On Non-Homogeneous Bi-Quadratic Diophantine Equation $4(x^2+y^2)-7xy = 19z^4$

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Abstract: *Five different methods of the non-zero integral solutions of the homogeneous biquadratic Diophantine equation with five unknowns $4(x^2+y^2)-7xy = 19z^4$ are determined. Some interesting relations among the special numbers and the solutions are exposed.*

Keywords: Quadratic, non-homogenous, integer solutions, special numbers, polygonal, and pyramidal numbers

Mathematics Subject Classification: 11D09

Notations used:
- $T_{m,n}$: Polygonal number of rank $n$ with sides $m$
- $p_{n}^m$ : Pyramidal number of rank $m$ with side $n$
- $G_n$ : Gnomonic number of rank $n$
- $f_{4,3}^r$ : Fourth dimensional figurate number of rank $r$, whose generating polygon is a Triangle
- $f_{4,4}^r$ : Fourth dimensional figurate number of rank $r$, whose generating polygon is a Square
- $f_{4,5}^r$ : Fourth dimensional figurate number of rank $r$, whose generating polygon is a Pentagon
- $f_{4,6}^r$ : Fourth dimensional figurate number of rank $r$, whose generating polygon is a Hexagon
- $f_{4,7}^r$ : Fourth dimensional figurate number of rank $r$, whose generating polygon is a Heptagon
- $f_{4,8}^r$ : Fourth dimensional figurate number of rank $r$, whose generating polygon is a Octagon

1. Introduction

The number theory is the queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In 2014, Jayakumar. P, Sangeetha. K, [12] have published a paper in finding the integer solutions of the homogeneous Biquaratic Diophantine equation $(x^3 - y^3)z = (W^2 - P^2)R^4$. In 2015, Jayakumar. P, Meena.J [14, 15] published two papers in finding integer solutions of the homogeneous Biquadratic Diophantine equation $(x^3 - y^3) = 26 (z^2 - w^2) R^2$ and $(x^4 - y^4) = 40 (z^4 - w^4) R^2$. Inspired by these, in this work, we are observed another interesting five different methods of the non-zero integral solutions of the non homogeneous biquadratic Diophantine equation with three unknowns $4(x^2 + y^2) - 7xy = 19z^4$. Further, some elegant properties among the special numbers and the solutions are exposed.

2. Description of Method

Consider biquadratic Diophantine equation

$$4(x^2+y^2) - 7xy = 19z^4 \quad (1)$$

We solved (3) through various choices and the different methods of solutions of (1) are obtained as follows.

2.1 Method: I

Consider (3) as $u^2 + 15v^2 = 15z^4 + 4z^4$ and take it as in the form of ratio as

$$\frac{u + 2z^2}{15(z^2 + v)} = \frac{z^2 - v}{u - 2z^2} = \frac{a}{b} \neq 0 \quad (4)$$

(4) is equivalent to the system of equations as

$$6u - 15av + (2b - 15a)z^2 = 0$$

and

$$- au - bv + (2a + b)z^2 = 0 \quad (5)$$

By the cross multiplication method, the above equations yields as

$$u = 30a^2 - 2b^2 = 30ab$$

$$v = 15a^2 + b^2 + 4ab \quad (7)$$

$$z^2 = 15a^2 + b^2$$

If we take $a = 2pq$, $b = 15p^2 - q^2$ in (7) and using (2), then we find

$$x = x(p, q) = - 225p^4 - q^4 + 90p^2q^2 + 1020p^3q - 68pq^3$$
\[ y = y(p, q) = -675p^4 + 3p^4 + 270p^2q + 780p^3q - 52pq \]

This gives us the non-zero different integer values to (1)

**Observations**

1. \[ x(p, 1) + 1350 \int_{4,6} P \int_{p,5} + 4375T_{=4} + G_{=2051} = 0 (\text{Mod } 2) \]

2. \[ y(p, 1) + 18 \int_{4,6} P \int_{p,5} - 319T_{=4} - G_{=2051} = 0 (\text{Mod } 2) \]

3. \[ x(p, 1) + 24 \int_{4,6} P \int_{p,5} - 2T_{=4} + 58 P_{=4} + 157T_{=4} - G_{=2051} = 0 (\text{Mod } 2) \]

4. \[ y(p, 1) + 16200 \int_{4,6} P \int_{p,5} + 960P_{=4} + 2865T_{=4} + G_{=2051} = 0 (\text{Mod } 2) \]

5. \( x(p, p) \) is a perfect square.

**2.2 Method: II**

Instead of (4), the form of ratio as

\[
\frac{u + 2v^2}{z^2 - v} = \frac{15(z^2 + v^2)}{u - 2z^2}, \quad b \neq 0
\]

The procedure following is same as the method -I, the relating integer solutions to (1) are found as

\[ x = x(p, q) = 675p^2 + 3q^4 - 270p^2q + 780pq - 52pq \]

\[ y = y(p, q) = 225p^4 + q^4 + 90p^2q + 1020pq - 68pq \]

\[ z = z(p, q) = 15p^2 + q^2 \]

**Observations:**

1. \( x(p, 1) + 36 \int_{4,6} P \int_{p,5} + 221T_{=4} + G_{=2051} = 0 (\text{Mod } 2) \]

2. \( x(1, q) + 16200 \int_{4,6} P \int_{p,5} + 3375T_{=4} + 20400 P_{=4} - 5700T_{=4} + G_{=2051} = 0 (\text{Mod } 2) \]

3. \( y(1, q) + 24 \int_{4,6} P \int_{p,5} + 234T_{=4} + 165 P_{=4} + 157T_{=4} - G_{=2051} = 0 (\text{Mod } 2) \)

4. \( y(q, 1) - 1350 \int_{4,6} P \int_{p,5} - 690 P_{=4} + 4375T_{=4} + G_{=2051} = 0 (\text{Mod } 2) \)

5. \( z(1, 1) \) is a Nasty Number.

**2.3 Method: III**

Let us take

\[ 19 = (2 + i \sqrt{5}) (2 - i \sqrt{5}) \]

Take z as

\[ z = z(a, b) = a^2 + 15b^2 \]

Using (9) and (10) is (3) and applying factorization process, define

\[ u + i \sqrt{5}v = (2 + i \sqrt{5})(a + i \sqrt{5}b)^2 \]

This gives us

\[ u^2 = 2a^4 + 450b^4 - 180a^2b^2 - 60a^3b + 90ab^3 \]

\[ v^2 = a^2 + 225b^4 - 90a^2b^2 + 8a^3b - 120ab^3 \]

Using (11), in (2), the relating integer solutions to (1) are found as

\[ x = x(a, b) = 3a^4 + 675b^4 - 270a^2b^2 - 52ab + 780ab^3 \]

\[ y = y(a, b) = 225a^4 + q^4 + 90a^2q + 1020a^2q - 68aq^3 \]

\[ z = z(a, b) = 15a^2 + q^2 \]

**Observations:**

1. \( x(1, A) + y(1, A) - 10800 \int_{4,6} P \int_{p,5} + 3600P_{A} + 3060T_{A}A + G_{=2051} = 0 (\text{Mod } 2) \)

2. \( x(1, A) - y(1, A) - 10800 \int_{4,6} P \int_{p,5} + 1800T_{A}A + 13080 P_{A} - 3210T_{A}A + G_{=2051} = 0 (\text{Mod } 2) \)

3. \( x(1, A) - 16200 \int_{4,6} P \int_{p,5} + 6540P_{A} + 4425T_{=4} + G_{=2051} = 0 (\text{Mod } 2) \)

4. \( y(1, A) - 1350 \int_{4,6} P \int_{p,5} + 690P_{A} + 885T_{=4} + G_{=2051} = 0 \)

5. \( 6z(1, 0) \) is a Nasty Number.

**2.4 Method: IV**

In place of (9) take 19 as

\[ 19 = \frac{(17 + i \sqrt{5})(17 - i \sqrt{5})}{16} \]

The procedure following is same as the method -III, the relating integer solutions to (1) are found as

\[ u = \frac{1}{4} [17a^4 + 3825b^4 + 1530a^2b^2 + 900ab^3 - 60a^2b] \]

\[ v = \frac{1}{4} [a^4 + 225b^4 - 90a^2b^2 - 102ab^3 + 68a^2b] \]

In true of (2), the values x and y are

\[ x = \frac{1}{4} [18a^4 + 405b^4 - 1620a^2b^2 - 120ab^3 + 8a^2b] \]

\[ y = \frac{1}{4} [16a^4 + 3650b^4 - 1440ab^3 + 1920ab^3 - 128ab^3] \]

Since our intention is to find integer solutions, taking a as 4a and b as 4b in (4),(15) and (16), the relating parametric integer values of (1) are found as

\[ x = x(A, B) = 576A^4 + 129600B^4 - 51840A^2B^2 + 256A^2B - 3840AB^3 \]

\[ y = y(A, B) = 512A^4 + 115200B^4 - 46080A^2B^2 - 4096A^2B + 61440AB^3 \]

\[ z = 16A^2 + 240B^2 \]

**Observations:**

1. \( x(1, n) + y(1, n) - 1468800 \int_{4,6} P \int_{p,5} + 1353600P_{n} - 89280T_{A} + G_{=2051} = 1 (\text{Mod } 2) \)

2. \( x(1, n) - y(1, n) - 346500 \int_{4,6} P \int_{p,5} - 418560P_{n} - 2880T_{A} + G_{=2051} = 1 (\text{Mod } 2) \)

3. \( x(q, 1) - 13824 \int_{4,6} P \int_{p,5} + 2304T_{q} + 15616P_{q} + 48064T_{A} + G_{=2051} = 1 (\text{Mod } 2) \)

4. \( y(q, 1) - 6144 \int_{4,6} P \int_{p,5} + 12288P_{q} + 42496T_{q} - G_{=2051} = 0 (\text{Mod } 5) \)

5. \( z(A, A) \) is a perfect square.

**2.5 Method: V**

Let us take (3) as \( u^2 + 15v^2 = 19z^2 + 1 \) \( \text{and take } 1 = \frac{(1 + i \sqrt{5})(1 - i \sqrt{5})}{16} \)

Using (9) and (10) in (13) and applying factorization process, define

\[ u + i \sqrt{5}v = (2 + i \sqrt{5})(a + i \sqrt{5}b)^2 \]

It furnishes us

\[ u = \frac{1}{4} [-13a^4 - 2925b^4 + 1170ab^3 + 2700ab^3 - 180ab^3] \]

\[ v = \frac{1}{4} [3a^4 + 675b^4 - 270ab^3 + 780ab^3 - 52ab^3] \]

In sight of (2), the values of x and y as
\[ x = x(a, b) = 1 - 10a^4 - 2250b^4 + 4900a^2b^2 + 3480ab^3 - 232a b \]  
(21)

\[ y = y(a, b) = 1 - 16a^4 - 3600a^2b^2 + 1440ab^3 + 1920ab^2 - 128a b \]  
(22)

As our intention is to find integer solutions, taking \( a = 4A \) and \( b = 4B \) in (4), (21) and (22), the relating parametric integer values of (1) are found as:

\[ x = x(a, B) = -320A^4 - 7200A^2B^2 + 28800A^2B^2 - 7424A^2B + 111360AB^2 \]

\[ y = y(a, B) = -512A^4 - 11520B^4 + 46080A^2B^2 - 4096A^2B + 61440AB^2 \]

\[ z = 16A^4 + 240B^2 \]

Observations:

1. \[ (x(1, B) + y(1, B) - 2246400 + G_{19296B} \equiv 1 \text{ (Mod 2)} \]

2. \[ (x(1, A) + 7680 f_{4,2} + 49664T_{4,4A} - G_{56640A} \equiv 1 \text{ (Mod 2)} \]

3. \[ (x(1, A) + 3072 f_{4,2} + 5120 p_3^5 - 8x_12T_{4,4A} - G_{3072A} \equiv 1 \text{ (Mod 2)} \]

4. \[ 4,4 \text{ } f_{4,4} \text{ } x_{4,4} + 245760 p_3^4 + 110400T_{4,4A} + G_{44464A} = 0 \text{ (Mod 2)} \]

5. If \( z(1, 0) \) is a nasty number, then \( 8 \)

3. Conclusion

In this paper, we have observed various process of determining infinitely a lot of non-zero different integer values to the non-homogeneous bi-quadratic Diophantine equation \( 4(x^2 + y^2) - 7xy = 19z^2 \). One may try to find non-negative integer solutions of the above equations together with their similar observations.

References


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