$$\frac{n}{2} b_1^2 b_2^{\frac{(n+1)(n-2)}{2}} + \frac{n}{2} b_2^{\frac{n(n-1)}{2}} \quad (3.1.15)$$

Combining 3.7 and 3.15 we have the cycle index formula as;

$$Z_{D_n, X^{[2]}} = \frac{1}{2n} \left[\sum_{\substack{d|n \\ 2|d}} \phi(d) \left(b_d^{\frac{n(n-1)}{d}} \right) + \sum_{\substack{d|n \\ 2\nmid d}} \phi(d) \left(b_d^{\frac{n(n-1)}{d}} \right) + \frac{n}{2} b_1^2 b_2^{\frac{(n+1)(n-2)}{2}} + \frac{n}{2} b_2^{\frac{n(n-1)}{2}} \right] \quad (3.1.16)$$

3.2 Case 2: If n is odd.

We first consider the cyclic part $Z_{C_n} = \frac{1}{n} \sum_{d|n} \phi(d) t_d^{\frac{n}{d}}$

In this case d must be odd because it is a divisor of an odd number.

If the elements in the pair come from a common cycle, then from 2.1 we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{\frac{n}{d}(d-1)} \quad (3.2.1)$$

If the elements come from different cycles of length d then from 2.4 we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{2d\binom{\frac{n}{d}}{2}} = b_d^{\frac{n(n-d)}{d}} \quad (3.2.2)$$

Combining (3.2.1) and (3.2.2) we have

$$b_d^{\frac{n(n-d)}{d} + \frac{n}{d}(d-1)} = b_d^{\frac{n(n-1)}{d}} \quad (3.2.3)$$

Therefore the cycle index of C_n acting on $X^{[2]}$ when n is odd is given by;

$$\frac{1}{n} \left[\sum_{d|n} \phi(d)\, b_d^{\frac{n(n-1)}{d}} \right] \quad (3.2.4)$$

To study the monomials induced by the reflection symmetries we note that all the reflection symmetries of a regular n-gon with n odd have their lines of symmetry passing through a vertex and an edge.

We now consider the monomials induced by the part $t_1 t_2^{\frac{n-1}{2}}$.

If the two elements in a pair come from a common cycle then from 2.2 we have;

$$t_2^{\frac{n-1}{2}} \longrightarrow b_2^{\frac{n-1}{2}} \quad (3.2.5)$$

If the two elements come from different cycles then we have two cases.

(a) One element from a cycle of length one and the other from a cycle of length two. Then from (2.3) we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow b_2^{n-1} \quad 3.2.6(a)$$

(b) If each come from a different cycle of length two then from 2.4 we have;

$$t_2^{\frac{n-1}{2}} \longrightarrow b_2^{4\binom{\frac{n-1}{2}}{2}} = b_2^{\frac{n^2-4n+3}{2}} \quad 3.2.6(b).$$

Combining (3.2.5), 3.2.6(a) and 3.2.6(b) we have;

$$b_2^{4\binom{\frac{n-1}{2}}{2} + n - 1 + \frac{n-1}{2}} = b_2^{\frac{n(n-1)}{2}}$$

But from 2.5(b) we have n monomials of the form $t_1 t_2^{\frac{n-1}{2}}$ and hence n monomials will be induced giving;

$$n\, b_2^{\frac{n(n-1)}{2}} \quad (3.2.7)$$

Combining (3.2.4) and (3.2.7) we have the cycle index formula as;

$$Z_{D_n, X^{[2]}} = \frac{1}{2n} \left[\sum_{d|n} \phi(d)\, b_d^{\frac{n(n-1)}{d}} + n\, b_2^{\frac{n(n-1)}{2}} \right] \quad (3.2.8)$$

4. Conclusion

The cycle index formulas of D_n acting on ordered pairs are given as;

$$Z_{D_n, X^{[2]}} = \frac{1}{2n} \left[\sum_{\substack{d|n \\ 2|d}} \phi(d) \left(b_d^{\frac{n(n-1)}{d}} \right) + \sum_{\substack{d|n \\ 2\nmid d}} \phi(d) \left(b_d^{\frac{n(n-1)}{d}} \right) + \frac{n}{2} b_1^2 b_2^{\frac{(n+1)(n-2)}{2}} + \frac{n}{2} b_2^{\frac{n(n-1)}{2}} \right] \text{ if n is even}$$

And

$$Z_{D_n, X^{[2]}} = \frac{1}{2n} \left[\sum_{d|n} \phi(d)\, b_d^{\frac{n(n-1)}{d}} + n\, b_2^{\frac{n(n-1)}{2}} \right] \quad \text{if n is odd}$$

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