Fixed Coefficients for Certain Subclass of Univalent Functions using Hypergeometric Function

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Abstract: ***When object of the present paper is to determine coefficient estimates, distortion bounds, closure theorems and extreme points for functions*** $f(z)$ ***belonging to a new subclass of uniformly starlike functions.***

Keywords: Univalent, Uniformly starlike function, Hypergeometric function.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \qquad (1)$$

which are analytic in the open unit disk $\Upsilon=\{z\in \mathrm{X}:|z|<1\}$. Further, by Σ we shall denote the class of functions $f\in \mathrm{A}$ which are univalent in Υ.

For $f\in \mathrm{A}$ given by (1) and $g(z)$ given by

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n, \qquad (2)$$

their convolution (or Hadamard product), denoted by $(f*g)$, is defined as

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_n b_n z^n, \quad (z\in\Upsilon). \qquad (3)$$

Note that $f*g\in \mathrm{A}$.

A function $f\in \mathrm{A}$ is said to be in $\beta-US(\alpha)$, the class of β - uniformly starlike functions of order α, $0\le\alpha<1$, if satisfies the condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\beta\left|\frac{zf'(z)}{f(z)}-1\right|+\alpha \quad (\beta\ge 0), \qquad (4)$$

and a function $f\in \mathrm{A}$ is said to be in $\beta-UC(\alpha)$, the class of β- uniformly convex functions of order α, $0\le\alpha<1$, if satisfies the condition

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta\left|\frac{zf''(z)}{f'(z)}\right|+\alpha \quad (\beta\ge 0). \qquad (5)$$

Uniformly starlike and uniformly convex functions were first introduced by [1, 2] and then studied by various authors [3, 4]. It is known that $\beta-US(\alpha)$ or $\beta-UC(\alpha)$ if and only if $1+\frac{zf''(z)}{f'(z)}$ or $\frac{zf'(z)}{f(z)}$, respectively, takes all the values in the conic domain $R_{\beta,\alpha}$ which is included in the right half plane given by

$$R_{\beta,\alpha}=\omega=u+iv\in C: u>\beta\sqrt{(u-1)^2+v^2}+\alpha,$$
$$\beta\ge 0 \quad and \quad 0\le\alpha<1 \qquad (6)$$

Denote by $P_{\beta,\alpha}(\beta\ge 0, 0\le\alpha<1)$ of functions p, such that $p\in P_{\beta,\alpha}$, where P denotes well known class of Caratheodory functions. The function $P_{\beta,\alpha}$ maps the unit disk conformally onto the domain $R_{\beta,\alpha}$ such that $1\in R_{\beta,\alpha}$ and $\partial R_{\beta,\alpha}$ is a curve defined by the equality

$$\partial R_{\beta,\alpha}=\omega=u+iv\in C: u^2=\left(\beta\sqrt{(u-1)^2+v^2}+\alpha\right)^2,$$
$$\beta\ge 0 \quad and \quad 0\le\alpha<1. \qquad (7)$$

From elementary computations we see that (7) represents conic sections symmetric about the real axis. Thus $R_{\beta,\alpha}$ is an elliptic domain for $\beta>1$, a parabolic domain for $\beta=1$, a hyperbolic domain for $0<\beta<1$ and the right half plane $u>\alpha$, for $\beta=0$.

In [5], Sakaguchi (1959) defined the class S_s of starlike functions with respect to symmetric points as follows:

Let $f\in \mathrm{A}$. Then f is said to be starlike with respect to symmetric points in Υ if and only if

$$Re\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\}>0 \quad (z\in\Upsilon). \qquad 8)$$

Recently, Owa et. al. (2007) [6] defined and studied the class $S_s(\alpha,t)$,

$$Re\left\{\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right\}>\alpha \quad (z\in U),$$

where $0\le\alpha<1, |t|\le 1, t\ne 1.$ (9)

In 2008, Selvaraj and Karthikeyan [7] defined the following operator $D_\lambda^m(\alpha_1,\beta_1)f:\Upsilon\to\Upsilon$ by

$$D_\lambda^0(\alpha_1;\beta_1)f(z)=f(z)*G_{q,s}(\alpha_1,\beta_1;z),$$
$$D_\lambda^1(\alpha_1;\beta_1)f(z)=(1-\lambda)(f(z)*G_{q,s}(\alpha_1,\beta_1;z))$$
$$+\lambda z(f(z)*G_{q,s}(\alpha_1,\beta_1;z))'$$
$$D_\lambda^m(\alpha_1;\beta_1)f(z)=D_\lambda^1(D_\lambda^{m-1}(\alpha_1;\beta_1)f(z)), \quad (10)$$

where $m\in\mathrm{N}_0=\mathrm{N}\cup\{0\}$ and $\lambda\ge 0$.

If $f\in\mathrm{A}$, then from (10) we may easily deduce that

$$D_\lambda^m(\alpha_1;\beta_1)f(z)$$
$$=z+\sum_{n=2}^{\infty}[1+(n-1)\lambda]^m\Gamma_n(\alpha_1)a_nz^n \quad (11)$$

where $\Gamma_n(\alpha_1)=\dfrac{(\alpha_1)_{n-1}\ldots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_s)_{n-1}}\dfrac{1}{(n-1)!}.$

Special cases of the operator $D_\lambda^m(\alpha_1;\beta_1)f$ includes various other linear operators which were considered in many earlier work on the subject of analytic and univalent functions. If we let $m=0$ in $D_\lambda^m(\alpha_1;\beta_1)f$, we have

$$D_\lambda^0(\alpha_1;\beta_1)f(z)=\mathrm{H}_q^1(\alpha_1;\beta_1)f(z)$$

where $\mathrm{H}_{q,s}^1(\alpha_1;\beta_1)$ is Dziok-Srivastava operator for functions in A (see [8]) and for $q=2,s=1,\alpha_1=\beta_1,\alpha_2=1$ and $\lambda=1$, we get the operator introduced by Salagean (1983)([9]).

Definition 1.1 *A function $f(z)\in\mathrm{A}$ is said to be in the class $k-\Upsilon\Sigma_s(\lambda,\alpha_1,\beta_1,\gamma,t)$ if for all $z\in\Upsilon$,*

$$Re\left[\frac{(1-t)z\left(D_\lambda^m(\alpha_1,\beta_1)f(z)\right)'}{D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)}-\gamma\right]$$
$$\ge k\left|\frac{(1-t)z\left(D_\lambda^m(\alpha_1,\beta_1)f(z)\right)'}{D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)}\right| \quad (12)$$

for $\lambda\ge 0$, $k,m\ge 0$, $|t|\le 1$, $t\ne 1$ and $0\le\gamma<1$.

Furthermore, we say that a function $f(z)\in k-\Upsilon\Sigma_s(\lambda,\alpha_1,\beta_1,\gamma,t)$ is in the subclass $k-\Upsilon\Sigma\mathrm{T}_s(\lambda,\alpha_1,\beta_1,\gamma,t)$ if $f(z)$ is of the following form: $f(z)=z-\sum_{n=2}^{\infty}a_nz^n \quad (a_n\ge 0, n\in N).$ (13)

The main objective of this paper is to study the coefficient estimates, extreme points, distortion bounds and closure properties for $f(z)\in k-\Upsilon\Sigma\mathrm{T}_s(\lambda,\alpha_1,\beta_1,\gamma,t)$ by fixing second coefficients.

Similar other classes of univalent functions with fixed second coefficients have been extensively studied by Aouf (1997)[10, 11], S. M. Khairnar et. al., (2010), [12], Darwish (2008)[13], and others see [14].

2. Coefficient Estimate

Lemma 2.1 *Let $\omega=u+iv$. Then $Re\,\omega\ge\alpha$ if and only if $|\omega-(1+\alpha)|\le|\omega+(1+\alpha)|$.*

Lemma 2.2 *Let $\omega=u+iv$ and α,γ are real numbers. Then $Re\,\omega>\alpha|\omega-1|+\gamma$ if and only if $Re\{\omega(1+\alpha e^{i\theta})-\alpha e^{i\theta}\}>\gamma$.*

Theorem 2.3 *Let the function $f(z)$ be defined by (13). Then $f(z)\in k-\Upsilon\Sigma\mathrm{T}_s(\lambda,\alpha_1,\beta_1,\gamma,t)$ if and only if*

$$\sum_{n=2}^{\infty}C_n(m,\lambda,k,\gamma)a_n\le(1-\gamma), \quad (14)$$

where
$$C_n(m,\lambda,k,\gamma)=(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)\left[n(k+1)-(k+\gamma)u_n\right],$$

$\lambda\ge 0$, $k,m\ge 0$, $0\le\gamma<1$, $|t|\le 1$, $t\ne 1$ and $u_n=1+t+t^2+\cdots+t^n$.

The result is sharp for the function $f(z)$ is given by

$$f(z)=z-\frac{1-\gamma}{C_n(m,\lambda,k,\gamma)}z^n.$$

Proof. By definition $f(z)\in k-\Upsilon\Sigma\mathrm{T}_s(\lambda,\alpha_1,\beta_1,\gamma,t)$ if and only if the condition (12) is satisfied.

Then by Lemma 2.1, we have

$$Re\left[\frac{(1-t)z\left(D_\lambda^m(\alpha_1,\beta_1)f(z)\right)'}{D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)}\left(1+ke^{i\theta}\right)-ke^{i\theta}\right]$$
$$\ge\gamma, \quad -\pi<\theta\le\pi.$$

It is also written as

$$Re\left\{\frac{(1-t)z\left(D_\lambda^m(\alpha_1,\beta_1)f(z)\right)'\left(1+ke^{i\theta}\right)}{D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)}\right\}$$
$$-\frac{ke^{i\theta}D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)}{D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)}\ge\gamma. \quad (15)$$

Let $A(z)=(1-t)z\left(D_\lambda^m(\alpha_1,\beta_1)f(z)\right)'\left(1+ke^{i\theta}\right)$
$$-ke^{i\theta}\left(D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)\right)$$

and $B(z)=D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)$.

From Lemma 2.1 and Lemma 2.2,

$|A(z)+(1-\gamma)B(z)|\ge|A(z)-(1+\gamma)B(z)|$ *for* $0\le\gamma<1$.

Using a simple computation,

$|A(z)+(1-\gamma)B(z)|\ge$

$$|1-t|\left\{(2-\gamma)|z|-\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)\left[n+(1-\gamma)u_n\right]a_n|z|^n\right.$$

$$\left.-k\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)[n-u_n]a_n|z|^n\right\}.$$

Also, $|A(z)-(1+\gamma)B(z)|\le$

$$|1-t|\left\{\gamma|z|+\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)\left[n-(1+\gamma)u_n\right]a_n|z|^n\right.$$

$$\left.+k\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)[n-u_n]a_n|z|^n\right\}.$$

Now $\quad |A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|$

$$\ge\left\{2(1-\gamma)|z|-2\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)\left[n(k+1)-(k+\gamma)u_n\right]a_n|z|^n\right\}$$

$$\ge 0.$$

or

$$\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)\left[n(k+1)-(k+\gamma)u_n\right]a_n\le(1-\gamma).$$

which gives the desired estimation.

Conversely, suppose that (14) holds. Then we must show that

$$Re\left\{\frac{(1-t)z\left(D_\lambda^m(\alpha_1,\beta_1)f(z)\right)'\left(1+ke^{i\theta}\right)-ke^{i\theta}\left[D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)\right]}{D_\lambda^m(\alpha_1,\beta_1)f(z)-D_\lambda^m(\alpha_1,\beta_1)f(tz)}\right\}\ge\gamma.$$

Choosing the values of z on the positive real axis where $0\le z=r<1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma)-\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)\left(n(1+ke^{i\theta})-u_n(\gamma+ke^{i\theta})\right)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)u_na_nz^{n-1}}\right\}$$

$$\ge 0.$$

Since $Re(-e^{i\theta})\ge-|e^{i\theta}|=-1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma)-\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)\left(n(1+k)-(\gamma+k)u_n\right)a_nr^{n-1}}{1-\sum_{n=2}^{\infty}(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)u_na_nr^{n-1}}\right\}$$

$$\ge 0.$$

Letting $r\to1^-$, we have desired conclusion.

Corollary 2.4 *Let the function* $f(z)$ *defined by (13) be in the class* $k-\Upsilon\Sigma\mathrm{T}_s(\lambda,\alpha_1,\beta_1,\gamma,t)$, *then*

$$a_n\le\frac{(1-\gamma)}{(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)[n(k+1)-(k+\gamma)u_n]}$$
$$\le\frac{(1-\gamma)}{C_n(m,\lambda,k,\gamma)}\qquad(n\ge2).\tag{16}$$

where
$C_n(m,\lambda,k,\gamma)=(1+(n-1)\lambda)^m\Gamma_n(\alpha_1)[n(k+1)-(k+\gamma)u_n]$, $\lambda\ge0$, $k,m\ge0$ and $0\le\gamma<1$.

Setting $n=2$ in (16), we have

$$a_2\le\frac{(1-\gamma)(\beta_1)}{(1+\lambda)^m[(2+k-\gamma)-t(\gamma+k)](\alpha_1)}\le\frac{(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}.\tag{17}$$

where $C_2(m,\lambda,k,\gamma)=(1+\lambda)^m[(2+k-\gamma)-t(\gamma+k)]$.

Let $k-\Upsilon\Sigma\mathrm{T}_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$ denote the class of functions $f(z)$ in $k-\Upsilon\Sigma\mathrm{T}_s(\lambda,\alpha_1,\beta_1,\gamma,t)$ of the form

$$f(z)=z-\frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}z^2-\sum_{n=3}^{\infty}a_nz^n$$
$$(a_n\ge0),0\le c\le1.\tag{18}$$

Remark 2.1 *If* $\alpha_1=\beta_1=1$, *then the result is reduced to the class* $k-\widetilde{\Upsilon}\Sigma_s(\lambda,\mu,\gamma,t)$ *studied by Murat Caglar and Orhan [15]. Take* $t=0$, $\beta=0$ *and* $\lambda=1$, *this result is reduced into the class* $T(n,\alpha)$ *studied by Aouf [10]. If* $t=0$, $\beta=0$, $\alpha_1=\beta_1=1$ *and* $\lambda=1$, *then the result was reduced into the class* $UCT(\alpha,\beta)$ *by Khairanar (2010)[12].*

Theorem 2.5 *Let the function* $f(z)$ *be defined by (18). Then* $f(z)$ *in* $k-\Upsilon\Sigma\mathrm{T}_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$ *if and only if*

$$\sum_{n=3}^{\infty}C_n(m,\lambda,k,\gamma)a_n\le(1-\gamma)(1-c).\tag{19}$$

Proof. Substituting

$$a_2=\frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)},\qquad 0\le c\le1,$$ in (14) and simplifying we get the result.

Corollary 2.6 *Let the function* $f(z)$ *defined by (18) be in the class* $k-\Upsilon\Sigma\mathrm{T}_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$, *then*

$$a_n\le\frac{(1-\gamma)(1-c)}{C_n(m,\lambda,k,\gamma)}\qquad(n\ge3)\quad 0\le c\le1.\tag{20}$$

3. Extreme Points

Theorem 3.1 *Let*

$$f_2(z)=z-\frac{c(1-\gamma)(\beta_1)}{C_2(c,m,\lambda,k,\gamma)(\alpha_1)}z^2 \tag{21}$$

and

$$f_n(z)=z-\frac{c(1-\gamma)(\beta_1)}{C_2(c,m,\lambda,k,\gamma)(\alpha_1)}z^2-\frac{(1-\gamma)(1-c)}{C_n(m,\lambda,k,\gamma)}z^n \tag{22}$$

for $n=3,4,\cdots$. Then $f(z)$ is in the class $k-\mathrm{Y\Sigma T}_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$ if and only if it can be expressed in the form $f(z)=\sum_{n=2}^{\infty}\mu_n f_n(z)$, (23)

where $\mu_n \geq 0$ and $\sum_{n=2}^{\infty}\mu_n=1$.

Proof. We suppose that $f(z)$ can be expressed in the form (23). Then we have

$$f_n(z)=z-\frac{c(1-\gamma)(\beta_1)}{C_2(c,m,\lambda,k,\gamma)(\alpha_1)}z^2-\frac{(1-\gamma)(1-c)\mu_n}{C_n(m,\lambda,k,\gamma)}z^n. \tag{24}$$

Since $\sum_{n=3}^{\infty}\frac{(1-\gamma)(1-c)\mu_n}{C_n(m,\lambda,k,\gamma)}\frac{C_n(m,\lambda,k,\gamma)}{(1-\gamma)}$ (25)

$\leq(1-c)(1-\mu_2)$

$\leq(1-\gamma)$

it follows from (14) that $f(z)$ is in the class $k-\mathrm{Y\Sigma T}_s(c,\lambda,\alpha_1,\beta_1,\gamma)$. Conversely, suppose that $f(z)$ defined by (18) is in the class $k-\mathrm{Y\Sigma T}_s(c,\lambda,\alpha_1,\beta_1,\gamma)$. Then, by using (20), we get

$$a_n \leq \frac{(1-c)(1-\gamma)}{C_n(m,\lambda,k,\gamma)} \qquad (n\geq 3). \tag{26}$$

Setting

$$\mu_n=\frac{C_n(m,\lambda,k,\gamma)}{(1-c)(1-\gamma)}a_n \qquad (n\geq 3) \tag{27}$$

and $\mu_2=1-\sum_{n=3}^{\infty}\mu_n$, (28)

we have (23). This completes the proof of the theorem.

Corollary 3.2 *The extreme points of the class* $k-\mathrm{Y\Sigma T}_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$ *are functions* $f_n(z)$, $n\geq 2$ *given by (3.1).*

4. Growth and Distortion Theorem

The following lemmas are required in our investigation of growth and distortion properties of the general class $k-\mathrm{Y\Sigma T}_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$.

Lemma 4.1 *Let the function* $f_3(z)$ *be defined by*

$$f3(z)=z-\frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}z^2-\frac{(1-\gamma)(1-c)}{C_3(m,\lambda,k,\gamma)}z^3, \tag{29}$$

For some n. Then for $0\leq r<1$ and $0\leq c\leq 1$,

$$\left|f_3(re^{i\theta})\right|\geq r-\frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r^2-\frac{(1-\gamma)(1-c)}{C_3(m,\lambda,k,\gamma)}r^3 \tag{30}$$

with equality for $\theta=0$. For either $0\leq c<c_0$ and $0\leq r\leq r_0$ or $c_0\leq c\leq 1$,

$$\left|f_3(re^{i\theta})\right|\leq r+\frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r^2-\frac{(1-\gamma)(1-c)}{C_3(m,\lambda,k,\gamma)}r^3 \tag{31}$$

with equality for $\theta=\pi$. Further, for $0\leq c\leq c_0$ and $r_0\leq r<1$,

$$\left|f_3(re^{i\theta})\right|\leq r\left\{\left(1+\frac{c^2(1-\gamma)\beta_1^2C_3(m,\lambda,k,\gamma)}{2(1-c)\left(C_2(m,\lambda,k,\gamma)\right)^2\alpha_1^2}\right)\right.$$
$$+\left(\frac{2c^2(1-\gamma)^2\beta_1^2}{\left(C_2(m,\lambda,k,\gamma)\right)^2\alpha_1^2}+\frac{2(1-c)(1-\gamma)(\beta_1)}{\left(C_3(m,\lambda,k,\gamma)\right)^2}\right)r^2$$
$$\left.+\left(\frac{(1-c)^2(1-\gamma)^2}{\left(C_3(m,\lambda,k,\gamma)\right)^2}+\frac{c^2(1-c)(1-\gamma)^3\left(\beta_1^2\right)}{2\left(C_2(m,\lambda,k,\gamma)\right)^2\left(C_3(m,\lambda,k,\gamma)\alpha_1^2\right)}\right)r^4\right\}^{\frac{1}{2}} \tag{32}$$

with equality for

$$\theta=cos^{-1}\left(\frac{c(1-c)(1-\gamma)(\beta_1)r^2-c(C_3(m,\lambda,k,\gamma))(\beta_1)}{4(1-c)(C_2(m,\lambda,k,\gamma))(\alpha_1)r}\right) \tag{33}$$

where

$$c_0=\frac{\left[(1-\gamma)(\beta_1)-4\left(C_2(m,\lambda,k,\gamma)\right)(\alpha_1)-C_3(m,\lambda,k,\gamma)(\beta_1)\right]+c_0^*}{2(1-\gamma)(\beta_1)}, \tag{34}$$

$$c_0^*=\left\{\left[(1-\gamma)(\beta_1)-4\left(C_2(m,\lambda,k,\gamma)\right)(\alpha_1)-C_3(m,\lambda,k,\gamma)(\beta_1)\right]^2\right.$$
$$\left.+16C_2(m,\lambda,k,\gamma)(1-\gamma)(\alpha_1)(\beta_1)\right\}^{\frac{1}{2}}$$

and $r_0=\dfrac{-2(1-c)(C_2(m,\lambda,k,\gamma)(\alpha_1))+r_0^*}{c(1-c)(1-\gamma)(\beta_1)}$, (35)

$$r_0^*=\left\{4(1-c)^2\left(C_2(m,\lambda,k,\gamma)\right)^2(\alpha_1^2)+c^2(1-c)(1-\gamma)C_3(m,\lambda,k,\gamma)(\beta_1^2)\right\}^{\frac{1}{2}}.$$

Proof. We employ the same technique as used by Silverman and Silvia (1981)([16]), since

$$\frac{\partial\left|f3(re^{i\theta})\right|^2}{\partial\theta}=2(1-\gamma)r^3\sin\theta\times$$

$$\left[\frac{c(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}-\frac{c(1-c)(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)C_3(m,\lambda,k,\gamma)(\alpha_1)}r^2+\frac{4(1-c)}{C_3(m,\lambda,k,\gamma)}r\cos\theta\right] \quad (36)$$

we can see that

$$\frac{\partial\left|f_3(re^{i\theta})\right|^2}{\partial\theta}=0$$

for $\theta_1=0$, $\theta_2=\pi$ and

$$\theta_3=\cos^{-1}\left(\frac{c(1-c)(1-\gamma)(\beta_1)r^2-c(\beta_1)C_3(m,\lambda,k,\gamma)}{4(1-c)C_2(m,\lambda,k,\gamma)(\alpha_1)r}\right) \quad (37)$$

Since θ_3 is a valid root only when $-1\le\cos\theta_3\le 1$, we have a third root if and only if $r_0\le r<1$ and $0\le c<c_0$. Thus the results of Lemma 4.1 follow upon comparing the extremal values $\left|f_3(re^{i\theta_k})\right|$, $(k=1,2,3)$ on the appropriate intervals.

Lemma 4.2 *Let the function* $f_n(z)$ $(n\ge 4)$ *be defined by (22). Then*

$$\left|f_n\left(re^{i\theta}\right)\right|\le\left|f_4(-r)\right| \qquad (n\ge 4). \quad (38)$$

Proof. Since

$$f_n(z)=z-\frac{c(1-\gamma)(\beta_1)}{C_2(c,m,\lambda,k,\gamma)(\alpha_1)}z^2-\frac{(1-\gamma)(1-c)}{C_n(m,\lambda,k,\gamma)}z^n$$

and $\frac{(1-\gamma)(1-c)}{C_n(m,\lambda,k,\gamma)}r^n$ is a decreasing function of n, we have

$$\left|f_n\left(re^{i\theta}\right)\right|\le r+\frac{c(1-\gamma)(\beta_1)}{C_2(c,m,\lambda,k,\gamma)(\alpha_1)}r^2+\frac{(1-\gamma)(1-c)}{C_n(m,\lambda,k,\gamma)}r^4=-f_4(-r),$$

which proves (38).

Theorem 4.3 *Let the function* $f(z)$ *defined by (18) be in the class* $k-\Upsilon\Sigma\mathrm{T}_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$. *Then, for* $0\le r<1$,

$$\left|f\left(re^{i\theta}\right)\right|\ge r-\frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r^2-\frac{(1-c)(1-\gamma)}{C_3(m,\lambda,k,\gamma)}r^3 \quad (39)$$

with equality for $f_3(z)$ at $z=r$ and

$$\left|f\left(re^{i\theta}\right)\right|\le\max\left\{\max_{\theta}\left|f_3(re^{i\theta})\right|,-f_4(-r)\right\} \quad (40)$$

where $\max_\theta\left|f_3(re^{i\theta})\right|$ is given by Lemma 4.1.

The proof of this theorem is obtained by comparing the bounds given by Lemma 4.1 and Lemma 4.2. Putting $c=1$ in theorem 4.3, we obtain the following corollary.

Corollary 4.4 *Let the function* $f(z)$ *defined by (13) be in the class* $k-\Upsilon\Sigma\mathrm{T}_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$. *Then, for* $0\le r<1$,

$$r-\frac{(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r^2\le|f(z)|\le r+\frac{(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r^2. \quad (41)$$

The result is sharp for the function

$$f(z)=z-\frac{(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}z^2. \quad (42)$$

Putting $c=1$ and $k=0$ in theorem 4.3, we obtain the following corollary.

Corollary 4.5 *Let the function* $f(z)$ *defined by (13) be in the class* $k-\Upsilon\Sigma\mathrm{T}_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$. *Then, for* $0\le r<1$,

$$r-\frac{(1-\gamma)(\beta_1)}{C_2(m,\lambda,0,\gamma)(\alpha_1)}r^2\le|f(z)|\le r+\frac{(1-\gamma)(\beta_1)}{C_2(m,\lambda,0,\gamma)(\alpha_1)}r^2. \quad (43)$$

The result is sharp for the function

$$f(z)=z-\frac{(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}z^2. \quad (44)$$

Remark 4.1 *If* $\alpha_1=\beta_1=1$, $m=1$ *and* $t=-1$ *then the above result (corollary 4.4) is reduced to the class* $S_s(\lambda,k,\beta)$ *studied by C.Selvaraj et.al. (2009)[17],*

$$r-\frac{(1-\gamma)}{2(1+k)(1+\lambda)}r^2\le|f(z)|\le r+\frac{(1-\gamma)(\beta_1)}{2(1+k)(1+\lambda)}r^2. \quad (45)$$

Lemma 4.6 *Let the function* $f_3(z)$ *be defined by (29). Then, for* $0\le r<1$ *and* $0\le c\le 1$,

$$\left|f_3'(re^{i\theta})\right|\ge 1-\frac{2c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r-\frac{3(1-\gamma)(1-c)}{C_3(m,\lambda,k,\gamma)}r^2 \quad (46)$$

with equality for $\theta=0$. For either $0\le c<c_1$ and $r_1\le r\le 1$,

$$\left|f_3'(re^{i\theta})\right|\le 1+\frac{2c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r-\frac{3(1-\gamma)(1-c)}{C_3(m,\lambda,k,\gamma)}r^3 \quad (47)$$

with equality for $\theta=\pi$. Furthermore, for $0\le c\le c_1$ and $r_1\le r<1$,

$$\left|f_3'(re^{i\theta})\right|\le\left\{\left(1+\frac{c^2(1-\gamma)\beta_1^2C_3(m,\lambda,k,\gamma)}{3(1-c)\left(C_2(m,\lambda,k,\gamma)\right)^2\alpha_1^2}\right)+\left(\frac{4c^2(1-\gamma)^2\beta_1^2}{\left(C_2(m,\lambda,k,\gamma)\right)^2\alpha_1^2}+\frac{6(1-c)(1-\gamma)(\beta_1)}{\left(C_3(m,\lambda,k,\gamma)\right)^2}\right)r^2+\left(\frac{9(1-c)^2(1-\gamma)^2}{\left(C_3(m,\lambda,k,\gamma)\right)^2}+\frac{6c^2(1-c)(1-\gamma)^3(\beta_1^2)}{2\left(C_2(m,\lambda,k,\gamma)\right)^2\left(C_3(m,\lambda,k,\gamma)\alpha_1^2\right)}\right)r^4\right\}^{\frac{1}{2}} \quad (48)$$

with equality for

$$\theta = cos^{-1}\left(\frac{3c(1-c)(1-\gamma)(\beta_1)r^2 - c(C_3(m,\lambda,k,\gamma))(\beta_1)}{6(1-c)(C_2(m,\lambda,k,\gamma))(\alpha_1)r}\right) \quad (49)$$

where

$$c_1 = \frac{3[(1-\gamma)(\beta_1) - 6(C_2(m,\lambda,k,\gamma))(\alpha_1) - C_3(m,\lambda,k,\gamma)(\beta_1)] + c_1^*}{6(1-\gamma)(\beta_1)}, \quad (50)$$

$$c_1^* = \{3[(1-\gamma)(\beta_1) - 6C_2(m,\lambda,k,\gamma)(\alpha_1) - C_3(m,\lambda,k,\gamma)(\beta_1)]^2 + 72C_2(m,\lambda,k,\gamma)(1-\gamma)(\alpha_1)(\beta_1)\}^{\frac{1}{2}}$$

and $r_1 = \dfrac{-6(1-c)(C_2(m,\lambda,k,\gamma)(\alpha_1)) + r_1^*}{6c(1-c)(1-\gamma)(\beta_1)}$, (51)

$$r_1^* = \{36(1-c)^2(C_2(m,\lambda,k,\gamma))^2(\alpha_1^2) + 12c^2(1-c)(1-\gamma)C_3(m,\lambda,k,\gamma)(\beta_1^2)\}^{\frac{1}{2}}.$$

The proof is omitted.

Theorem 4.7 *Let the function $f(z)$ defined by (18) be in the class $k-Y\Sigma T_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$. Then, for $0 \le r < 1$,*

$$\left|f_3'(re^{i\theta})\right| \ge 1 - \frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r - \frac{(1-c)(1-\gamma)}{C_3(m,\lambda,k,\gamma)}r^2 \quad (52)$$

with equality for $f_3'(z)$ at $z = r$ and

$$\left|f_3'(re^{i\theta})\right| \le \max\left\{\max_\theta \left|f_3'(re^{i\theta})\right|, -f_4'(-r)\right\} \quad (53)$$

where $\max_\theta \left|f_3'(re^{i\theta})\right|$ is given by Lemma 4.6.

5. Radii of Starlikeness and Convexity

Theorem 5.1 *Let the function $f(z)$ defined by (18) be in the class $k-Y\Sigma T_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$. Then $f(z)$ is starlike of order ρ $(0 \le \rho < 1)$ in the disc $|z| = r_1(c,\rho,m,\lambda,k,\gamma)$, where $r_1(c,\rho,m,\lambda,k,\gamma)$ is the largest value for which*

$$\frac{c(1-\gamma)(2-\rho)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r + \frac{(1-\gamma)(1-c)(n-\rho)}{C_n(m,\lambda,k,\gamma)}r^{n-1} \le 1-\rho \qquad (n \ge 3). \quad (54)$$

The result is sharp with the extremal function

$$f_n(z) = z - \frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}z^2 - \frac{(1-\gamma)(1-c)}{C_n(m,\lambda,k,\gamma)}z^n, \quad (55)$$

Proof. It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1-\rho \quad (0 \le \rho < 1)$$

for $|z| = r_1(c,\rho,m,\lambda,k,\gamma)$. We note that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r + \sum_{n=3}^{\infty}(n-1)a_n r^{n-1}}{1 - \frac{c(1-\gamma)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r - \sum_{n=3}^{\infty}a_n r^{n-1}} \le 1-\rho, \quad (56)$$

$|z| \le r$ if and only if

$$\frac{c(1-\gamma)(2-\rho)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r + \sum_{n=3}^{\infty}(n-\rho)a_n r^{n-1} \le 1-\rho \quad (57)$$

Since $f(z)$ is in the class $k-Y\Sigma T_s(c,\lambda,\alpha_1,\beta_1,\gamma)$, from (14) we may take

$$a_n = \frac{(1-c)(1-\gamma)}{C_n(m,\lambda,k,\gamma)}\mu_n \qquad (n = 3,4,5,\ldots) \quad (58)$$

where $\mu_n \ge 0$ and $\sum_{n=3}^{\infty}\mu_n \le 1$.

For each fixed r, we choose the positive integer $n_0 = n_0(r)$ for which $\dfrac{(n_0-\rho)}{C_{n_0}(m,\lambda,k,\gamma)}$ is maximal. Then it follows that

$$\sum_{n=3}^{\infty}(n-\rho)a_n r^{n-1} \le \frac{(1-c)(1-\gamma)(n_0-\rho)}{C_{n_0}(m,\lambda,k,\gamma)}r^{n_0-1}. \quad (59)$$

Hence $f(z)$ is starlike is of order ρ in $|z| = r_1(c,\rho,m,\lambda,k,\gamma)$ provided that

$$\frac{c(1-\gamma)(2-\rho)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r + \frac{(1-\gamma)(1-c)(n_0-\rho)}{C_{n_0}(m,\lambda,k,\gamma)}r^{n_0-1} \le 1-\rho. \quad (60)$$

We find the value $r_1 = r_1(c,\rho,m,\lambda,k,\gamma)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{c(1-\gamma)(2-\rho)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r_0 + \frac{(1-\gamma)(1-c)(n_0-\rho)}{C_{n_0}(m,\lambda,k,\gamma)}r_0^{n_0-1} = 1-\rho. \quad (61)$$

Then this value r_0 is the radius of starlikeness of order ρ for functions $f(z)$ belonging to the class $k-Y\Sigma T_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$.

In similar manner, we can prove the following theorem concerning the radius of convexity of ρ for functions in the class $k-Y\Sigma T_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$.

Theorem 5.2 *Let the function $f(z)$ defined by (18) be in the class $k-Y\Sigma T_s(c,\lambda,\alpha_1,\beta_1,\gamma,t)$. Then $f(z)$ is convex of order ρ, $(0 \le \rho < 1)$ in the disc*

$|z| = r_2(c,\rho,m,\lambda,k,\gamma)$, *where* $r_2(c,\rho,m,\lambda,k,\gamma)$ *is the largest value for which*

$$\frac{2c(1-\gamma)(2-\rho)(\beta_1)}{C_2(m,\lambda,k,\gamma)(\alpha_1)}r + \frac{(1-\gamma)(1-c)n(n-\rho)}{C_n(m,\lambda,k,\gamma)}r^{n-1} \le 1-\rho \qquad (n \ge 3). \tag{62}$$

The result is sharp for the function $f(z)$ given by (55).

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