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Second Hankel Determinant for Multivalent Spirallike and Convex Functions of Order α

M. Elumalai[1], C.Selvaraj[2]

[1]Department of Mathematics, Presidency College, Chennai, Tamil Nadu, India

Abstract: ***The objective of this paper is to obtain an upper bounded to the second Hankel determinant*** $|a_{p+1}a_{p+3}-a_{p+2}^2|$ ***for*** $\alpha-$ ***spiral starlike and convex*** $\alpha-$ ***spiral function of*** f ***and using Teoplitz determinants.***

Keywords: Analytic functions, multivalent functions, α -spiral starlike functions, convex α -spiral function, upper bound, second Hankel determinant, positive real function, Toeplitz determinants.

1. Introduction and Definitions

Let A_p denote the class of functions analytic in U and having the power series expansion

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k \qquad (p\in \mathbb{N}=\{1,2,3,...\}) \qquad (1.1)$$

in the open unit disc $U=\{z:|z|<1\}$. Let S be the subclass of $A_1=A$, consisting of univalent functions.

In 1976, Noonan and Thomas defined the q^{th} Hankel determinant of f for $q\geq 1$ and $k\geq 1$ as

$$H_q(k)=\begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+q-1} & a_{k+q} & \cdots & a_{k+2q-2} \end{vmatrix}. \qquad (1.2)$$

This determinant has been considered by several authors in the literature. For example, Noonan and Thomas [21] studied about the second Hankel determinant of a really mean p -valent functions. Noor [22] determined the rate of growth of $H_q(k)$ as $k\to\infty$ for functions in U with bounded boundary rotation. Ehrenborg [8] considered the Hankel determinant of exponential polynomials. In [16], Layman considered Handel transform and obtained integrating properties.

Also, the Hankel determinant has been studied by various authors including Hayman [13] and Pommerenke [25]. We observe that $H_2(1)$ is nothing but the classical Fekete-Szegö functional. Then Fekete-Szegö further generalizes the estimate $|a_3-\mu a_2^2|$, where μ is real and $f\in U$. Ali [3] finds sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $\gamma_3-t\gamma_2^2$, where t is real. For our discussion in this paper, we consider the Hankel determinant for the case $q=2$ and $k=2$, known as second Hankel determinant,

$$H_2(2)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2. \qquad (1.3)$$

Janteng, Halim and Darus [14] have determined the functional $|a_2a_4-a_3^2|$ and found a sharp bound for the functions f in the subclass RT of U, consisting of functions whose derivative has a positive real part, studied by Mac Gregor [17]. In this work, he has shown that if $f\in RT$, then $|a_2a_4-a_3^2|\leq\frac{4}{9}$. In [12], the authors obtained the second Hankel determinant and sharp upper bounds for the familiar subclasses namely, starlike and convex functions denoted by ST and CV of U and have shown that $|a_2a_4-a_3^2|\leq 1$ and $|a_2a_4-a_3^2|\leq\frac{1}{8}$, respectively. Mishra and Gochhayat [18] have obtained sharp bound to the non-linear functional $|a_2a_4-a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha,\rho)$ $\left(0\leq\rho\leq p, 0\leq\lambda<p, |\alpha|<\frac{\pi}{2p}\right)$. Similarly, the same coefficient inequality is calculated for certain subclasses of analytic functions by many authors, see e.g. [1], [4], [5], [10-12], [18], [19], [25], [27-35].

Motivated by the earlier works obtained by different authors in this direction, we in the present paper, seek upper bound of the functional $|a_2a_4-a_3^2|$ for functions f belonging to the classes $SP_p(\alpha)$ and $CVSP_p(\alpha)$, defined as follows:

Definition 1.1
A function $f\in A_p$ given by (1.1) is said to be $p-$ valently $\alpha-$ spiral if it satisfies the inequality

$$\mathrm{Re}\frac{1}{p}\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\}\geq 0, \qquad \forall\ (z\in U), \qquad |\alpha|\leq\frac{\pi}{2p}. (1.4)$$

We denote this class of functions by $SP_p(\alpha)$. Note that the class $SP_p(\alpha)$ reduces to $SP_1(\alpha)=:SP(\alpha)$, the class of $\alpha-$ spiral functions introduced by spacek [30] and when $p=1$ and $\alpha=0$, it is ST, the class of starlike functions.

Definition 1.2
A function $f\in A_p$ is said to be convex $\alpha-$ spiral, where $\left(|\alpha|\leq\frac{\pi}{2p}\right)$, if it satisfies the condition

$$\mathrm{Re}\frac{1}{p}\left\{e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\geq 0, \qquad \forall\ z\in U. \qquad (1.5)$$

The class of convex $\alpha-$ spiral functions introduced by Rotertson is denoted by $CVSP_p(\alpha)$.
It is observed when $\alpha=0$, $CVSP_p(0)=CV$.

Further, from the Definition 1.1 and 1.2, it is observed that, the Alexander type theorem [2] becomes true for the classes $SP_p(\alpha)$ and $CVSP_p(\alpha)$, stated as follow.

$$f(z)\in CVSP_p(\alpha) \quad \text{if and only if} \quad \frac{zf'(z)}{p}\in SP_p(\alpha).$$

Some preliminary Lemmas needed for proving our results are as follows :

2. Preliminary Results

Let A be the family of all functions h analytic in U, for which $Re\{h(z)\}>0$ and

$$h(z)=1+\sum_{n=1}^{\infty}c_n z^n, \quad \forall\ z\in \mathrm{U}. \tag{2.1}$$

Lemma 2.1.
[9] If $h\in \mathrm{A}$, then $|c_k|\le 2$, for each $k\ge 1$.

Lemma 2.2.
[10] The power series for h given in (2.1) converges in the unit disc U to a function in A if and only if the Toeplitz determinants.

$$D_k=\begin{vmatrix} 2 & c_1 & c_2 & \dots & c_k \\ c_{-1} & 2 & c_1 & \dots & c_{k-1} \\ \vdots & \vdots & \vdots & \vdots & \\ c_{-k} & c_{-k+1} & c_{-k+2} & \dots & 2 \end{vmatrix}, \quad k=1,2,3,\dots$$

and $c_{-k}=\overline{c_k}$, are all non-negative. These are strictly positive except for $h(z)=\sum_{k=1}^{m}\rho_k h_0 e^{it_k z}$, $\rho_k>0$, t_k real and $t_k\ne t_j$, for $k\ne j$, in this case $D_k>0$ for $k<(m-1)$ and $D_k=0$ for $k\ge m$.

This necessary and sufficient condition due to Carathéodory and Toeplitz can be found in [10]. We may assume without restriction that $c_1>0$ and on using Lemma 2.2, for $k=2$ and $k=3$ respectively, we get

$$D_2=\begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c_1} & 2 & c_1 \\ \overline{c_2} & \overline{c_1} & 2 \end{vmatrix}=\left[8+2\mathrm{Re}\{c_1^2 c_2\}-2|c_2|^2-4c_1^2\right]\ge 0,$$

which is equivalent to

$$2c_2=\left\{c_1^2+x(4-c_1^2)\right\}, \quad \text{for some} \quad x, \quad |x|\le 1. \tag{2.2}$$

$$D_3=\begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c_1} & 2 & c_1 & c_2 \\ \overline{c_2} & \overline{c_1} & 2 & c_1 \\ \overline{c_3} & \overline{c_2} & \overline{c_1} & 2 \end{vmatrix}.$$

Then $D_3\ge 0$ is equivalent to

$$\begin{aligned}(4c_3-4c_1c_2+c_1^3)(4-c_1^2)+c_1(2c_2-c_1^2)^2 \\ \le 2(4-c_1^2)^2-2|2c_2-c_1^2|^2 .\end{aligned} \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$\begin{aligned}4c_3=c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2 \\ +2(4-c_1^2)(1-|x|^2)z\end{aligned} \tag{2.4}$$

for some real value of x, with $|x|\le 1$.

3. Main Result

Theorem 3.1.

If $f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k\in SP_p(\alpha)\left(-\frac{\pi}{2p}\le\alpha\le\frac{\pi}{2p}\right)$, then $|a_{p+1}a_{p+3}-a_{p+2}^2|\le p^2\cos^2\alpha$.

Proof:

Since $f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k\in SP_p(\alpha)$, from the definition (1.1), there exists an analytic function $h\in \mathrm{A}$ in the unit disk U with $h(0)=1$ and $Re\{h(z)\}>0$ such that

$$\begin{aligned}e^{-i\alpha}\left\{\frac{zf'(z)}{pf(z)}\right\}=h(z)\Rightarrow\left\{e^{-i\alpha}zf'(z)+ip\sin\alpha f(z)\right\} \\ =p\cos\alpha\{f(z)\times h(z)\}.\end{aligned} \tag{3.1}$$

Replacing $f(z)$, $f'(z)$ by their equivalent p-valent expressions and also the equivalent expression for $h(z)$ in series in (3.1), we have

$$e^{-i\alpha}z\left\{pz^{p-1}+\sum_{k=p+1}^{\infty}ka_k z^{k-1}\right\}+ip\sin\alpha\left\{z^p+\sum_{k=p+1}^{\infty}a_k z^k\right\}$$

$$=p\cos\alpha\left[\left\{z^p+\sum_{k=p+1}^{\infty}a_k z^k\right\}\times\left\{1+\sum_{n=1}^{\infty}c_n z^n\right\}\right].$$

Upon simplification, we obtain

$$\begin{aligned}e^{-i\alpha}(a_{p+1}z^p+2a_{p+2}z^{p+1}+\dots)=p\cos\alpha(c_1z^p \\ +(c_2+c_1a_{p+1})z^{p+1}+(c_3+c_2a_{p+1}+c_1a_{p+2})z^{p+2}+\dots).\end{aligned} \tag{3.2}$$

Equating the coefficients of like powers of z^p, z^{p+1} and z^{p+2} respectively in (3.2), we have

$$a_{p+1}e^{-i\alpha}=c_1p\cos\alpha,$$

$$2a_{p+2}e^{-i\alpha}=(c_2+c_1a_{p+1})p\cos\alpha$$

$$3a_{p+3}e^{-i\alpha}=(c_3+c_2a_{p+1}+c_1a_{p+2})p\cos\alpha.$$

After simplifying, we get

$$\begin{aligned}a_{p+1}&=e^{i\alpha}c_1p\cos\alpha,\\ a_{p+2}&=\frac{e^{i\alpha}}{2}(c_2+c_1^2e^{i\alpha}\ p\cos\alpha)p\cos\alpha\\ a_{p+3}&=\frac{e^{i\alpha}}{6}(c_3+3c_1c_2e^{i\alpha}\ p\cos\alpha+c_1^3e^{i\alpha}\ p^2\cos^2\alpha)p\cos\alpha.\end{aligned} \tag{3.3}$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from (3.3) in the second Hankel functional $|a_{p+1}a_{p+3}-a_{p+2}^2|$ for the function $f\in SP_p(\alpha)$, we have

$$\begin{aligned}|a_{p+1}a_{p+3}-a_{p+2}^2|=|e^{i\alpha}c_1p\cos\alpha \\ \times\frac{e^{i\alpha}}{6}\{2c_3+3c_1c_2e^{i\alpha}p\cos\alpha+c_1^3e^{2i\alpha}p\cos^2\alpha\}p\cos\alpha \\ -\frac{e^{2i\alpha}}{4}\{c_2+c_1^2e^{i\alpha}\ p\cos\alpha\}^2\ p^2\cos^2\alpha|.\end{aligned}$$

Using the facts $|xa+yb|\le|x||a|+|y||b|$, where x, y, a and b are real numbers and $|e^{ni\alpha}|=1$, upon simplification, we obtain

$$|a_{p+1}a_{p+3}-a_{p+2}^2|\le\frac{p^2\cos^2\alpha}{12}\times|4c_1c_3-3c_2^2-c_1^4p^2\cos^2\alpha|. \tag{3.4}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively, we have

$$|4c_1c_3-3c_2^2-c_1^4p^2\cos^2\alpha|=|4c_1 \times\frac{1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z\} -3\times\frac{1}{4}\{c_1^2+x(4-c_1^2)\}^2-c_1^4p^2\cos^2\alpha|.$$

Using the fact $|x|\le 1$, upon simplification, we obtain

$$4|4c_1c_3-3c_2^2-c_1^4p^2\cos^2\alpha|\le|(1-4p^2\cos^2\alpha)c_1^4+8c_1(4-c_1^2) +2c_1^2(4-c_1^2)|x|-(c_1+2)(c_1+6)(4-c_1^2)|x|^2|.$$

Since $c_1\in[0,2]$, using the result $(c_1+a)(c_1+b)\ge(c_1-a)(c_1-b)$, where $a,b\ge 0$ on the right-hand side of the above inequality we get

$$4|4c_1c_3-3c_2^2-c_1^4p^2\cos^2\alpha|\le|(1-4p^2\cos^2\alpha)c_1^4 +8c_1(4-c_1^2)+2c_1^2(4-c_1^2)|x|-(c_1-2)(c_1-6)(4-c_1^2)|x|^2|. \tag{3.5}$$

Choosing $c_1=c, c\in[0,2]$, applying triangle inequality and replacing $|x|$ by δ on the right hand side of (3.5) we obtain

$$4|4c_1c_3-3c_2^2-c_1^4p^2\cos^2\alpha|\le|(4p^2\cos^2\alpha-1)c_1^4+8c_1(4-c_1^2) +2c_1(4-c_1^2)\delta+(c_1-2)(c_1-6)(4-c_1^2)\delta^2|. =F(c,\delta),\text{ with } 0\le\delta=|x|\le 1, \tag{3.6}$$

where

$$F(c,\delta)=(4p^2\cos^2\alpha-1)c_1^4+8c_1(4-c_1^2)+2c_1(4-c_1^2)\delta +(c_1-2)(c_1-6)(4-c_1^2)\delta^2. \tag{3.7}$$

Now the function $F(c,\delta)$ is maximized on the closed square $[0,2]\times[0,1]$. Differentiating $F(c,\delta)$ in (3.7), partially with respect to δ, we get

$$\frac{\partial F}{\partial\delta}=2[c^2+(c-2)(c-6)\delta](4-c^2) \tag{3.8}$$

for $0\le\delta\le 1$, for fixed c with $0\le c\le 2$, from (3.8) we observe that $\frac{\partial F}{\partial\delta}>0$.

Consequently, $F(c,\delta)$ is an increasing function of δ and hence cannot have maximum value at any point in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$, we have

$$\max_{0\le\delta\le 1}F(c,\delta)=F(c,1)=G(c). \tag{3.9}$$

Upon simplifying the relation (3.7) and (3.9) we obtain

$$G(c)=4(p^2\cos^2\alpha-1)c^4+48. \tag{3.10}$$

Differentiation yields:

$$G'(c)=16(p^2\cos^2\alpha-1)c^3. \tag{3.11}$$

From the expression (3.11), we observe that $G'(c)\le 0$ from all values of c in the interval $0\le c\le 2$ and for a fixed valued of α with $(-\frac{\pi}{2p}\le\alpha\le\frac{\pi}{2p})$. Therefore, $G(c)$ is a monotonically decreasing function of c in the interval $[0,2]$. So, that its maximum value occurs at $c=0$. From (3.10), we get

$$\max_{0\le c\le 2}G(0)=48. \tag{3.12}$$

After simplifying the expressions (3.6) and (3.12) we obtain

$$|4c_1c_3-3c_2^2-c_1^4p^2\cos^2\alpha|\le 12. \tag{3.13}$$

Upon simplifying the expressions (3.4) and (3.13), we get

$$|a_{p+1}a_{p+3}-a_{p+2}^2|\le p^2\cos^2\alpha. \tag{3.14}$$

Choosing $c_1=c=0$ and selecting $x=-1$ in (2.2) and (2.4), we find that $c_2=-2$ and $c_3=0$. Substituting these values in (3.13), it is observed that equality is attained which shows that our result is sharp. This completes the proof of our Theorem 3.1.

Choosing $p=1$ from (3.14) following

Corollary 3.2.

[37] If $f(z)\in SP(\alpha)$, then $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$

$|a_2a_4-a_3^2|\le\cos^2\alpha.$

For the choice of $p=1$ and $\alpha=0$ from (3.14) following

Corollary 3.3.

If $f(z)\in SP(\alpha)$, then

$|a_2a_4-a_3^2|\le 1.$

This inequality is sharp and concides with that of Janteng, Halim and Darus [14].

Theorem 3.4.

If $f(z)\in CVSP_p(\alpha)$ $(|\alpha|\le\frac{\pi}{2p})$, then

$$|a_{p+1}a_{p+3}-a_{p+2}^2| \le\frac{p^4\left\{\begin{array}{l}6(1+2p\cos\alpha+p^2\cos^2\alpha)+(p+1)(p+3)\\ \quad(p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha)\end{array}\right\}}{(p+1)(p+2)^2(p+3)\left\{\begin{array}{l}2(p^2+4p+1)\\ \quad+(p^2+4p+7)p^2\sec^2\alpha\end{array}\right\}}.$$

Proof:

Since $f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k\in CVSP_p(\alpha)$, from the definition (1.2), there exists an analytic function $h\in A$ in the unit disk U with $h(0)=1$ and $Re\{h(z)\}>0$ such that

$$\frac{1}{p}\left[e^{-i\alpha}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right]=h(z)\Leftrightarrow\{e^{-i\alpha}\{f'(z)+zf''(z)\}+ip\sin\alpha f'(z)\} =p\cos\alpha\{f'(z)\times h(z)\} \tag{3.15}$$

Replacing $f'(z)$, $f''(z)$ and $h(z)$ with their equivalent series expressions in (3.1), we have

$$e^{-i\alpha}\left\{pz^{p-1}+\sum_{k=p+1}^{\infty}ka_kz^{k-1}\right\}+z\left\{p(p-1)z^{p-2}+\sum_{k=p+1}^{\infty}k(k-1)a_kz^{k-2}\right\} +ip\sin\alpha\left\{z+\sum_{k=2}^{\infty}a_kz^k\right\} =p\cos\alpha\left[\left\{pz^{p-1}+\sum_{k=p+1}^{\infty}ka_kz^{k-1}\right\}\times\left\{1+\sum_{n=1}^{\infty}c_nz^n\right\}\right].$$

Upon simplification, we obtain

$$e^{-i\alpha}\{(p+1)a_{p+1}z^p+2(p+2)a_{p+2}z^{p+1}+3(p+3)a_{p+3}z^{p+2}+\ldots\} =p\cos\alpha[pc_1z^p+\{pc_2+(p+1)c_1a_{p+1}\}z^{p+1} +\{pc_3+(p+1)c_2a_{p+1}+(p+2)c_1a_{p+2}\}z^{p+2}+\ldots]. \tag{3.16}$$

Equating the coefficients of like powers of z^p, z^{p+1} and z^{p+2} respectively in (3.16), we have

$$(p+1)a_{p+1}e^{-i\alpha}=pc_1p\cos\alpha,$$

$$2(p+2)a_{p+2}e^{-i\alpha}=\{pc_2+(p+1)c_1a_{p+1}\}p\cos\alpha,$$

$$3(p+2)a_{p+3}e^{-i\alpha}=\{pc_3+(p+1)c_2a_{p+1}+(p+2)c_1a_{p+2}\}p\cos\alpha$$

After simplifying, we get

$$a_{p+1}=\frac{e^{i\alpha}}{p+1}c_1\ p^2\cos\alpha$$

$$a_{p+2}=\frac{e^{i\alpha}}{2(p+2)}(c_2+c_1^2e^{i\alpha}\ p\cos\alpha)\ p^2\cos\alpha \qquad (3.17)$$

$$a_{p+3}=\frac{e^{i\alpha}}{6(p+3)}(c_3+3c_1c_2e^{i\alpha}\ p\cos\alpha+c_1^3e^{2i\alpha}\ p^2\cos^2\alpha)\ p^2\cos\alpha$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from (3.17) in the second Hankel functional $|a_{p+1}a_{p+3}-a_{p+2}^2|$ for the function $f\in CVSP_p(\alpha)$, applying the same procedure as described in Theorem
3.1, upon simplification, we obtain

$$|a_{p+1}a_{p+3}-a_{p+2}^2|=\frac{p^4\cos^2\alpha}{24(p+1)(p+2)^2(p+3)}|8(p+2)^2c_1c_3+12p\,c_1^2c_2\cos\alpha -6(p+1)(p+3)c_2^2-(p^2+4p+7)2p^2c_1^4\cos^2\alpha|.$$

The above expression is equivalent to

$$|a_{p+1}a_{p+3}-a_{p+2}^2|=\frac{p^4\cos^2\alpha}{12(p+1)(p+2)^2(p+3)}\times\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right| \qquad (3.18)$$

where

$$d_1=4(p+2)^2\ ,\ d_2=6p\cos\alpha\ ,\ d_3=-3(p+1)(p+3)=-3(p^2+4p+3) \qquad (3.19)$$

$$d_4=-(p^2+4p+7)p^2\cos^2\alpha.$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from lemma 2.2 in the right hand side of (3.18), we have

$$\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|=\Big|d_1c_1\times\frac{1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2 +2(4-c_1^2)(1-|x|^2)z\}+d_2c_1^2\times\frac{1}{2}\{c_1^2+x(4-c_1^1)\} +d_3\times\{c_1^2+x(4-c_1^2)\}^2+d_4c_1^4\Big|.$$

After simplifying, we get

$$4\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|=|(d_1+2d_2+d_3+4d_4)c_1^4 +2d_1c_1(4-c_1^2)z+2d_1c_1(4-c_1^2)z+2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x| +\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\}(4-c_1^2)|x|^2\,z| \qquad (3.20)$$

Using the values of d_1, d_2, d_3 and d_4 from the relation (3.19), upon simplification, we obtain

$$d_1+2d_2+d_3+4d_4=p^2+4p+7+12p\cos\alpha -4(p^2+4p+1)p^2\cos^2\alpha$$

$$d_1=4(p+2)^2 \qquad (3.21)$$

$$d_1+d_2+d_3=p^2+4p+7-6p\cos\alpha.$$

$$(d_1+d_3)c_1^2+2d_1c_1-4d_3=(p^2+4p+7)c_1^2-8(p+2)^2c_1 +12(p+1)(p+3)\}. \qquad (3.22)$$

Consider

$$\{(p^2+4p+7)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\}=$$

$$(p^2+4p+7)\times\left[c_1^2+\frac{8(p+2)^2}{(p^2+4p+7)}c_1+\frac{12(p+1)(p+3)}{(p^2+4p+7)}\right]$$

$$=(p^2+4p+7)\times\left[\left\{c_1+\frac{4(p+2)^2}{(p^2+4p+7)}\right\}^2-\frac{16(p+2)^4}{(p^2+4p+7)^2}+\frac{12(p+1)(p+3)}{(p^2+4p+7)}\right]$$

Upon simplification, the above expression can be expressed as

$$\{(p^2+4p+7)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\}=(p^2+4p+7) \times\left[\left\{c_1+\frac{4(p+2)^2}{(p^2+4p+7)}\right\}^2-\left\{\frac{2\sqrt{p^4+8p^3+18p^2+8p+1}}{(p^2+4p+7)}\right\}^2\right].$$

$$\{(p^2+4p+1)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\}=(p^2+4p+7) \times\left[c_1+\left\{\frac{4(p+2)^2}{(p^2+4p+7)}+\frac{2\sqrt{p^4+8p^3+18p^2+8p+1}}{(p^2+4p+7)}\right\}\right] \times\left[c_1+\left\{\frac{4(p+2)^2}{(p^2+4p+7)}-\frac{2\sqrt{p^4+8p^3+18p^2+8p+1}}{(p^2+4p+7)}\right\}\right]. \qquad (3.23)$$

Since $c_1\in[0,2]$, using the results $(c_1+a)(c_1+b)\geq(c_1-a)(c_1-b)$, where $a,b\geq 0$ in the right hand side of (3.23)

$$\{(p^2+4p+1)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\} \geq\{(p^2+4p+1)c_1^2-8(p+2)^2c_1+12(p+1)(p+3)\}. \qquad (3.24)$$

From the relation (3.22) and (3.24), we obtain

$$-\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\}\geq-\{(p^2+4p+1)c_1^2 -8(p+2)^2c_1+12(p+1)(p+3)\}. \qquad (3.25)$$

Substituting the calculated values from (3.21) and (3.25) in the right hand side of the relation (3.20), we get

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\leq\big|\{p^2+4p+7+12p\cos\alpha -4(p^2+4p+1)p^2\cos^2\alpha\}c_1^4+2(p^2+4p+7+6p\cos\alpha)c_1^2(4-c_1^2)|x| -\{(p^2+4p+7)c_1^2-8(p+2)^2c_1+12(p+1)(p+3)\}(4-c_1^2)|x|^2\,z\big|. \qquad (3.26)$$

Choosing $c_1=c\in[0,2]$, applying Triangle inequality and replacing $|x|$ by μ in the right hand side of (3.20), it reduces to

$$=F(c,\delta),\ \text{for } 0\leq\delta=|x|\leq 1, \qquad (3.27)$$

where

$$F(c,\mu)=\big[\{p^2+4p+7+12p\cos\alpha-4(p^2+4p+1)p^2\cos^2\alpha\}c_1^4 +8(p+2)^2c_1(4-c_1^2)z+2(p^2+4p+7+6p\cos\alpha)c_1^2(4-c_1^2)\delta +\{(p^2+4p+7)c_1^2-8(p+2)^2c_1+12(p+1)(p+3)\}(4-c_1^2)\delta^2\big]$$

$$=F(c,\delta),\ \text{for } 0\leq\delta=|x|\leq 1. \qquad (3.28)$$

We assume that the upper bound for (3.27) occurs at an interior point of the set $\{(\delta,c):\delta\in[0,1] \text{ and } c\in[0,2]\}$. Differentiating $F(c,\delta)$ in (3.28) partially with respect to δ, we get

$$\frac{\partial F}{\partial\delta}=\big[2(p^2+4p+7+6p\cos\alpha)c^2(4-c^2) +2\{(p^2+4p+7)c^2-8(p+2)^2c_1+12(p+1)(p+3)\}(4-c^2)\delta\big]. \qquad (3.29)$$

For $0\leq\delta\leq 1$, for fixed c with $0\leq c\leq 2$ and $(-\frac{\pi}{2p}\leq\alpha\leq\frac{\pi}{2p})$,

from (3.29), we observe that $\frac{\partial F}{\partial\delta}>0$. Therefore, $F(c,\delta)$ is an increasing function of μ, which contradicts our assumption that the maximum value of it occurs at an interior point of the set $\{(\delta,c):\delta\in[0,1] \text{ and } c\in[0,2]\}$.

Further, for a fixed $c\in[0,2]$, we have

$$\max_{0\leq\delta\leq 1}F(c,\delta)=F(c,1)=G(c),\ \text{say}. \qquad (3.30)$$

From the relations (3.28) and (3.30), upon simplification, we obtain

$$G(c)=\big[-2\{p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha\}c^4 +48(1+p\cos\alpha)c^2+48(p+1)(p+3)\big]. \qquad (3.31)$$

$$G'(c) = -8\{p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha\}c^3 + 96(1+p\cos\alpha)c. \tag{3.32}$$

$$G''(c) = -24\{p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha\}c^2 + 96(1+p\cos\alpha). \tag{3.33}$$

The maximum or minimum value of $G(c)$ is obtained for the values of $G'(c)=0$. From the expression (3.32), we get

$$8\{p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha\}c^3 + 96(1+p\cos\alpha)c = 0. \tag{3.34}$$

We now discuss the following cases.

Case 1. If $c=0$, then from (3.33), we obtain

$$G''(c) = 96(1+p\cos\alpha) > 0, \text{ because } |\alpha| \le \frac{\pi}{2p}.$$

Therefore, by the second derivative test, $G(c)$ has a minimum value at $c=0$, which is ruled out.

Case 2. If $c \ne 0$, then from (3.34), we obtain

$$c^2 = \frac{12(1+p\cos\alpha)}{p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha}.$$

Using the value of c^2 given (3.35) in (3.33), after simplifying, we get

$$G''(c) = -192(1+p\cos\alpha) < 0, \text{ because } |\alpha| \le \frac{\pi}{2p}.$$

From the second derivative test, $G(c)$ has a maximum value at c, where c^2 is given by (3.35). From the expression (3.31), we have $G-$maximum value at c^2, after simplifying, it is given by

$$\max_{0\le c\le 2} G(c) = \frac{288(1+2p\cos\alpha+p^2\cos^2\alpha)+48(p+1)(p+3) + p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha}{p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha} \tag{3.36}$$

Considering only the maximum value of $G(c)$ at c, where c^2 is given by (3.35). From the expressions (3.27) and (3.36), upon simplification, we obtain

$$|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \le \frac{72(1+2p\cos\alpha+p^2\cos^2\alpha)+12(p+1)(p+3) + p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha}{p^2+4p+7+2(p^2+4p+1)p^2\cos^2\alpha} \tag{3.37}$$

From the expressions (3.18) and (3.37), after simplifying, we get

$$|a_{p+1}a_{p+3}-a_{p+2}^2| \le \frac{p^4\left\{\begin{array}{l}6(1+2p\cos\alpha+p^2\cos^2\alpha)+(p+1)(p+3)(p^2+4p+7\\ \qquad +2(p^2+4p+1)p^2\cos^2\alpha)\end{array}\right\}}{(p+1)(p+2)^2(p+3)\{2(p^2+4p+1)+(p^2+4p+7)p^2\sec^2\alpha\}} \tag{3.38}$$

This completes the proof of the theorem 3.4.

Choosing $p=1$ in (3.38) we have the following

Corollary 3.5.

[37] If $f(z)\in CVSP(\alpha)$, then

$$|a_2a_4-a_3^2| \le \frac{17(1+\cos^2\alpha)+2\cos\alpha}{144(1+\sec^2\alpha)}.$$

For the choice of $p=1$ and $\alpha=0$ in (3.38) we have the following

Corollary 3.6.

If $f(z)\in CV(\alpha)$, then

$$|a_2a_4-a_3^2| \le \frac{1}{8}.$$

This inequality is sharp and concides with that of Janteng, Halim and Darus [14].

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