On Non-Homogeneous Bi-Quadratic Diophantine Equation $4(x^2+y^2)-7xy = 19z^4$

Dr. P. Jayakumar[1], V. Pandian[2], R. Venkatraman[3]

[1]Professor of Mathematics, Periyar Maniammai University, Vallam, Thanajvur -613 403, T N, India

[2]Assistant Professor of Mathematics, A.V.V.M.Sri Pushpam College, Poondi -613 503, Thanajvur, TN, India

[3]Assistant Professor of Mathematics, SRM University Vadapalani Campus, Chennai - 600026.T.N, India

Abstract: ***Five different methods of the non-zero integral solutions of the homogeneous biquadratic Diophantine equation with five unknowns $4(x^2 + y^2) -7xy =19z^4$ are determined. Some interesting relations among the special numbers and the solutions are exposed***

Keywords: Quadratic, non-homogenous, integer solutions, special numbers, polygonal, and pyramidal numbers **2010**

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Notations used:

$T_{m,n}$: Polygonal number of rank n with sides m.

p_n^m : Pyramidal number of rank m with side n

G_n : Gnomonic number of rank n

$f_{4,3}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Triangle

$f_{4,4}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Square

$f_{4,5}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Pentagon

$f_{4,6}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Hexagon

$f_{4,7}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Heptagon

$f_{4,8}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Octagon

1. Introduction

The number theory is the queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In 2014, Jayakumar. P, Sangeetha. K, [12] have published a paper in finding the integer solutions of the homogeneous Biquaratic Diophantine equation $(x^3 - y^3) z = (W^2 – P^2)R^4$.

In 2015, Jayakumar. P, Meena.J [14, 15] published two papers in finding integer solutions of the homogeneous Biquaratic Diophantine equation $(x^4 – y^4) = 26 (z^2 – w^2) R^2$ and $(x^4 – y^4) = 40 (z^2 – w^2) R^2$ Inspired by these , in this work, we are observed another interesting five different methods of the non-zero integral solutions of the non homogeneous biquadratic Diophantine equation with three unknowns $4(x^2 + y^2) -7xy =19z^4$. Further, some elegant properties among the special numbers and the solutions are exposed.

2. Description of Method

Consider biqudratic Diophantine equation

$$4(x^2+ y^2) – 7xy = 19z^4 \quad (1)$$

Introduce the linear transformations

$$x = u + v,\ y = u – v \quad (2)$$

Using (2) in (1), this gives to

$$u^2 + 15v^2 = 19z^4 \quad (3)$$

We solved (3) through various choices and the different methods of solutions of (1) are obtained as follows.

2.1 Method: I

Consider (3) as $u^2 + 15v^2 = 15z^4 +4z^4$
and take it as in the form of ratio as

$$\frac{u+2z^2}{15(z^2+v)} = \frac{z^2-v}{u-2z^2} = \frac{a}{b},\ b \neq 0 \quad (4)$$

(4) is equivalent to the system of equations as

$$6u – 15av + (2b -15a)z^2 = 0 \quad (5)$$

$$- au – bv + (2a + b)z^2 = 0 \quad (6)$$

By the cross multiplication method, the above equations yields as

$$\left.\begin{aligned} u &= 30a^2 - 2b^2 + 30ab \\ v &= -15a^2 + b^2 + 4ab \\ z^2 &= 15a^2 + b^2 \end{aligned}\right\} \quad (7)$$

If we take a =2pq, b =15p^2 – q^2 in (7) and using (2), then we find

$x = x (p, q) = - 225p^4 – q^4+ 90p^2q^2 + 1020p^3q – 68pq^3$

$y = y(p, q) = -675p^4 - 3p^4 + 270p^2q^2 + 780p^3q - 52pq^3$
$z = z(p, q) = 15p^2 + q^2$
This gives us the non- zero different integer values to (1)

Observations

1. $x(p, 1) + 1350\, f_{4,6}^{p} - 3390\, p_p^5 + 1155T_{4,p} + G_{3,4p} \equiv 0 \pmod 2$
2. $y(1, p) + 18\, f_{4,6}^{p} + 80\, p_p^{\ 5} - 319T_{4,p} - G_{290p} \equiv 0 \pmod 2$
3. $x(1, p) + 24\, f_{4,6}^{p} - 2T_{4,p2} + 58\, p_p^5 - 157T_{4,p} - G_{511p} \equiv 0 \pmod 2$
4. $y(p,1) + 16200\, f_{4,6}^{p} - 9660\, p_p^5 - 2865T_{4,p} - G_{1999p} \equiv 0 \pmod 2$
5. z (p, p) is a perfect square.

2.2 Method: II

Instead of (4) take the form of ratio as

$$\frac{u+2z^2}{z^2-v} = \frac{15(z^2+v)}{u-2z^2} = \frac{a}{b},\ b \neq 0 \qquad (8)$$

The procedure following is same as the method -I, the relating integer solutions to (1) are found as
$x = x(p, q) = 675p^4 + 3q^4 - 270p^2q^2 + 780p^3q - 52pq^3$
$y = y(p, q) = 225p^4 + q^4 - 90p^2q^2 + 1020p^3q - 68pq^3$
$z = z(p, q) = 15p^2 + q^2$

Observations:-

1. $.x(1, q) - 36\, f_{4,4}^{q} + 128\, p_p^5 + 221T_{4,q} + G_{387q} \equiv 0 \pmod 2$
2. $x(q, 1) - 16200\, f_{4,8}^{q} + 3375T_{4,q2} + 20040\, p_q^5 - 5700T_{4,q} - G_{1324q} \equiv 0 \pmod 2$
3. $y(1, q) - 24\, f_{4,7}^{q} + 4T_{4,q2} + 164\, p_q^5 + 15T_{4,q} - G_{511q} \equiv 0 \pmod 2$
4. $y(q, 1) - 1350\, f_{4,6}^{q} - 690\, p_q^{\ 5} + 885T_{4,q} + G_{34q} = 0$
5. $\frac{3}{8}$ z (1, 1) is a Nasty Number.

2.3 Method: III

Let us take

$$19 = (2 + i\sqrt{15})\,(2 - i\sqrt{15}) \qquad (9)$$

Take z as

$$z = z(a, b) = a^2 + 15b^2 \qquad (10)$$

Using (9) and (10) is (3) and applying factorization process, define
$u + i\sqrt{15}\, v = (2 + i\sqrt{15})\,(a + i\sqrt{15}\, b)^4$

This gives us
\u $= 2a^4 + 450b^4 - 180a^2b^2 - 60a^3b + 900ab^3$

$$v = a^4 + 225b^4 - 90a^2b^2 + 8a^3b - 120ab^3 \qquad (11)$$

Using (11) in (2), the relating integer solutions to (1) are found as $x = x(a, b) = 3a^4 + 675b^4 - 270a^2b^2 - 52a^3b + 780ab^3$ $y = y(a, b) = a^4 + 225b^4 - 90a^2b^2 - 68a^3b + 1020ab^3$ $z = a^2 + 15b^2$

Observations: 1. $x(1, A) + y(1, A) - 10800\, f_{4,4}^{A} + 3600\, p_A^5 + 3060_{4,A} + G_{960A} \equiv 1 \pmod 2$

2. $x(1, A) - y(1, A) - 10800\, f_{4,7}^{A} + 1800T_{4,A2} + 13080\, p_A^5 - 3210T_{4,A} - G_{3728A} \equiv 1 \pmod 2$
3. $x(1, A) - 16200\, f_{4,3}^{A} + 6540\, p_A^5 + 4425T_{4,4} + G_{2051A} \equiv 0 \pmod 2$
4. $y(1, A) - 1350\, f_{4,6}^{A} - 690\, p_A^5 + 885T_{4,A} + G_{34A} = 0$
5. 6z (1, 0) is a Nasty Number.

2.4 Method: IV

In place of (9) take 19 as

$$19 = \frac{(17 + i\sqrt{15})(17 - i\sqrt{15})}{16} \qquad (12)$$

The procedure following is same as the method -III, the relating integer solutions to (1) are found as

$$u = \frac{1}{4}\,[17a^4 + 3825b^4 + 1530a^2b^2 + 900ab^3 - 60a^3b] \qquad (13)$$

$$v = \frac{1}{4}\,[a^4 + 225b^4 - 90a^2b^2 - 1020ab^3 + 68a^3b] \qquad (14)$$

In true of (2), the values x and y are

$$x = \frac{1}{4}\,[18a^4 + 4050b^4 - 1620a^2b^2 - 120ab^3 + 8a^3b] \qquad (15)$$

$$y = \frac{1}{4}\,[16a^4 + 3600b^4 - 1440a^2b^2 + 19200ab^3 - 128a^3b] \qquad (16)$$

Since our intension is to find integer solutions, taking a as 4a and b as 4b in (4),(15) and (16), the relating parametric integer values of (1) are found as

$x = x(A, B) = 576A^4 + 129600B^4 - 51840A^2B^2 + 256A^3B - 3840AB^3$
$y = y(A, B) = 512A^4 + 115200B^4 - 46080A^2B^2 - 4096A^3B + 61440AB^3$
$z = 16A^2 + 240B^2$

Observations:

1. $x(1, n) + y(1, n) - 1468800\, f_{4,6}^{n} + 1353600\, p_n^5 - 89280T_{4,n} + G_{1920\,n} \equiv 1 \pmod 2$
2. $x(1, n) - y(1, n) - 345600\, f_{4,5}^{n} - 418560\, p_n^5 - 28800T_{4,n} + G_{12224n} \equiv 1 \pmod 2$
3. $x(q, 1) - 13824\, f_{4,7}^{n} + 2304\, t_{4,q2} + 15616\, p_q^5 + 48064T_{4,q} + G_{13344q} \equiv 1 \pmod 2$
4. $y(q, 1) - 6144\, f_{4,6}^{q} + 12288\, p_q^5 + 42496T_{4,q} - G_{30208q} \equiv 0 \pmod 5$
5. .z (A, A) is a perfect square.

2.5 Method: V

$$\text{Let us take (3) as } u^2 + 15v^2 = 19z^4 * 1 \qquad (17)$$

$$\text{Take 1 as } 1 = \frac{(1 + i\sqrt{15})(1 - i\sqrt{15})}{16} \qquad (18)$$

Using (9) (10) and (14) in (13) and applying factorization process, define
$(u + i\sqrt{15}\, v = (2 + i\sqrt{15})(a + i\sqrt{15}\, b)^2\, \frac{(1 + i\sqrt{15})}{4}$ It furnishes us

$$u = \frac{1}{4}\,[-13a^4 - 2925b^4 + 1170a^2b^2 + 2700ab^3 - 180a^3b] \qquad (19)$$

$$v = \frac{1}{4}\,[3a^4 + 675b^4 - 270a^2b^2 + 780ab^3 - 52a^3b] \qquad (20)$$

In sight of (2), the values of x and y as

$$x = x(a, b) = \frac{1}{4}[-10a^4 - 2250b^4 + 900a^2b^2 + 3480ab^3 - 232a^3b] \quad (21)$$

$$y = y(a, b) = \frac{1}{4}[-16a^4 - 3600a^3b + 1440a^2b^2 + 1920ab^3 - 128a^3b] \quad (22)$$

As our intension is to find integer solutions, taking a as 4A and b as 4B in (4), (21) and (22), the relating parametric integer values of (1) are found as

$x = x(A, B) = -320A^4 - 72000B^4 + 28800A^2B^2 - 7424A^3B + 111360AB^3$

$y = y(A, B) = -512A^4 - 115200B^4 + 46080A^2B^2 - 4096A^3B + 61440AB^3$

$z = 16A^2 + 240B^2$

Observations:

1 .$x(1, B) + y(1, B) - 2246400\, f^{B}_{4,4} + 1152000\, p^{5}_{B} + 285120T_{4,B} + G_{192960B} \equiv 1 \pmod 2$

2 .$x(A, 1) + 7680\, f^{A}_{4,3} - 18688\, p^{5}_{A} - 22976T_{4,A} - G_{56640A} \equiv 1 \pmod 2$

3. $y(A, 1) + 3072\, f^{A}_{4,6} + 5120\, p^{5}_{A} - 49664T_{4,A} - G_{3072A} \equiv 1 \pmod 2$

4. $x(1, A) - y(1, A) - 518400\, f^{A}_{4,4} + 245760\, p^{5}_{A} + 110400T_{4,A} + G_{44864A} \equiv 1 \pmod 2$

5. $\frac{3}{8} z(1, 0)$ is a Nasty number.

3. Conclusion

In this paper, we have observed various process of determining infinitely a lot of non-zero different integer values to the non-homogeneous bi-quadratic Diophantine equation $4(x^2 + y^2) - 7xy = 19z^4$. One may try to find non-negative integer solutions of the above equations together with their similar observations.

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Author Profile

Dr. P. Jayakumar received the B. Sc, M.Sc degrees in Mathematics from Madras University in 1980 and 1983 and the M. Phil, Ph.D degrees in Mathematics from Bharathidasan University , Thiruchirappalli in 1988 and 2010.Who is now working as Professor of Mathematics, Periyar Maniammai University, Vallam, Thanajvur-613 403,Tamil Nadu,India.

V. Pandian received the B.Sc, M.Sc,and MPhil degrees in Mathematics from Bharathidasan University, Thiruchirappalli in 2002, 2004 and 2006. Who is now working as Assistant Professor of Mathematics, A.V.V.M. Sri Pushpam College (Autonomous), Poondi -613 503, Thanajvur. T. N, India.

R. Venkatraman received the B.Sc, M.Sc,and MPhil degrees in Mathematics from Bharathidasan University, Thiruchirappalli in 2002, 2004 and 2006. Who is now working as Assistant Professor of Mathematics, SRM University Vadapalani Campus, Chennai - 600026.