]

**Proof.** Substituting

\[ a_n = \frac{c(1 - \gamma) \beta \gamma}{C_2(m, \lambda, k, \gamma) (\alpha_i)}, \]

simplifying we get the result.

**Corollary 2.6** Let the function \( f(z) \) defined by (18) be in the class \( k - \Sigma T \), then

\[ a_n \leq \frac{(1 - \gamma) |1 - c|}{C_2(m, \lambda, k, \gamma)} \quad (n \geq 3), \quad 0 \leq c \leq 1. \]
3. Extreme Points

Theorem 3.1 Let

\[ f_1(z) = z - \frac{c(1-\gamma)(\beta_1)}{C_1(c, m, \lambda, k, \gamma)(\alpha_1)} z^2 \]  

and

\[ f_n(z) = z - \frac{c(1-\gamma)(\beta_1)}{C_1(c, m, \lambda, k, \gamma)(\alpha_1)} z^2 - \frac{(1-\gamma)(1-c)}{C_n(m, \lambda, k, \gamma)(\alpha_1)} z^n \]  

for \( n = 3, 4, \ldots \). Then \( f(z) \) is in the class \( k - \Sigma T_{c, \lambda, \alpha_1, \beta_1, \gamma, t} \) if and only if it can be expressed in the form

\[ f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z), \]  

where \( \mu_n \geq 0 \) and \( \sum_{n=2}^{\infty} \mu_n = 1 \).

Proof. We suppose that \( f(z) \) can be expressed in the form (23). Then we have

\[ f_n(z) = z - \frac{c(1-\gamma)(\beta_1)}{C_1(c, m, \lambda, k, \gamma)(\alpha_1)} z^2 - \frac{(1-\gamma)(1-c)}{C_n(m, \lambda, k, \gamma)(\alpha_1)} z^n \]  

where \( \mu_n \geq 0 \) and \( \sum_{n=2}^{\infty} \mu_n = 1 \).

It follows from (14) that \( f(z) \) is in the class \( k - \Sigma T_{c, \lambda, \alpha_1, \beta_1, \gamma} \). Conversely, suppose that \( f(z) \) defined by (18) is in the class \( k - \Sigma T_{c, \lambda, \alpha_1, \beta_1, \gamma} \). Then, by using (20), we get

\[ a_n \leq \frac{(1-c)(1-\gamma)}{C_n(m, \lambda, k, \gamma)} (n \geq 3) \]  

and

\[ \mu_n = \frac{C_n(m, \lambda, k, \gamma)}{(1-c)(1-\gamma)} a_n \]  

where

\[ \sum_{n=2}^{\infty} \mu_n = 1 \]  

we have (23). This completes the proof of the theorem.

Corollary 3.2 The extreme points of the class \( k - \Sigma T_{c, \lambda, \alpha_1, \beta_1, \gamma, t} \) are functions \( f_n(z) \), \( n \geq 2 \) given by (3.1).

4. Growth and Distortion Theorem

The following lemmas are required in our investigation of growth and distortion properties of the general class \( k - \Sigma T_{c, \lambda, \alpha_1, \beta_1, \gamma, t} \).

Lemma 4.1 Let the function \( f_3(z) \) be defined by

\[ f_3(z) = z - c(1-\gamma)(\beta_1) \frac{z^2}{C_2(m, \lambda, k, \gamma)(\alpha_1)} - \frac{(1-\gamma)(1-c)}{C_3(m, \lambda, k, \gamma)} z^3, \]  

for some \( n \). Then for \( 0 \leq r < 1 \) and \( 0 \leq \theta \leq 1 \),

\[ \left| f_3(re^{i\theta}) \right| \geq r \left[ 1 - c(1-\gamma)(\beta_1) \frac{r^2}{C_2(m, \lambda, k, \gamma)(\alpha_1)} - \frac{(1-\gamma)(1-c)}{C_3(m, \lambda, k, \gamma)} r^3 \right] \]  

with equality for \( \theta = 0 \). For either \( 0 \leq c < c_0 \) and \( 0 \leq r < r_0 \) or \( c_0 \leq c \leq 1 \),

\[ \left| f_3(re^{i\theta}) \right| \leq r + c(1-\gamma)(\beta_1) \frac{r^2}{C_2(m, \lambda, k, \gamma)(\alpha_1)} - \frac{(1-\gamma)(1-c)}{C_3(m, \lambda, k, \gamma)} r^3 \]  

with equality for \( \theta = \pi \). Further, for \( 0 \leq c \leq c_0 \) and \( r_0 \leq r < 1 \),

\[ \left| f_3(re^{i\theta}) \right| \leq r + \frac{c(1-\gamma)(\beta_1)}{C_2(m, \lambda, k, \gamma)(\alpha_1)} r^2 - \frac{(1-\gamma)(1-c)}{C_3(m, \lambda, k, \gamma)} r^3 \]  

with equality for \( \theta = \pi \).
we can see that
\[
\frac{\partial}{\partial \theta} f_3(ke^{i\theta}) = 0
\]
for \(\theta_1 = 0, \theta_2 = \pi\) and
\[
\theta_3 = \cos^{-1}\left(\frac{1/c - (1-c)(1-\gamma)(\beta_1)}{4/c_{21} + 4/c_{22} + 4/c_{23} + 4/c_{24}}\right)
\]
Since \(\theta_3\) is a valid root only when \(-1 \leq \cos \theta_3 \leq 1\), we have a third root if and only if \(r \leq r < 1\) and \(0 \leq c < c_0\). Thus the results of Lemma 4.1 follow upon comparing the extremal values \(f_3(ke^{i\theta_k})\), \(k = 1, 2, 3\) on the appropriate intervals.

**Lemma 4.2** Let the function \(f_n(z)\) \((n \geq 4)\) be defined by (22). Then
\[
\left|f_n'(ke^{i\theta})\right| \leq |f_4'(-r)| \quad (n \geq 4).
\]
**Proof.** Since
\[
f_n(z) = z - \frac{c(1-\gamma)(\beta_1)}{C_2^n(m, \lambda, k, \gamma)(\alpha_1)}z^2 - \frac{(1-\gamma)(1-c)}{C_1^n(m, \lambda, k, \gamma)}z^n
\]
and \((1-\gamma)(1-c)\) is a decreasing function of \(n\), we have
\[
\left|f_n'(ke^{i\theta})\right| \leq r + \frac{c(1-\gamma)(\beta_1)}{C_2^n(m, \lambda, k, \gamma)(\alpha_1)}^2 + \frac{(1-\gamma)(1-c)}{C_1^n(m, \lambda, k, \gamma)}r^n = -f_4'(-r),
\]
which proves (38).

**Theorem 4.3** Let the function \(f(z)\) defined by (18) be in the class \(k - \Sigma \mathcal{T}_r(c, \lambda, \alpha_1, \beta_1, \gamma, t)\). Then, for \(0 \leq r < 1\),
\[
f(ke^{i\theta}) \geq r - \frac{c(1-\gamma)(\beta_1)}{C_2^n(m, \lambda, k, \gamma)(\alpha_1)}r^2 - \frac{(1-\gamma)(1-c)}{C_1^n(m, \lambda, k, \gamma)}r^n
\]
with equality for \(f_3(z)\) at \(z = r\) and
\[
f(ke^{i\theta}) \leq \max_{\theta} \left\{f_3(ke^{i\theta}) - f_4'(-r)\right\}
\]
where \(\max_{\theta} f_3(ke^{i\theta})\) is given by Lemma 4.1.

The proof of this theorem is obtained by comparing the bounds given by Lemma 4.1 and Lemma 4.2. Putting \(c = 1\) in theorem 4.3, we obtain the following corollary.

**Corollary 4.4** Let the function \(f(z)\) defined by (13) be in the class \(k - \Sigma \mathcal{T}_r(c, \lambda, \alpha_1, \beta_1, \gamma, t)\). Then, for \(0 \leq r < 1\),
\[
r - \frac{(1-\gamma)(\beta_1)}{C_1^n(m, \lambda, k, \gamma)(\alpha_1)}r^2 \leq |f(z)| \leq r + \frac{(1-\gamma)(\beta_1)}{C_1^n(m, \lambda, k, \gamma)(\alpha_1)}r^n.
\]
The result is sharp for the function
\[
f(z) = z - \frac{(1-\gamma)(\beta_1)}{C_2^n(m, \lambda, k, \gamma)(\alpha_1)}z^2.
\]
Putting \(c = 1\) and \(k = 0\) in theorem 4.3, we obtain the following corollary.

**Corollary 4.5** Let the function \(f(z)\) defined by (13) be in the class \(k - \Sigma \mathcal{T}_r(c, \lambda, \alpha_1, \beta_1, \gamma, t)\). Then, for \(0 \leq r < 1\),
\[
r - \frac{(1-\gamma)(\beta_1)}{C_2^n(m, \lambda, k, \gamma)(\alpha_1)}r^2 \leq |f(z)| \leq r + \frac{(1-\gamma)(\beta_1)}{C_1^n(m, \lambda, k, \gamma)(\alpha_1)}r^n.
\]
The result is sharp for the function
\[
f(z) = z - \frac{(1-\gamma)(\beta_1)}{C_1^n(m, \lambda, k, \gamma)(\alpha_1)}z^2.
\]

**Remark 4.1** If \(\alpha_1 = \beta_1 = 1, m = 1\) and \(t = -1\) then the above result (corollary 4.4) is reduced to the class \(S_c(\lambda, k, \beta)\) studied by C.Selvaraj et.al. (2009)[17].

**Lemma 4.6** Let the function \(f_3(z)\) be defined by (29). Then, for \(0 \leq r < 1\) and \(0 \leq c \leq 1\),
\[
|f_3'(ke^{i\theta})| \geq 1 - \frac{2c(1-\gamma)(\beta_1)}{C_2^n(m, \lambda, k, \gamma)(\alpha_1)}r - \frac{(1-\gamma)(1-c)}{C_1^n(m, \lambda, k, \gamma)}r^n
\]
with equality for \(f_3(z)\) at \(z = r\) and
\[
f_3'(ke^{i\theta}) \leq \max_{\theta} \left\{f_3'(ke^{i\theta}) - f_4'(-r)\right\}
\]
where \(\max_{\theta} f_3'(ke^{i\theta})\) is given by Lemma 4.1.
with equality for
\[ \theta = \cos^{-1}\left(\frac{3c(1-c)(1-\gamma)(\beta_1)^2-c(C_1(m, \lambda, k, \gamma))/(\alpha_1)}{6(1-c)(C_2(m, \lambda, k, \gamma)/(\alpha_1))}\right) \] 

where
\[ c_1 = \frac{3[(1-\gamma)/(\beta_1)-6C_1(m, \lambda, k, \gamma)/(\alpha_1)-C_1(m, \lambda, k, \gamma)/(\beta_1)]}{6(1-c)/(\alpha_1)} \]

\[ c_1^* = \left[3[(1-\gamma)/(\beta_1)-6C_1(m, \lambda, k, \gamma)/(\alpha_1)-C_1(m, \lambda, k, \gamma)/(\beta_1)]\right]^2 \]

and
\[ r_1 = \frac{-6(1-c)(C_1(m, \lambda, k, \gamma)/(\alpha_1)+r_1^*)}{6c(1-c)/(1-\gamma)/(\beta_1)} \]

The proof is omitted.

**Theorem 4.7** Let the function \( f(z) \) defined by (18) be in the class \( k - \Sigma T_s(c, \lambda, \alpha_1, \beta_1, \gamma, t) \). Then for \( 0 \leq r < 1 \),

\[ |f'(r e^{i\theta})| \geq 1 - \frac{c(1-\gamma)/(\beta_1)}{C_1(m, \lambda, k, \gamma)/(\alpha_1)} r - (1-c)/(1-\gamma) \]

with equality for \( f'(z) \) at \( z = r \) and
\[ |f'(r e^{i\theta})| \leq \max_{\alpha} |f'(r e^{i\theta})| - |f'(r e^{i\theta})| \]

where \( \max_{\alpha} |f'(r e^{i\theta})| \) is given by Lemma 4.6.

5. Radii of Starlikeness and Convexity

**Theorem 5.1** Let the function \( f(z) \) defined by (18) be in the class \( k - \Sigma T_s(c, \lambda, \alpha_1, \beta_1, \gamma, t) \). Then \( f(z) \) is starlike of order \( \rho \) \( (0 \leq \rho < 1) \) in the disc
\[ |z| = r_1(c, \rho, m, \lambda, k, \gamma) \]

where \( r_1(c, \rho, m, \lambda, k, \gamma) \) is the largest value for which
\[ \frac{c(1-\gamma)/(\beta_1)}{C_2(m, \lambda, k, \gamma)/(\alpha_1)} r + \frac{(1-\gamma)(1-c)/(n-\rho)}{c_n(m, \lambda, k, \gamma)} r^{n-1} \leq 1 - \rho \]

The result is sharp with the extremal function
\[ r_n(z) = z - \frac{c(1-\gamma)/(\beta_1)}{C_2(m, \lambda, k, \gamma)/(\alpha_1)} z^2 - \frac{(1-\gamma)(1-c)}{c_n(m, \lambda, k, \gamma)} z^n. \]

**Proof.** It suffices to show that
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad (0 \leq \rho < 1) \]

for \( |z| = r_1(c, \rho, m, \lambda, k, \gamma) \). We note that
\[ |zf'(z)/f(z)| - 1 = \frac{c(1-\gamma)/(\beta_1)}{C_1(m, \lambda, k, \gamma)/(\alpha_1)} \frac{r + \sum_{n=3}^{\infty} (n-1) a_n r^{n-1}}{1 - \frac{c(1-\gamma)/(\beta_1)}{C_1(m, \lambda, k, \gamma)/(\alpha_1)} \sum_{n=3}^{\infty} a_n r^{n-1}} \]
\[ \leq 1 - \rho, \]

(56)

Since \( f(z) \) is in the class \( k - \Sigma T_s(c, \lambda, \alpha_1, \beta_1, \gamma, t) \), from (14) we may take
\[ a_n = \frac{(1-c)(1-\gamma)}{C_1(m, \lambda, k, \gamma)/(\alpha_1)} \mu_n \quad (n = 3, 4, 5, \ldots) \]

where \( \mu_n \geq 0 \) and \( \sum_{n=3}^{\infty} \mu_n \leq 1 \).

For each fixed \( r \), we choose the positive integer
\[ n_0 = n_0(r) \]

for which \( \frac{n_0 - \rho}{C_1(m, \lambda, k, \gamma)/(\alpha_1)} \) is maximal. Then it follows that
\[ \sum_{n=3}^{\infty} (n-\rho) a_n r^{n-1} \leq \frac{c(1-\gamma)(1-c)/(n_0 - \rho)}{C_1(m, \lambda, k, \gamma)/(\alpha_1)} r^{n_0-1} \]

(59)

Hence \( f(z) \) is starlike of order \( \rho \) in \( |z| = r_1(c, \rho, m, \lambda, k, \gamma) \) provided that
\[ \frac{(1-\gamma)(1-c)/(n_0 - \rho)}{C_1(m, \lambda, k, \gamma)/(\alpha_1)} \mu_n \leq 1 - \rho. \]

(60)

We find the value \( r_1 = r_1(c, \rho, m, \lambda, k, \gamma) \) and the corresponding integer \( n_0(r_1) \) so that
\[ \frac{(1-\gamma)(1-c)/(n_0 - \rho)}{C_1(m, \lambda, k, \gamma)/(\alpha_1)} \mu_n = 1 - \rho. \]

(61)

Then this value \( r_0 \) is the radius of starlikeness of order \( \rho \) for functions \( f(z) \) belonging to the class \( k - \Sigma T_s(c, \lambda, \alpha_1, \beta_1, \gamma, t) \).

In similar manner, we can prove the following theorem concerning the radius of convexity of \( \rho \) for functions in the class \( k - \Sigma T_s(c, \lambda, \alpha_1, \beta_1, \gamma, t) \).

**Theorem 5.2** Let the function \( f(z) \) defined by (18) be in the class \( k - \Sigma T_s(c, \lambda, \alpha_1, \beta_1, \gamma, t) \). Then \( f(z) \) is convex of order \( \rho \), \( (0 \leq \rho < 1) \) in the disc
\[ z = r_2(c, \rho, m, \lambda, k, \gamma) \], where \( r_2(c, \rho, m, \lambda, k, \gamma) \) is the largest value for which
\[
\frac{2c(1 - \rho)(2 - \gamma)(\alpha)}{C_2(m, \lambda, k, \gamma)} + \frac{(1 - \rho)(1 - c)(n - \rho)}{C_n(m, \lambda, k, \gamma)} \leq 1 - \rho \quad (n \geq 3).
\]

The result is sharp for the function \( f(z) \) given by (55).

References